Spectroscopy and truncations from holography and exceptional dualities

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A mi madre, Blanca, Bea y a su ejemplo

Si je dors, qui me donnera la lune?

A. Camus, Caligula

Si no siempre entendidos, siempre abiertos,
o enmiendan, o fecundan mis asuntos;
i en músicos callados contrapuntos
al sueño de la vida hablan despiertos.

F. de Quevedo, Polimnia pp.314

Resumen

Esta tesis explora la relación entre teorías en diferentes dimensiones, centrándose en la física de las compactificaciones de teoría de cuerdas a $D = 4$. Estas teorías se consideran en el límite en que pueden ser descritas por supergravedad en $D = 11$ o por supergravedades de tipo II, y las soluciones estudiadas contienen un factor AdS_4 y son por tanto relevantes en holografía.

Los dos principales objetos de estudio son la espectroscopía de Kaluza-Klein (KK) y los truncamientos consistentes. La primera consiste en el estudio de las caractersticas de las torres infinitas de modos resultantes de la compactificación, cuyas propiedades están controladas por los flujos y la geometra del espacio interno. Las segundas son situaciones en las que estas torres pueden reducirse a un subconjunto finito de modos cuya dinámica viene dada por una supergravedad en cuatro dimensiones que puede ser embebida consistentemente en su contrapartida con dimensiones extra.

Tras discutir compactificaciones en toros como un ejemplo introductorio donde presentar los conceptos relevantes, analizaremos cada uno de estos temas en partes separadas. En la primera parte, explicamos los recientes progresos en la obtención de truncamientos consistentes mediante supersimetría y G-estructuras por un lado, y por otro basándonos en los grupos de dualidad que gobiernan las supergravedades en cuatro dimensiones gracias a las jerarquías tensoriales, las jerarquías de dualidades y *Exceptional Field Theory* (ExFT). La segunda parte aborda la espectroscopía KK y su importancia en holografía. Hasta hace muy poco, sobre soluciones inhomogéneas solamente existían herramientas para estudiar el sector de espín-2. Estas herramientas en combinación con teoría de grupos son utilizadas para examinar la configuración dual a la teoría de campos superconforme que aparece en el infrarrojo de una deformación relevante de la teoría en el interior de una pila de M2 branas. Posteriormente, se muestra cómo estos métodos se pueden extender a campos de menor espín mediante el marco proporcionado por ExFT, y su uso para analizar diferentes clases de soluciones en $D = 10$ y $D = 11$ con interés holográfico.

Abstract

This thesis explores the relation between theories in different dimensions, focusing on the physics of string theory compactifications down to $D = 4$. We consider the limit in which the higher-dimensional theories are described by $D = 11$ or type II supergravity, and the studied solutions contain an AdS_4 factor and are thus relevant for holography.

The two main objects of study are Kaluza-Klein (KK) spectroscopy and consistent truncations. The first consists in the study of the features of the infinite towers of modes resulting from the compactification, whose properties are controlled by the choice of fluxes and geometry on the internal space. The latter are situations in which we can reduce these towers to a finite subset of modes whose dynamics is given by a lower-dimensional supergravity which can be consistently embedded into the higher-dimensional counterpart.

After discussing the compactifications on tori as an introductory example to present the relevant concepts, we will analyse each topic in separate parts. In the first part, we explain how progress has been made in obtaining consistent truncations based on supersymmetry and G-structures, and on the duality groups governing the lower-dimensional supergravities thanks to the tensor and duality hierarchies and Exceptional Field Theory (ExFT). The second part addresses KK spectroscopy and its importance to holography. Until very recently, on non-homogeneous solutions only tools to study the spin-2 sector were available. These tools are combined with group theory to examine the configuration dual to the IR SCFT of a relevant deformation of the theory in the worldvolume of a stack of M2 branes. Subsequently, we show how these methods can be extended to lower-spin fields within the ExFT framework, and use them to analyse different classes of solutions in $D = 10$ and $D = 11$ of holographic interest.

Preface

This thesis is based on the following research articles that I published during my PhD, together with Mattia Cesàro, Kevin Dimmitt, Praxitelis Ntokos, and Óscar Varela.

[A] **G. Larios**, P. Ntokos, and O. Varela,
Embedding the SU(3) sector of SO(8) supergravity in $D = 11$,
Phys. Rev. **D100** (2019) no. 8 086021 [arXiv:1907.02087].

[B] **G. Larios** and O. Varela,
Minimal $D = 4$*,* $\mathcal{N} = 2$ *supergravity from* $D = 11$*: An M-theory free lunch*,
JHEP **10** (2019) 251, [arXiv:1907.11027].

[C] K. Dimmitt, **G. Larios**, P. Ntokos, and O. Varela,
Universal properties of Kaluza-Klein gravitons,
JHEP **03** (2020) 039, [arXiv:1911.12202].

[D] M. Cesàro, **G. Larios**, and O. Varela,
A Cubic Deformation of ABJM: The Squashed, Stretched, Warped, and Perturbed Gets Invaded,
JHEP **10** (2020) 041, [arXiv:2007.05172].

[E] M. Cesàro, **G. Larios**, and O. Varela,
Supersymmetric spectroscopy on $AdS_4 \times S^7$ *and* $AdS_4 \times S^6$,
JHEP **07** (2021) 094, [arXiv:2103.13408].

[F] M. Cesàro, **G. Larios**, and O. Varela,
The spectrum of marginally-deformed $\mathcal{N} = 2$ *CFTs with* AdS_4 *S-fold duals of type IIB*,
JHEP **12** (2021) 214, [arXiv:2109.11608].

During this period I also completed the following publication with Camille Eloy and Henning Samtleben,

[G] C. Eloy, **G. Larios**, and H. Samtleben,
Triality and the consistent reductions on $AdS_3 \times S^3$,
JHEP **01** (2022) 055, [arXiv:2111.01167],

and performed work in collaboration with Mattia Cesàro, Jim Liu, Emanuel Malek, Henning Samtleben, Christoph Uhlemann, Valentí Vall Camell and Óscar Varela. Some of these unpublished results are mentioned in the text, and they will appear in

[H] M. Cesàro, **G. Larios**, and O. Varela,
The spectrum of $\mathcal{N} = 1$ *S-fold families in type IIB*, *to appear.*

[I] M. Cesàro, **G. Larios**, and O. Varela,
Uplifting the $\mathrm{SO}(3) \times \mathbb{Z}_2$ *sector of SO(8) supergravity in* $D = 11$,
to appear.

[J] **G. Larios**, J. Liu, E. Malek, H. Samtleben, C. Uhlemann, V. Vall Camell,
AdS_6 *spectra from Exceptional field theory*, *to appear.*

Agradecimientos & Acknowledgements

This thesis would not have been possible without the constant support of many people. I will doubtlessly start thanking Óscar, whose continuous encouragement and inspiration have been truly invaluable. Among many other things, you have taught me to appreciate and how to do high quality science, and I will try never forget that I should always avoid epicycles.

I also want to express my dearest gratitude to Praxitelis and Kevin for that many super interesting discussions and lovely projects with sleepless trans-oceanic nights, and very specially to Mattia. È stato un onore godere della tua eccezionale amicizia, lavorare con te e fare in modo che gli spectra non siano fantasmi.

My home has been very important during these four years. IFT is the best place one could think of to carry out research. I have benefited a lot from every comment of Pepe Barbón, María José Rodriguez, Roberto Emparan, Karl Landsteiner, Matteo Baggioli and Daniel Areán and the inquiries of César Gómez at the Holoclubs, and from every conversation with Carlos Shahbazi and Tomás Ortín about (super)gravity. From Ángel Uranga and Carlos Pena, I value as highly all the physics I have learnt as much as the importance and joy of outreach (and how mentioning the albigensian crusade or the taxicab number can be a most enjoyable way of conveying many concepts). Among the people who form or used to form the IFT, I want to particularly mention the cohort of fellow students who have struggled with me all these years. Muchas gracias, Guille, Fer, Francesca, Uga y Gallego por las comidas en el CBM hablando de lo humano y lo divino. Muy en especial a Álvaro, por haberme acompañado en esos primeros meses en los que todo sonaba a chino, y a Judit, Salva, Ángel y Raquel por tantos cafés como sonrisas. También merecen una mención destacada Martín, de quien he aprendido desde complejidad cuántica a cómo dar un discurso de graduación o hacer que un journal club no implosione; mis (brevemente) compañeros de la cuarta planta David, Roberta y Sergio; y David, con quien estoy seguro de que volveré a compartir despacho y discutir encarnizadamente sobre si x idioma debería perder y consonantes o z acentos. Sin duda, hablando del IFT, tengo que cerrar con quienes hacen que todo vaya como la seda. Muchas gracias, Isabel, Rebeca, Mónica y Emilio por todo vuestro trabajo que hace posible el de todos los demás.

Part of this thesis has been done away from Madrid, and I want to express my deep gratitude to the people who have made it possible. My time at ENS de Lyon was really enjoyable despite its unfortunate stop due to the pandemic, and I want to thank Henning Samtleben for his hospitality and all the inspiring conversations we have had since then. Aussi je veux exprimer ma reconaissance à Camile Eloy. Il a été une vraie joie travailler et apprendre de toi. My time in Ann Arbor was equally lovely, and this was because the real kindness of Jim Liu. Deep recognition is also for Christoph Uhlemann, Leo Pando-Zayas, Henriette Elvang and all the students there, specially Marina and Robbie, for making me feel part of the group from the very first moment. In these two stays, there were also people who made me feel literally at home while I was away. Merci beaucoup, Claire, pour ta douce gentillesse essayant de comprendre mon pauvre français, and to the Kunselmans. If everybody were the way you are, the world would be a much better place.

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No puedo sino terminar dando gracias a una brujita que debería aparecer como coautora, ya que le debo cada palabra aquí escrita. $\alpha\varepsilon\acute{\iota}$.

Contents

Invitation

Modern physics is based on two solid pillars describing the four known forces of Nature. On the one hand, at large distances everything is dominated by gravity. This is the case because the other forces, although individually much stronger, can be both repulsive and attractive, and over large distances their net effects cancel. At the classical level, this interaction is described by Einstein's General Relativity, which identifies gravity with the geometry of the spacetime in which the rest of the matter lives.

On the other hand, microscopic phenomena belong to the quantum realm, whose most powerful description is quantum field theory. This language is the relevant tool to describe the physics over several orders of magnitud in distance and energy, from quantum computers to particle physics. In the theory of fundamental interactions, the main exponent is the Standard Model, which combines the forces describing atoms and nuclei into a single framework.

Both theories have been tested to agree with experiments at unprecedented levels. For quantum field theory, most predictions of the Standard Model have been verified. This ranges from the existence of the Higgs boson and the value of its mass [1, 2], to the value of quantities like the anomalous magnetic dipole of the electron, which has been checked to differ from the experimental data in less than one part in a billion [3]. Similarly, on the gravity side both the prediction of exotic phenomena and the precision checks have been verified. Among the recently achieved experimental findings predicted decades in advance, the photography of black holes by the Event Horizon Telescope [4] and detection of gravitational waves by the LIGO and Virgo spectrometers [5] stand out.

However, the puzzle of reality is not complete. There are certain extreme situations that require the combination of gravity and quantum mechanics, such as the physics of black holes or the entire universe close to the Big Bang. Nevertheless, General Relativity and quantum mechanics cannot be easily reconciled. Furthermore, apart from pieces that do not cope well with one another, there are entire sectors which are missing. The matter described by the Standard Model constitutes only 5% of what we think the Universe is currently formed by, with the remaining 95% being components that we only know by their gravitational interaction, which we call dark matter and energy [6, 7]. The theory that would allow us to understand these unknowns is known as Quantum Gravity, and finding its precise shape is one of the main objectives of theoretical physics today.

String theory [8–13] is currently our best candidate for such a theory. Its name originates from its weakly coupled limit, in which the relevant dynamical objects are one-dimensional cords characterised by a tension $T = (2\pi\alpha')^{-1}$. The theory on the worldsheet that these objects sweep as they propagate leads to an emergent notion of spacetime and includes gravity, which was not a priori required to be part of the theory. Surprisingly, quantum consistency of the two-dimensional theories describing these tiny strings is extremely restrictive, demanding that the strings propagate in ten dimensions, and providing only five examples, named type I, IIA, IIB, Heterotic $\mathrm{E}_8 \times \mathrm{E}_8$ and Heterotic SO(32), whose spectrum includes fermions. These five theories turn out to have the precise spectrum and interactions to enjoy a very powerful symmetry relating bosons and fermions, known as supersymmetry. Moreover, these superstring theories are strongly suspected to be UV-finite, and their low energy limits are described by ten-dimensional supergravities, which are theories combining General Relativity with supersymmetry.

A surprising fact is that all these theories, despite constructed independently, know from one another. For example, type IIA string theory compactified on a circle of radius R yields exactly the same predictions as type IIB string theory on a circle of radius α'/R, and type IIB string theory at weak coupling is expected to be equivalent to its strongly coupled regime. These relations, known respectively as T- and S-dualities, are the fundamental examples of string dualities, which provide links connecting all five string theories.

Another case in which string theory is understood follows from taking two consecutive limits. If one takes the strongly coupled limit of type IIA string theory, one can infer the existence of a theory in eleven dimensions which contains membranes but no strings. This theory, known as M-theory, therefore has the unique eleven-dimensional supergravity as its low-energy limit. Eleven-dimensional supergravity compactified on a circle successfully recovers type IIA supergravity. Moreover, if we compactified it on an interval, we would recover the dynamics of heterotic $\mathrm{E}_8 \times \mathrm{E}_8$ supergravity. All these interrelations suggest that the different string theories and eleven dimensional supergravity are only different limits of a single theory, that is often referred to as a whole as M-theory as well. A pictorial representation of this duality network can be found in figure I.1a.[1] This figure also shows a distinguishing feature of eleven-dimensional supergravity and the type II theories: they enjoy as much supersymmetry as a theory can have, whilst the other three

[1] Away from the boundaries in figure I.1a, it is expected that different extended objects apart from strings contribute in equal footing. This has led some authors to refer to the complete theory as "The theory formerly known as strings" (TTFKAS) [14, 15]. We will however use the names string/M-theory understood lato sensu.

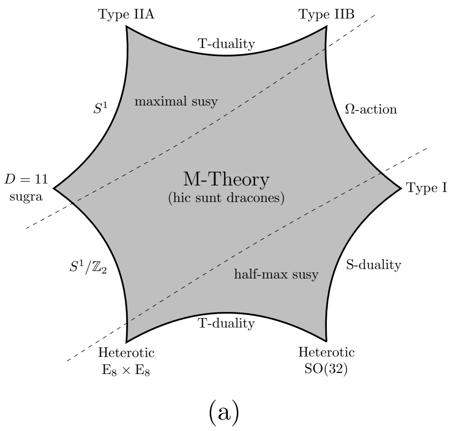


Figure I.1: (a) All the different supersymmetric string theories and eleven-dimensional supergravity, and their web of dualities leading to the notion of M-theory. The edges of the star contain information about the relation between adjacent nodes, with Ω denoting the orientifold action and all the other links mentioned in the text. (b) Schematic picture of the AdS/CFT correspondence, where the operators in the boundary CFT are dual to fields propagating in the AdS bulk.

theories are only half-maximal. In this work, we will devote ourselves to the maximal case.

Another duality found in string theory has deserved much attention in the recent years. It has a different flavour as compared to the other string dualities, as instead of relating different string theories, it equates string theory to quantum field theories without gravity. This duality is usually referred to as AdS/CFT correspondence or gauge/gravity duality, and it is holographic in nature: it states that certain string theory solutions with a $d + 1$ dimensional anti-de Sitter (AdS) factor are exactly equivalent to field theories with conformal symmetry (CFTs) in one dimension less sitting at the conformal boundary of AdS, as schematically represented in figure I.1b. The utility of this duality is manifold. From the computational side, given that it relates the limit in which we have good control of one theory with the one in which we hardly have tools to address the other, it provides new directions to approach situations where perturbation theory breaks down. Prominent examples of these applications are the use of simple gravity models to study the quark-gluon plasma in the Standard Model or systems in condensed matter with strongly correlated electrons, as high-temperature superconductors. On the other hand, it has also provided insight into deep conceptual issues in black hole evaporation, providing convincing evidence that evolution must be unitary in spite of the naïve expectation from Hawking's evaporation.

Apart from the abundance of dualities, string theory is characterised by the presence of extra dimensions. Contrary to previous proposals displaying this feature, in string theory the dimension of spacetime is not an arbitrary parameter, but the aforementioned quantum consistency requirements demand a precise value. To make contact with physics in lower dimensions, it is therefore necessary to specify what happens with the extra ones. The most common mechanism is known as compactification and lies in assuming that spacetime can be divided into external and internal spaces, and the latter form a compact manifold.

From a lower-dimensional perspective, the complete dynamics of the higher-dimensional theory can be reformulated in terms of objects that only depend on the external coordinates. Each of the different higher dimensional fields can be written as an infinite sum of lower-dimensional ones times specific functions describing vibrations in the internal manifold. For each vibrational mode, the associated fields will carry a different mass. As we will discuss in detail, each of these massive excitations, known as Kaluza-Klein (KK) modes, enjoys a very precise understanding in the the CFT side if we choose our external manifold to be anti-de Sitter.

Nonetheless, for most purposes, we would like to restrict ourselves to a finite subset of this infinite tower of Kaluza-Klein excitations. This reduction is non-trivial, as one cannot freely set some modes to zero while keeping others and still satisfy the required equations of motion. When it is possible, one obtains a consistent truncation that guarantees that every solution of the lower-dimensional theory, usually a gauged supergravity which combines General Relativity with gauge theories that generalise the Standard Model, provides a solution of the higher-dimensional counterpart. However, despite desirable, until very recently only a handful of non-trivial examples of consistent truncations were known.

The research in this work is framed in the intersection between holographic and string dualities, and focuses on their relevance in the physics of the Kaluza Klein modes and the existence of consistent truncations. To motivate a bit further the actual problems that this work addresses and in order to introduce some concepts in the simplest context, let us finish this invitation with a discussion of the most paradigmatic case where they make appearance: the toroidal compactification of $D = 11$ supergravity.

$D = 11$ supergravity on T^n

Let us recall the field content and dynamics of $D = 11$ supergravity [16], following the conventions of [17]. The bosonic field content includes the metric $ds^2_{11} = g_{MN}dx^M dx^N$, $M = 0, \ldots, 10$, and a three-form potential $A_{(3)}$ with four-form field strength $G_{(4)} = dA_{(3)}$. The fermionic sector is simply comprised by a Majorana gravitino ψ_M. In the following, we will consider configurations in which the gravitino vanishes, and the remaining fields are governed by the field equations

$$
\begin{aligned}
dG_{(4)} &= 0\,, \\
d \star_{11} G_{(4)} + \frac{1}{2} G_{(4)} \wedge G_{(4)} &= 0\,, \\
R_{MN} - \frac{1}{12}\left[G_{MPQR} G_N{}^{PQR} - \frac{1}{12} G^2 g_{MN}\right] &= 0\,,
\end{aligned}
\tag{I.1}
$$

where the first relation is the Bianchi identity for $G_{(4)}$, and the other two are the equations of motion that stem from the Lagrangian

$$
\mathcal{L}_{11} = R\,\mathrm{vol}_{11} - \frac{1}{2}\, G_{(4)} \wedge \star_{11} G_{(4)} - \frac{1}{6}\, A_{(3)} \wedge G_{(4)} \wedge G_{(4)}\,. \tag{I.2}
$$

The complete action (including fermions) is invariant under local supersymmetry. On bosonic backgrounds, the variations of the metric and three-form identically vanish. On the other hand, the gravitino variation reads

$$
\delta_\epsilon \Psi_M = \nabla_M \epsilon + \frac{1}{288}\left(\Gamma_M{}^{SPQR} - 8 \delta_M^S \Gamma^{PQR}\right) G_{SPQR}\epsilon\,, \tag{I.3}
$$

where ϵ is Majorana supersymmetry generator and $\Gamma^{A_1 \dots A_n}$ are the Dirac matrices and their antisymmetrised products. In (I.3), these matrices appear contracted with a local orthonormal frame $e^A{}_M$ in terms of which the metric can be written as

$$
g_{MN} = \eta_{AB}\, e^A{}_M e^B{}_N\,, \tag{I.4}
$$

with η_{AB} the mostly plus eleven-dimensional Minkowski metric. In flat indices, $A = 0, \dots, 10$, the Dirac matrices satisfy the Clifford algebra

$$
\{\Gamma_A, \Gamma_B\} = 2\,\eta_{AB}\,, \tag{I.5}
$$

and

$$
\Gamma_0 \dots \Gamma_{10} = 1\,. \tag{I.6}
$$

As argued before, the spacetimes of interest in string theory are most often factorised into a d-dimensional external space and an internal n-dimensional compact manifold. In the eleven dimensional case, $n = 11 - d$. The simplest example of this is the internal space having the topology of a torus,

$$
T^n = \underbrace{S^1 \times \cdots \times S^1}_{n}\,, \tag{I.7}
$$

so that the coordinates can be split into $x^M = (x^\mu,\, y^m)$, with $\mu = 0, \dots, d$ and $m = 1, \dots, n$, and the y^n chosen to be periodic. The eleven-dimensional fields can then be written in terms of indices that break the manifest higher-dimensional Lorentz invariance as

$$
\begin{array}{ccc}
\mathrm{GL}(d+n, \mathbb{R}) & \to & \mathrm{GL}(d, \mathbb{R}) \times \mathrm{SL}(n, \mathbb{R}) \\
\{x^M\} & \to & \{x^\mu,\ y^m\}\,,
\end{array}
\tag{I.8}
$$

in terms of which the bosonic fields can be written as

$$\begin{aligned}
d\hat{s}_{11}^2 &= \Delta^{-1} g_{\mu\nu} dx^\mu dx^\nu + g_{mn}(dy^m + B^m)(dy^n + B^n)\,, \\
\hat{A}_{(3)} &= \tfrac{1}{6} A_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho + \tfrac{1}{2} A_{\mu\nu m} dx^\mu \wedge dx^\nu \wedge (dy^m + B^m) \\
&\quad + \tfrac{1}{2} A_{\mu mn} dx^\mu \wedge (dy^m + B^m) \wedge (dy^n + B^n) \\
&\quad + \tfrac{1}{6} A_{mnp} (dy^m + B^m) \wedge (dy^n + B^n) \wedge (dy^p + B^p)\,,
\end{aligned} \tag{I.9}$$

with $\Delta(x, y)$ a function introduced for later convenience. We will refer to this way of writing higher-dimensional fields in terms of fields with a lower-dimensional tensor structure as a KK factorisation or KK ansatz. Notice that we have not restricted any coordinate dependence, and all fields are understood to depend on the full set of coordinates. As the internal coordinates label the circle directions in (I.7), we can encode this dependence as an expansion of the fields into Fourier modes, *e.g.*

$$\hat{A}_{\mu\nu\rho}(x, y) = \sum_{k \in \mathbb{Z}^n} \hat{A}^{(k)}_{\mu\nu\rho}(x)\, e^{i k_m y^m}\,. \tag{I.10}$$

Thus, as mentioned before, we can trade higher-dimensional fields into an infinite number of lower-dimensional ones once we know a complete basis of functions on the internal space.

The lower-dimensional fields, known as KK modes, inherit their properties from the with which they appear. For example, in our toroidal compactification the Fourier basis is organised in representations of $\mathrm{U}(1)^n$, and the fields therefore carry $\mathrm{U}(1)^n$-charges. In this case, the functions are also harmonic, and this endows the different KK fields with d-dimensional masses dictated by them. Schematically,

$$\partial^M \partial_M \phi(x, y) = \sum_k (\partial^\mu \partial_\mu - |k|^2) \phi^{(k)}(x)\, e^{i k_m y^m} \tag{I.11}$$

for ϕ any of the fields in (I.9). In more general compactifications, it is therefore convenient to require that the functions controlling the KK tower also be harmonic. These masses and charges for the full set of modes are interesting from different perspectives. First, from a phenomenological point of view, these modes could correspond to actual particles beyond the Standard Model in large compact dimensions scenarios. In our case, our main interest in them is going to originate from the AdS/CFT correspondence, as will be explained in more detail in chapter 4.

In many situations, we nevertheless want to restrict ourselves to a finite subset of fields within infinite-dimensional KK tower, *i.e.* we want to split the entire set as $\{\varphi_{\text{kept}}, \phi_{\text{trunc}}\}$ and set $\phi_{\text{trunc}} = 0$. Obtaining a truncation such that every solution of the equations for the reduced set of fields provides a solution of the full set of equations requires that the fields retained do not source the truncated ones. From an action perspective, this means that there is no term linear in ϕ_{trunc} in the Lagrangian.

On generic grounds, this is a very subtle requirement. However, in the toroidal compactification the underlying group theory makes it possible to truncate all fields in (I.10) with $k_m \neq 0$, while retaining the ones with zero charge under $\mathrm{U}(1)^n$. Consistency is guaranteed because we are keeping all singles under this group while discarding every non-singlet, and it is not possible to source non-singlets out of singlets for any group. One can also check that this truncation is compatible with supersymmetry, and therefore, through the ansatz (I.9), one can obtain solutions of $D = 11$ supergravity out of solutions of maximal supergravities in lower dimensions. In fact, the action of maximal supergravity with no gauging in $D = 4$ was obtained out of (I.2) by means of reduction on T^7 [18]. Let us finish this invitation with a brief recollection of how these supergravities are obtained and their duality groups identified.

If we reduce on a single circle, (I.9) simplifies into

$$\begin{aligned} d\hat{s}^2_{11} &= e^{\phi/6} g_{\mu\nu} dx^\mu dx^\nu + e^{-4\phi/3}(dy + B)^2 \,, \\ \hat{A}_{(3)} &= \tfrac{1}{6} A_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho + \tfrac{1}{2} A_{\mu\nu\, 1}\, dx^\mu \wedge dx^\nu \wedge (dy + B) \,, \end{aligned} \tag{I.12}$$

where we have chosen the warp factor $\Delta = e^{-\phi/6}$ so that (I.2) leads to a ten-dimensional action in the Einstein frame. This is the choice we make throughout this work, which more generally amounts to choosing Δ in (I.9) as

$$\Delta = \left(\frac{\det g_{mn}}{\det \mathring{g}_{mn}}\right)^{1/(d-2)} , \tag{I.13}$$

with $\mathring{g}_{mn}$ the metric obtained when setting all scalars to zero. From a ten-dimensional perspective, the M-theory fields in (I.12) are given in terms of a ten-dimensional metric, a vector, a scalar, a three form and a two-form. As is well-known, this is the bosonic field content of type IIA supergravity in $D = 10$. These fields carry a definite weight under $\mathrm{GL}(1, \mathbb{R})$, and the scalar can be thought to parametrise this scaling. Let us also point out that the two-forms in (I.12) are physical and cannot be set to zero despite the gauge freedom of $\hat{A}_{(3)}$, as they can be thought as Wilson lines on S^1. For the nine-dimensional theory obtained by reduction of $D = 11$ supergravity on T^2, we have the same types of forms, now transforming in representations of $\mathrm{GL}(2, \mathbb{R})$, and the scalars can be thought as coordinates on $\mathrm{GL}(2, \mathbb{R})/\mathrm{SO}(2)$.

More generally, from a d-dimensional perspective and once the y-dependence

d	n	$\mathrm{E}_{n(n)}$	$K(\mathrm{E}_{n(n)})$
8	3	$\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(3,\mathbb{R})$	$\mathrm{SO}(2) \times \mathrm{SO}(3)$
7	4	$\mathrm{SL}(5,\mathbb{R})$	$\mathrm{SO}(5)$
6	5	$\mathrm{Spin}(5,5)$	$\mathrm{SO}(5) \times \mathrm{SO}(5)$
5	6	$\mathrm{E}_{6(6)}$	$\mathrm{USp}(8)$
4	7	$\mathrm{E}_{7(7)}$	$\mathrm{SU}(8)$
3	8	$\mathrm{E}_{8(8)}$	$\mathrm{SO}(16)$

Table 1: Maximal supergravity $\mathrm{E}_{n(n)}$ duality groups and their respective maximal compact subgroups.

in (I.9) is dropped, the fields entering the metric and three-form are

$$\begin{aligned} g_{\mu\nu}(x) \quad &: \quad \text{metric}\,, \\ B_\mu^m(x),\ A_{\mu np}(x) \quad &: \quad \text{vectors}\,, \\ A_{\mu\nu p}(x),\ A_{\mu\nu\rho}(x) \quad &: \quad \text{two- and three-forms}\,, \\ g_{mn}(x),\ A_{mnp}(x) \quad &: \quad \text{scalars}\,. \end{aligned} \tag{I.14}$$

These fields can be understood to furnish different $\mathrm{GL}(11-d,\mathbb{R})$ representations, as for the previous reductions. However, the complete duality groups for the maximal supergravities arising from toroidal reduction of M-theory are bigger than this for $d \leq 8$, as hinted by the appearance of the axionic scalar A_{mnp} outside the $\mathrm{GL}(n,\mathbb{R})/\mathrm{SO}(n)$ coset. The duality group in this cases enhances to the $\mathrm{E}_{n(n)}$ family, where the notation denotes that they are the maximally non-compact (split) real form of the associated complex algebras. The associated scalar manifolds are then cosets of these duality groups by their maximal compact subgroup. A summary of these groups is given in table 1.

For $d \leq 6$, some of these forms need to be Hodge-dualised into lower rank fields to complete representations of $\mathrm{E}_{n(n)}$. In particular, for $d = 5$

$$\begin{aligned} g_{\mu\nu}(x) \quad &: \quad \text{metric}\,, \\ B_\mu^m(x),\ A_{\mu np}(x),\ \tilde{A}_{\mu p}(x) \quad &: \quad \text{vectors}\,, \\ g_{mn}(x),\ A_{mnp}(x),\ \tilde{A}_0(x) \quad &: \quad \text{scalars}\,, \end{aligned} \tag{I.15}$$

with

$$d\tilde{A}_p = *dA_p\,, \qquad d\tilde{A}_0 = *dA \tag{I.16}$$

in terms of the two- and three-forms in (I.14), and $*$ the $d = 5$ Hodge operator. Here, $p = 1, \ldots, 6$, and (I.15) therefore amounts to 27 vectors in the $\mathbf{6} \oplus \mathbf{15'} \oplus \mathbf{6}$ of $\mathrm{SL}(6,\mathbb{R})$, and 42 scalars in the $\mathbf{20} \oplus \mathbf{1}$ plus $\mathrm{GL}(6,\mathbb{R})/\mathrm{SO}(6)$, as appropriate for their respective branchings from the $\mathbf{27'}$ of $\mathrm{E}_{6(6)}$ and the $\mathrm{E}_{6(6)}/\mathrm{USp}(8)$ scalar manifold.

Correspondingly, in $d = 4$ the fields are

$$\begin{aligned} g_{\mu\nu}(x) \quad &: \quad \text{metric}\,, \\ B_\mu^m(x),\ A_{\mu np}(x) \quad &: \quad \text{vectors}\,, \\ g_{mn}(x),\ A_{mnp}(x),\ \tilde{A}_p(x) \quad &: \quad \text{scalars}\,, \end{aligned} \tag{I.17}$$

with $p = 1, \ldots, 7$. In this case, the three-form $A_{\mu\nu\rho}$ in (I.14) does not carry degrees of freedom, and $\tilde{A}_p$ are the axions dual to the two-form $A_{\mu\nu p}$. This totals 70 scalars in $\mathrm{GL}(7,\mathbb{R})/\mathrm{SO}(7)$ plus the $\mathbf{35} \oplus \mathbf{7}$ of $\mathrm{SL}(7,\mathbb{R})$, as appropriate for fields in the $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$ coset. The counting of the vectors is more subtle due to electric-magnetic duality in even dimensions. In (I.17), and thus in the action, there are 28 of them in the $\mathbf{7'} \oplus \mathbf{21}$, which is half of the number required for $\mathrm{E}_{7(7)}$ covariance. The missing 28 vectors can be understood as the magnetic duals of the ones appearing here. We will dwell at length into this point in chapter 3.

In this simple example we have observed the appearance of towers of KK modes upon compactification, and the possibility of setting to zero an infinite number of these fields while retaining a finite choice of them in a way consistent with the equations of motion of $D = 11$ supergravity. Although these two topics are very much interrelated, we have decided to treat them separately in the following for the sake of clarity. The analysis of consistent truncations will be carried out in Part I in the light of supersymmetry, dualities, and the holography of relevant deformations in the dual CFTs. In Part II, we will analyse perturbations on top of solutions of type II and eleven-dimensional supergravity. Many of these higher-dimensional solutions have been constructed by uplifting solutions of the consistent truncations discussed in the former part, and the analysis of the spectrum takes advantage of this fact and from a duality-covariant reformulation of the higher-dimensional theories known as Exceptional Field Theory. We will end this dissertation with some comments on applications and future directions and relegate some further discussion on technical details to five appendices.

Part I

Consistent Truncations

Chapter 1

Introduction: The road to consistency

The construction of String/M-theory solutions with non-trivial fluxes and reduced (super-)symmetry at the ten- or eleven-dimensional level is a monumental task. Moreover, in many cases we are not interested in the dynamics of the full set of modes, but only in a restricted subset, as might be the reduction to lower-dimensional configurations discussed in the invitation. In general, however, the reduction of the equations of motion to such a finite subset of modes, despite desirable, is very far from trivial due to the highly non-linear nature of the equations.

Traditionally, group theory has enjoyed a prominent rôle in these truncations, as already encountered in the toroidal case and discussed above (I.12). This idea in the T^n reduction can be generalised to other group manifolds in what are known as deWitt reductions [19]. If the internal manifold is chosen to be a non-abelian Lie group G, *e.g.* $G = \mathrm{SU}(2) \simeq S^3$, the isometry group is $G_{\mathrm{left}} \times G_{\mathrm{right}}$, with one factor acting on the group element from the left and the other from the right. Using the same logic as before, we can compactify a higher-dimensional supergravity on this manifold and expand the higher-dimensional fields into a tower of KK modes carrying representations of $G_{\mathrm{left}} \times G_{\mathrm{right}}$[1] and discard all modes which transform non-trivially under one of the factors, say G_{right}. The retained fields, in spite of being singles under G_{right}, need not be singlets under G_{left}, and this leads to interesting gauged supergravities in lower dimensions.

One could also try to perform this exercise on other seemingly simple spaces with non-trivial symmetry groups, such as spheres other than S^1 or S^3 (as these two spheres are the only cases that are group manifolds).

[1] Strictly speaking, the KK modes furnish representations of the symmetry group preserved by both the supergravity metric and fluxes, which is generically smaller than the isometry group. Here, for simplicity, we assume that the (possibly zero) fluxes do not break any symmetries.

From a group theoretic perspective, an n-sphere can be understood as a homogeneous space

$$S^n \simeq \frac{\mathrm{SO}(n+1)}{\mathrm{SO}(n)}\,. \tag{1.1}$$

and on a maximally symmetric solution all fields can be taken to transform in representations of $\mathrm{SO}(n+1)$. If we keep all singlets under any subgroup of the isometry group, the truncation is again guaranteed to be consistent [20]. We could try to perform a truncation retaining fields that transform non-trivially under this $\mathrm{SO}(n+1)$ so that it becomes the gauge group of the lower dimensional supergravity. However, this kind of truncations, know as Pauli reductions [21, 22], are generically inconsistent as the coordinate dependence does not factorise [23, 24]. The problem is easy to see: starting from the ansatz (I.9), the external components of the Einstein equation are, in general,

$$R_{\mu\nu} - \tfrac{1}{2}\, g_{\mu\nu} R = \tfrac{1}{2}\, Y^{IJ}\, T(A)_{IJ\,\mu\nu} - \Lambda\, g_{\mu\nu} \tag{1.2}$$

for a solution dependent cosmological constant, as we have chosen Δ so that (1.2) is in Einstein frame. Here, $T(A)_{IJ\,\mu\nu}$ is the appropriate stress-energy tensor for the Yang-Mills field strengths associated to the gauge vectors A^I, and the tensor Y^{IJ} is a theory-dependent combination of the internal Killing vectors and their derivatives. Here, the internal index I now runs over $\mathbf{adj} \oplus \ldots$ representations of the gauge group. In simple terms, the obstacle is that the tensor $Y^{IJ}(y)$ generically depends on the internal coordinates.

Nevertheless, in a number of cases one succeeds in this remarkable exercise, with the truncations to gauged maximal supergravities in $D = 4$ and 5 being some prominent instances [25–28]. When this is possible, the lower-dimensional gauged supergravities can be regarded as efficient ways of probing the higher-dimensional counterparts, as every stationary point of their potentials will correspond, by the consistency of the truncation, to a higher-dimensional solution.

These consistent truncations to lower-dimensional maximal supergravities of $D = 11$ or type II supergravities can be understood as turning on the supergravity multiplet on top of maximally supersymmetric configurations of the higher-dimensional theories. From this point of view, the KK ansatz in (I.9) can be understood as a proper embedding of the (suitably dualised) fields of maximal $D = 4$ supergravity on top of the vacuum

$$d\hat{s}^2_{11} = ds^2(\mathbb{R}^{1,3}) + \delta_{mn} dy^m\, dy^n\,, \qquad F_{(4)} = 0\,. \tag{1.3}$$

Similarly, we can ask whether it is possible to embed other supergravities on top more non-trivial solutions with a lower amount of preserved supersymmetry. If the solution upon which we want to build can be found in the potential of a maximal supergravity, the corresponding embedding of the associated $\mathcal{N} < 8$ supergravity also follows from the maximal truncation from a suitable dismissal of part of the $\mathcal{N} = 8$ fields. On the other hand, we expect these truncations to exist also on top of solutions which do not arise from a lower-dimensional maximal supergravity.

In the two remaining chapters of this part, we are going to be interested in truncations based on configurations that also split the eleven-dimensional spacetime as

$$\begin{array}{ccc} \mathrm{GL}(11,\mathbb{R}) & \to & \mathrm{GL}(4,\mathbb{R}) \times \mathrm{SL}(7,\mathbb{R}) \\ \{x^M\} & \to & \{x^\mu,\ y^m\}\,, \end{array} \tag{1.4}$$

with $\mu = 0, \ldots, 3$ and $m = 1, \ldots, 7$ in line with (I.8). These configurations are going to be taken to be maximally symmetric solutions of M-theory such that the external factor is four-dimensional anti-de Sitter, AdS_4, and the seven-dimensional internal space will be required to be compact. Then, instead of (1.3), we will consider an eleven-dimensional metric of the form

$$d\hat{s}^2_{11} = e^{2A} ds^2(\mathrm{AdS}_4) + g_{mn}(y) dy^m\, dy^n\ , \tag{1.5}$$

with $ds^2(\mathrm{AdS}_4)$ having unit radius and $A(y)$ a warp factor which only depends on the internal coordinates. The four-form accordingly factorises as

$$G_{(4)} = m\,\mathrm{vol}_4 + \tfrac{1}{4!} F_{mnpq}\, dy^m \wedge dy^n \wedge dy^p \wedge dy^q\ , \tag{1.6}$$

with the Freund-Rubin term involving the volume form associated to the corresponding external metric, and the magnetic component, F, carrying dependence on the internal coordinates only.

When we turn on the four-dimensional fields, they deform (1.5) and (1.6) in different ways. The four-dimensional metric simply replaces the vacuum $ds^2(\mathrm{AdS}_4)$ factor. In turn, $D = 4$ the scalars modify the warping, the Freund-Rubin factor and the internal metric and four-form. Finally, as happened for the maximal toroidal reduction in (I.9), the lower-dimensional vectors fibre the internal manifold over the external space and can also enter through their field strengths in the eleven-dimensional four-form.

Under the 4+7 dimensional splitting, fermions also need to be repackaged. As mentioned in the invitation, the two relevant spinors in M-theory are the gravitino, Ψ_M, and the supersymmetry generator, ϵ. Both of them are Majorana spinors in $D = 11$, with the former also carrying a vector index. The Majorana spinor index becomes under (1.4) a product of $D = 4$ and $D = 7$ spinor indices. Accordingly, it is useful to decompose the elements of the Clifford algebra as

$$\Gamma_\alpha = \rho_\alpha \otimes \mathbb{1}\ , \qquad\qquad \Gamma_a = \rho_5 \otimes \gamma_a \tag{1.7}$$

in terms of $D = 4$ Dirac matrices, ρ_α, $\alpha = 0, \ldots, 3$, with $\rho_5 = i\rho_0\rho_1\rho_2\rho_3$; and their $D = 7$ counterparts γ_a, $a = 1, \ldots, 7$. These matrices satisfy the Clifford algebras

$$\{\rho_\alpha, \rho_\beta\} = 2\,\eta_{\alpha\beta}\ , \qquad \{\gamma_a, \gamma_b\} = 2\,\delta_{ab}\ , \tag{1.8}$$

with $\eta_{\alpha\beta}$ the four-dimensional mostly plus Minkowski metric and δ_{ab} the Euclidean metric. The chirality matrix satisfies $(\rho_5)^2 = \mathbb{1}$.

When a $D = 11$ fermion is expressed in this spitting, the $D = 11$ Majorana index $\hat{A} = 1, \ldots, 32$ factorises as a product $\hat{A} = (\hat{a}, I)$, with $\hat{a} = 1, \ldots, 4$ and $I = 1, \ldots, 8$ Dirac indices in $D = 4$ and $D = 7$, respectively. The latter can also be thought as a fundamental SU(8) index thanks to the inclusion

$$\begin{array}{ccccc} \mathrm{SU}(8) & \supset & \mathrm{SO}(8) & \supset & \mathrm{SO}(7) \\ \mathbf{8} & \rightarrow & \mathbf{8}_s & \rightarrow & \mathbf{8}\,, \end{array} \tag{1.9}$$

as required by the $D = 4$ R-symmetry of the $\mathcal{N} = 8$ theory.

Currently, the main tools to obtain these consistent truncations are either based on the supersymmetry of the higher-dimensional background solution and how it constrains the string theory configurations, or rely on the structure of the lower-dimensional truncated theory and take advantage of its duality group. In the first approach, the fundamental tool are G-structures [29]. These can be understood as a preferred basis of p-forms built out of the preserved Killing spinors on which to expand the supersymmetric solutions. This language is very well suited to classification efforts, and was used in [17] to obtain the form of all solutions of M-theory with an AdS_4 factor preserving $\mathcal{N} = 2$. In chapter 2, we review these notions and the construction in [17] to then build new consistent truncations of $D = 11$ supergravity to $D = 4$ $\mathcal{N} = 2$ minimal gauged supergravity on a topological S^7.

Regarding the approach based on dualities, it is explored in chapter 3, where the fully-fledged uplift to $D = 11$ of the maximal SO(8)-gauged supergravity in $D = 4$ is presented. This uplift is performed by making use of the formulation of the gauged supergravity theory in a $\mathrm{E}_{7(7)}$ covariant language by means of the tensor hierarchy [30]. We further focus on the solutions present in the SU(3)-invariant subsector of the gauged supergravity and their uplift to eleven dimensions. The consistency of the truncation of M-theory on S^7 is checked for these solutions at the level of the $D = 11$ four-form equations, and having the detailed form of this solutions proves instrumental in Part II. Moreover, as the full dynamics of the $D = 4$ theory is kept, this also allows us to construct the long-sought minimal $\mathcal{N} = 2$ consistent truncation about the Corrado-Pilch-Warner (CPW) solution [31].

Chapter 2

Supersymmetry and G-structures

It has long been known that supersymmetry is a powerful tool in string and field theory. In this chapter, we will review how to use it to classify configurations in M-theory which preserve $\mathcal{N} = 2$ in four dimensions using the notion of G-structures in section 2.1. This technique was used in [17] to construct a general class of AdS_4 vacua, which we review in section 2.2. After this recap, we will construct consistent truncations around the solutions in section 2.2 down to $\mathcal{N} = 2$ minimal supergravity, in line with the general conjecture in [32].

2.1 Holonomy and G-structures

Requiring that a configuration is supersymmetric amounts to finding a spinor generator ϵ such that

$$\delta_\epsilon B = 0 \, , \qquad \delta_\epsilon F = 0 \, , \tag{2.1}$$

for every bosonic, B, and fermionic, F, field. In the case of bosonic configurations, where all fermionic fields vanish, the first equality in (2.1) is automatic, as the right-hand side of the supersymmetric variation depends on the fermions. In M-theory, the second amounts to the vanishing of (I.3), which depends on the background geometry and fluxes. Requiring that globally well-defined non-vanishing solutions to this equation exist constrains severely the properties of the vacuum.

Before any further structure is introduced, the fibre bundle of all possible frames on a d-dimensional manifold, known as the frame bundle, is patched with $\mathrm{GL}(d, \mathbb{R})$ transformations. If some globally well-defined non-vanishing section is introduced, we can choose the subset of frames adapted to it. For instance, if an orientation is introduced, we can restrict to frames whose wedge has the same sign as the non-vanishing volume form; or, if a metric is present, we can take frames whose elements are orthonormal with respect to it. For each of these restricted frames, the structure group is reduced, being $SL(d, \mathbb{R})$ in the first case and $O(d)$ in the second. Further, these restrictions are compatible, and for oriented Riemannian manifolds, the structure group is $SO(d)$, or a subgroup thereof in case other sections such as our Killing spinors are introduced.

For configurations without flux [33], (I.3) reduces to requiring the existence of a spinor parallel transported with respect to the Levi-Civita connection

$$\nabla_M \epsilon = 0\,. \tag{2.2}$$

In this case, the manifold must not only be Ricci-flat from (I.1),[1] but also admit a metric with special holonomy, which is a very restrictive topological requirement. These metrics have been classified [33, 34], and for the static case the problem reduces to classifying Calabi-Yau 5-folds, as the metric is locally isometric to a product $\mathbb{R} \times CY_5$.

In flux compactifications, where the four-form does not vanish identically, spinors are not covariantly constant with respect to the Levi-Civita connection, but the torsion-full connection appearing in (I.3)

$$\hat{\nabla}_M \equiv \nabla_M + \frac{1}{288}\left(\Gamma_M{}^{SPQR} - 8\delta_M^S \Gamma^{PQR}\right) G_{SPQR}\,. \tag{2.3}$$

The existence of a globally well-defined non-vanishing spinor implies that the manifold admits a reduced structure group G [35] under which this spinor is left invariant. In the seven-dimensional case, the relevant structure groups are therefore those subgroups of SO(7) for which the spinorial representation of SO(7) contains singlets. The number of internal Killing spinors associated to the reduced structure group follows the chain [35]

$$\begin{array}{ccccccc} \mathrm{SO}(7) & \supset & \mathrm{G}_2 & \supset & \mathrm{SU}(3) & \supset & \mathrm{SU}(2) \\ \mathbf{8} & \rightarrow & \mathbf{1}+\mathbf{7} & \rightarrow & 2\times\mathbf{1}+\mathbf{3}+\bar{\mathbf{3}} & \rightarrow & 4\times\mathbf{1}+2\times\mathbf{2}\,. \end{array} \tag{2.4}$$

It is useful to consider the p-forms that can be constructed as bilinears of these Killing spinors. Group theoretically, they follow the rule

$$\begin{array}{ccccccccc} \mathrm{SO}(7) & \supset & \mathrm{G}_2 & \supset & \mathrm{SU}(3) & \supset & \mathrm{SU}(2) \\ \mathbf{7} & \rightarrow & \mathbf{7} & \rightarrow & \mathbf{1}+\mathbf{3}+\bar{\mathbf{3}} & \rightarrow & 3\times\mathbf{1}+2\times\mathbf{2}\;, \\ \mathbf{21} & \rightarrow & \mathbf{7}+\mathbf{14} & \rightarrow & \mathbf{1}+2\times(\mathbf{3}+\bar{\mathbf{3}})+\mathbf{8} & \rightarrow & 6\times\mathbf{1}+6\times\mathbf{2}+\mathbf{3}\;, \\ \mathbf{35} & \rightarrow & \mathbf{1}+\mathbf{7}+\mathbf{27} & \rightarrow & \begin{array}{c} 3\times\mathbf{1}+2\times(\mathbf{3}+\bar{\mathbf{3}}) \\ +\mathbf{6}+\bar{\mathbf{6}}+\mathbf{8} \end{array} & \rightarrow & 10\times\mathbf{1}+8\times\mathbf{2}+3\times\mathbf{3}\;. \end{array} \tag{2.5}$$

with vectors and six-forms transforming in the **7**, two- and five-forms in the **21**, and three- and four-forms in the **35** of the Riemannian SO(7). This

[1] Note that Ricci-flatness is not an integrability condition for the existence of covariantly constant spinors in the case of Lorentzian manifolds.

	Λ^1	Λ^2	Λ^3
G_2	$-$	$-$	φ
SU(3)	η	J	Ω
SU(2)	E^I	J^I	$-$

Table 2.1: G-invariant forms for structure groups inside SO(7).

means that a G_2-structure is characterised by a globally defined three-form φ (and its Hodge dual $\psi = *\varphi$). Similarly, an SU(3)-structure is characterised by a one-form η, a two-form J and a complex three-form Ω (the third invariant three-form in (2.5) is simply $\eta \wedge J$). Finally, an SU(2)-structure is characterised by a triplet of one-forms E^I and a triplet of two-forms J^I. In this case, there is no non-trivial three-form, but only $E \wedge E \wedge E$ and $E^I \wedge J^J$.

Parallel transport of ϵ with respect to $\hat{\nabla}$ in (2.3) can be characterised in terms of the tensor $\hat{\nabla} - \nabla$ with information about the fluxes [35, 36]. This tensor can be identified as the torsion of the $\hat{\nabla}$ connection, which is an element of $\Lambda^1 \otimes \Lambda^2$. Given the isomorphism $\Lambda^2 \simeq \mathfrak{so}(7) = \mathfrak{g} \oplus (\mathfrak{so}(7)/\mathfrak{g})$, with $\mathfrak{g}$ the Lie algebra of the structure group G and $\mathfrak{so}(7)/\mathfrak{g}$ its complement inside $\mathfrak{so}(7)$, the action of $\hat{\nabla} - \nabla$ when acting on G-invariant objects is given by the intrinsic torsion $T^{(G)}$ in $\Lambda^1 \otimes (\mathfrak{so}(7)/\mathfrak{g})$. The different G-modules in $\Lambda^1 \otimes (\mathfrak{so}(7)/\mathfrak{g})$ are known as torsion classes. Their presence characterises the supergravity solution, as it is possible to express the supergravity fields in terms of them and the invariant forms [36, 37].

Let us briefly introduce the different torsion classes for the reduced structures in (2.4), and how they relate the differentials of the invariant p-forms to the forms themselves. For G_2-structures,

$$\Lambda^1 \otimes (\mathfrak{so}(7)/\mathfrak{g}_2) = \mathbf{7} \otimes \mathbf{7} = \mathbf{1} + \mathbf{7} + \mathbf{14} + \mathbf{27} \ , \tag{2.6}$$

and combining this information with (2.5), we find

$$d\varphi = \tau_1^0 \psi + \tau_7^1 \wedge \varphi + \tau_{27}^3 \wedge (\varphi \lrcorner \psi) \ , \tag{2.7}$$

with τ_m^k being k-form representatives of the m-th torsion class in (2.6). Similarly, for the SU(3) and SU(2) structures we have

$$\begin{aligned} \Lambda^1 \otimes (\mathfrak{so}(7)/\mathfrak{su}(3)) &= (\mathbf{1} + \mathbf{3} + \bar{\mathbf{3}}) \otimes [\mathbf{1} + 2 \times (\mathbf{3} + \bar{\mathbf{3}})] \\ &= 5 \times \mathbf{1} + 5 \times (\mathbf{3} + \bar{\mathbf{3}}) + 2 \times (\mathbf{6} + \bar{\mathbf{6}}) + 4 \times \mathbf{8} \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \Lambda^1 \otimes (\mathfrak{so}(7)/\mathfrak{su}(2)) &= (3 \times \mathbf{1} + 2 \times \mathbf{2}) \otimes (6 \times \mathbf{1} + 6 \times \mathbf{2}) \\ &= 30 \times \mathbf{1} + 30 \times \mathbf{2} + 12 \times \mathbf{3} \ , \end{aligned} \tag{2.9}$$

with the derivatives $d\eta$, dJ and $d\Omega$ given in terms of the 23 torsion classes in (2.8), and dE^I, dJ^I in terms of the 72 in (2.9). Detailed expressions can be found, *e.g.* in [37].

2.2 Classification of AdS_4 $\mathcal{N} = 2$ solutions

The class of background geometries that we are going to focus on was studied in [17]. These correspond to warped products $\mathrm{AdS}_4 \times Y_7$ preserving two supersymmetries and the maximal symmetry of the AdS_4 factor. As such, the have factorised $D = 11$ metric and four-form as given in (1.5) and (1.6)

$$g_{11} = e^{2\Delta}\left(g_{\mathrm{AdS}_4} + g_7\right) , \qquad G_{(4)} = m \ \mathrm{vol}(\mathrm{AdS}_4) + F_{(4)} , \tag{2.10}$$

where m is a constant and the function $e^{2\Delta}$, the Riemannian metric g_7 and the four-form $F_{(4)}$ are all defined on the internal manifold Y_7. In the remainder of this chapter, we follow [17] in defining g_{AdS_4} to be of radius $L_{\mathrm{AdS}_4} = \frac{1}{2}$ so that its Ricci tensor is -12 times the metric. In (2.10), $\mathrm{vol}(\mathrm{AdS}_4)$ is the volume form of g_{AdS_4}.

The preserved $\mathcal{N} = 2$ supersymmetry means that there exist two independent AdS_4 spinors ψ^i satisfying the lower dimensional Killing equation and of positive chirality

$$\nabla_\mu \psi^i = \rho_\mu (\psi^i)^c , \qquad \rho_5 \psi^i = \psi^i . \tag{2.11}$$

The superscript c here and in the following stands for charge conjugate with the standard conventions of [17], both for four- and seven-dimensional spinors. To promote these spinors to Killing spinors of the eleven-dimensional configuration, they must be interwoven with internal spinors χ. This still allows freedom to use the SU(3) or SU(2) structures discussed in section 2.1.

In the case relevant for SU(3) structures, the eleven-dimensional Majorana spinor ϵ can be decomposed as

$$\epsilon = \left(\psi^1 + (\psi^2)^c\right) \otimes \chi + m.c. . \tag{2.12}$$

with χ the complex spinor in (2.4) and $m.c.$ here and henceforth denoting charge conjugation of the terms shown. This class of solutions was analysed in [37], who showed that the only solutions are unwarped Freund-Rubin compactifications with a Sasaki-Einstein internal manifold. On the contrary, we will consider the SU(2) factorisation

$$\epsilon = \sum_i \psi_i \otimes e^{\Delta/2} \chi_i + (\psi_i)^c \otimes e^{\Delta/2} \chi_i^c , \tag{2.13}$$

where the two linearly independent Dirac spinors χ_i, $i = 1, 2$ on Y_7 correspond to the four singlets in (2.4), and the factors $e^{\Delta/2}$ have been introduced as in [17] for convenience. Combining (2.11) with this ansatz implies that χ_i are subject to the constraints

$$\begin{aligned} &\frac{1}{2}\partial_n \Delta \gamma^n \chi_i - \frac{ime^{-3\Delta}}{6}\chi_i + \frac{e^{-3\Delta}}{288} F_{bcde}\gamma^{bcde}\chi_i + \chi_i^c = 0 , \\ &\nabla_m \chi_i + \frac{ime^{-3\Delta}}{4}\gamma_m \chi_i - \frac{e^{-3\Delta}}{24}\gamma^{cde} F_{mcde}\chi_i - \gamma_m \chi_i^c = 0 , \end{aligned} \tag{2.14}$$

imposed by the Killing spinor equation. Indices $a, b, c, \ldots = 4, \ldots, 10$ and $m, n, p, \ldots = 4, \ldots, 10$ respectively are M_7 global and local indices, γ_a and γ_m respectively denote the seven-dimensional Dirac matrices and their contraction with a local frame, ∇_m is the covariant derivative compatible with g_7 acting on spinors.

A number of bilinears in χ_i can be constructed that correspond to the triplet of orthonormal one-forms, E_1, E_2, E_3, and two-forms, J_1, J_2, J_3 of the local SU(2) structure in table 2.1. One of the one-forms, E_1, is dual to a Killing vector ξ of g_7 that also preserves the four-form flux $F_{(4)}$. This vector thus generates the Reeb-like $\mathcal{N}=2$ direction. Local coordinates ψ, τ and ρ can be introduced on Y_7 so that the Killing vector is $\xi = 4\partial_\psi$, and the one-forms become

$$
\begin{aligned}
E_1 &= \frac{1}{4}\|\xi\|(d\psi + \mathcal{A})\,, \qquad E_2 = \frac{e^{-3\Delta}}{4\sqrt{1-\|\xi\|^2}}d\rho\,, \\
E_3 &= \frac{6}{m}\,\frac{\rho\|\xi\|}{4\sqrt{1-\|\xi\|^2}}(d\tau + \mathcal{A})\,,
\end{aligned}
\tag{2.15}
$$

where $\|\xi\|$ is the norm of ξ with respect to g_7,

$$
\|\xi\|^2 = \frac{e^{-6\Delta}}{36}\left(m^2 + 36\rho^2\right)\,, \tag{2.16}
$$

and $\mathcal{A}$ is a local one-form such that $\mathcal{L}_\xi \mathcal{A} = 0$ and $i_\xi \mathcal{A} = 0$.

The metric on M_7 can now be written as

$$
g_7 = g_{\mathrm{SU}(2)} + E_1^2 + E_2^2 + E_3^2\,, \tag{2.17}
$$

with $g_{\mathrm{SU}(2)}$ a metric on the local four-dimensional space where the two-forms J_I, $I = 1, 2, 3$, are defined. The SU(2) structure group rotates the frame of this four-dimensional metric, and is embedded into its spin group as $\mathrm{SO}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)'$. Using this $\mathrm{SU}(2)'$, we can choose a frame such that the two-forms take on canonical expressions[2]

$$
J_3 = e^{45} + e^{67}\,, \qquad \Omega = J_1 + iJ_2 = (e^4 + ie^5) \wedge (e^6 + ie^7)\,. \tag{2.18}
$$

In particular, J_I are self-dual with respect to the Hodge star associated to $g_{\mathrm{SU}(2)}$ and obey $J_I \wedge J_J = 2\,\mathrm{vol}(g_{\mathrm{SU}(2)})\,\delta_{IJ}$. The derivatives of these SU(2)-invariant forms can be shown [17] to combine into the following torsion conditions

$$
e^{-3\Delta} d\left[\|\xi\|^{-1}\left(\frac{m}{6}E_1 + e^{3\Delta}|S|\sqrt{1-\|\xi\|^2}E_3\right)\right] = 2\left(J_3 - \|\xi\| E_2 \wedge E_3\right)\,, \tag{2.19}
$$

[2] We label the $D=11$ frame so that $g_4 + g_7 = -e^0 \otimes e^0 + \sum_{i=1}^{10} e^i \otimes e^i$, with $e^0, \ldots, e^3$ associated to AdS_4, $e^4, \ldots, e^7$ to $g_{\mathrm{SU}(2)}$, and $e^8 = E_1$, $e^9 = E_2$, $e^{10} = E_3$.

$$d\left(\|\xi\|^2 e^{9\Delta} J_2 \wedge E_2\right) - e^{3\Delta}|S| d\left(\|\xi\| e^{6\Delta}|S|^{-1} J_1 \wedge E_3\right) = 0\,, \tag{2.20}$$

$$d\left(e^{6\Delta} J_1 \wedge E_2\right) + e^{3\Delta}|S| d\left(\|\xi\| e^{3\Delta}|S|^{-1} J_2 \wedge E_3\right) = 0\,, \tag{2.21}$$

where $S \equiv \rho e^{-3\Delta} e^{i(\psi-\tau)}$ is a zero-form bilinear. These determine the internal four-form as

$$F_{(4)} = \frac{1}{\|\xi\|} E_1 \wedge d\left(e^{3\Delta}\sqrt{1-\|\xi\|^2} J_1\right) - m\frac{\sqrt{1-\|\xi\|^2}}{\|\xi\|} J_1 \wedge E_2 \wedge E_3\,, \tag{2.22}$$

and the differential of the one-form $\mathcal{A}$ as

$$d\mathcal{A} = \frac{4me^{-3\Delta}}{3\|\xi\|^2}\left[J_3 + \left(3\|\xi\| - \frac{4}{\|\xi\|}\right) E_2 \wedge E_3\right]\,. \tag{2.23}$$

The supersymmetric configuration (2.10) with (2.15)–(2.23) solves the Bianchi identities and equations of motion of $D=11$ supergravity [17], as discussed in section 2.1. In particular, it is straightforward to check that the four-form (2.22) is closed, using the differential relations (2.19)–(2.21). In fact, the two distinct contributions to the four-form can be checked to be separately closed.

An interesting case of the general class of configurations of [17] was also studied in that reference, where the vector ∂_τ along the coordinate τ becomes an isometry of the internal metric g_7. This vector can never become a symmetry of $F_{(4)}$, though, unlike the Reeb vector $\xi = 4\partial_\psi$, which preserves the entire $D=11$ configuration. For this subclass, it is convenient to rescale the coordinate ρ by a constant factor as $r \equiv \frac{6}{m}\rho$ and introduce a function $f(r)$ such that

$$J_I = \frac{m}{24} e^{-3\Delta} f(r)\, \mathbb{J}_I\ ,\ I = 1,2,3\ , \qquad (1+r^2)(d\tau + \mathcal{A}) = f(r)(d\tau + A_{\mathrm{KE}})\ , \tag{2.24}$$

where the one-form A_{KE} and the triplet of two-forms $\mathbb{J}_I$ are r–independent and defined on the four-dimensional space with metric $g_{\mathrm{SU}(2)}$. The latter becomes, up to an overall r-dependent factor (cf. (2.28)), a Kähler-Einstein metric g_{KE} with canonical normalisation $\mathrm{Ric}_{\mathrm{KE}} = 6\, g_{\mathrm{KE}}$. The torsion conditions (2.19)–(2.21) reduce to

$$dA_{\mathrm{KE}} = 2\,\mathbb{J}_3\,, \qquad d\left(\mathbb{J}_1 + i\mathbb{J}_2\right) = 3i(\mathbb{J}_1 + i\mathbb{J}_2) \wedge (d\tau + A_{\mathrm{KE}})\,, \tag{2.25}$$

together with the following ordinary differential equations (ODEs) for $f(r)$,

$$f' = -\frac{1}{2} r\, \alpha^2 f\,, \qquad \frac{(r\alpha' - r^2\alpha^3)f}{\sqrt{1+(1+r^2)\alpha^2}} = -3\,, \tag{2.26}$$

where a prime denotes derivative with respect to r. The warp factor can be expressed in terms of α as $e^{6\Delta} = \left(\frac{m}{6}\right)^2\left(1 + r^2 + \alpha^{-2}\right)$. The first equation in (2.25) signals the two-form $\mathbb{J}_3$ as the Kähler-Einstein form and $A_{\rm KE}$ as a potential for it. Finally, the one-forms (2.15) become, using (2.16),

$$
\begin{aligned}
E_1 &= \frac{\alpha\sqrt{1+r^2}}{4\sqrt{1+(1+r^2)\alpha^2}}\left[d\psi - d\tau + \frac{f}{1+r^2}(d\tau + A_{\rm KE})\right] , \\
E_2 &= \frac{1}{4}\alpha\, dr \, , \qquad\qquad E_3 = \frac{1}{4}\frac{r\,\alpha\, f}{\sqrt{1+r^2}}(d\tau + A_{\rm KE}) \, .
\end{aligned}
\tag{2.27}
$$

In terms of these objects, the solutions simplify down to

$$
\begin{aligned}
g_{11} &= e^{2\Delta}\Bigg\{ g_{\rm AdS_4} + \frac{\alpha\, f}{4\sqrt{1+(1+r^2)\alpha^2}}\, g_{\rm KE} + \frac{\alpha^2}{16}\Bigg[dr^2 + \frac{r^2 f^2}{1+r^2}(d\tau + A_{\rm KE})^2 \\
&\qquad + \frac{1+r^2}{1+(1+r^2)\alpha^2}\left(d\psi - d\tau + \frac{f}{1+r^2}\,(d\tau + A_{\rm KE})\right)^2\Bigg]\Bigg\} \, , \\
\hat{F}_{(4)} &= h_1(r)(d\psi - d\tau)\wedge dr\wedge \mathbb{J}_1 + h_2(r)(d\psi - d\tau)\wedge(d\tau + A_{\rm KE})\wedge \mathbb{J}_2 \\
&\quad + h_3(r)(d\tau + A_{\rm KE})\wedge dr \wedge \mathbb{J}_1 \, ,
\end{aligned}
\tag{2.28}
$$

where we have defined the following shorthand functions of r

$$
\begin{aligned}
h_1(r) &= \frac{m^2}{3^2\cdot 2^6}\left(\alpha^{-1}e^{-3\Delta} f\right)' \, , \qquad\qquad h_2(r) = -\frac{m^2}{3\cdot 2^6}\left(\alpha^{-1}e^{-3\Delta} f\right) \, , \\
h_3(r) &= \frac{m^2}{3^2\cdot 2^7}\frac{f}{1+r^2}\left[2\left(\alpha^{-1}e^{-3\Delta} f\right)' - 3r\,\alpha^2\left(\alpha^{-1}e^{-3\Delta} f\right)\right] \, .
\end{aligned}
\tag{2.29}
$$

Explicit instances in this subclass of geometries are obtained for each solution $f(r)$ of the ODE system (2.26). Two such solutions were discussed in [17]. The first one, analytic, is obtained by setting [17]

$$
f(r) = 3\left(2 - \frac{r}{\sqrt{2}}\right) , \qquad \alpha(r) = \sqrt{\frac{2}{2\sqrt{2}r - r^2}} \, , \tag{2.30}
$$

with $r\in[0,\ 2\sqrt{2}]$. This reproduces the $\mathcal{N}=2$ AdS$_4$ solution first obtained by Corrado, Pilch and Warner (CPW) [31] by other methods. A second, numerical, solution to the ODE system (2.26) was obtained in [17] (see also [38]). This AdS$_4$ solution was argued [17] to dominate holographically the low-energy physics of a relevant deformation of the Aharony-Bergman-Jafferis-Maldacena (ABJM) [39] field theory defined on a stack of planar M2-branes, which is cubic in the adjoint $\mathcal{N}=2$ chiral fields. Its physical rôle is thus similar to the CPW solution, which is related to an analogue, quadratic, deformation in the chirals. We will describe further and spend much attention to this solution in chapter 5.

2.3 Consistent truncations

General idea and a simple example

For many supersymmetric AdS_d configurations of string theory, it was found to be possible to build Kaluza-Klein truncation ansätze that retained the full dynamics of the d-dimensional stress-energy supermultiplet [25, 40–43]. This dynamics is captured by a (possibly gauged) supergravity in d dimensions with as many supersymmetries as the base solution and involving only the supermultiplet containing the graviton field, which we refer as the minimal supergravity. This led Gauntlett and Varela [32] to conjecture that such consistent truncations always exist, and their reduction ansatz can be constructed using the G-structure machinery reviewed above. This conjecture was recently proved by Cassani, Josse, Petrini and Waldram [44] with arguments from generalised geometry [45–56].

As a simple example, we can consider the aforementioned $\mathcal{N}=2$ AdS_4 solutions of M-theory with SU(3) structure. They take the form

$$\begin{aligned} ds^2_{11} &= \tfrac{1}{4}ds^2(\mathrm{AdS}_4) + ds^2(\mathrm{SE}_7) \\ G &= \tfrac{3}{8}\,\mathrm{vol}(\mathrm{AdS}_4)\ , \end{aligned} \tag{2.31}$$

with

$$ds^2(\mathrm{SE}_7) = (d\psi+\sigma)^2 + ds^2(M_6)\ , \tag{2.32}$$

where the Sasaki-Einstein manifold is normalised as $\mathrm{Ric}(\mathrm{SE}_7) = 6ds^2(\mathrm{SE}_7)$. Here ψ is the coordinate along the Reeb direction corresponding to the R-symmetry of the dual $\mathcal{N}=2$ SCFT, and $ds^2(M_6)$ is locally Kähler-Einstein with Kähler form J such that $d\sigma = 2J$.

The supergravity multiplet in $D=4$ $\mathcal{N}=2$ contains two gravitini and a massless vector apart from the graviton. This vector can be understood as the gauge field of the U(1) R-symmetry, under which the gravitini are charged. The ansatz (2.31) can be modified to accommodate the bosonic degrees of freedom of the lower dimensional supergravity as [32, 57].

$$\begin{aligned} ds^2_{11} &= \tfrac{1}{4}d\bar{s}^2_4 + (d\psi+\sigma+\tfrac{1}{4}\bar{A})^2 + ds^2(M_6) \\ G &= \tfrac{3}{8}\,\overline{\mathrm{vol}}_4 - \tfrac{1}{4}\,\bar{\star}_4\,\bar{F}\wedge J\ , \end{aligned} \tag{2.33}$$

where $\overline{\mathrm{vol}}_4$ and $\bar{\star}_4$ are now associated to the arbitrary $d\bar{s}^2_4$ metric, and the graviphoton $\bar{A}$ enters both through its gauge-invariant field strength and gauging shifts along the U(1) Reeb direction. It is straightforward to show that the new ansatz satisfies the equations of motion (I.1) provided that the four-dimensional fields satisfy

$$d\bar{F}=0\ , \qquad d\,\bar{\star}_4\bar{F}=0\ , \qquad \bar{R}_{\mu\nu} = -3g^2\bar{g}_{\mu\nu} + \tfrac{1}{2}\big(\bar{F}_{\mu\sigma}\bar{F}_{\nu}{}^{\sigma} - \tfrac{1}{4}\bar{g}_{\mu\nu}\,\bar{F}_{\rho\sigma}\bar{F}^{\rho\sigma}\big)\ , \tag{2.34}$$

which are precisely the equations of motion of minimal $D = 4$ $\mathcal{N} = 2$ supergravity stemming from the bosonic Lagrangian

$$\mathcal{L} = \bar{R}\,\overline{\mathrm{vol}}_4 - \tfrac{1}{2}\bar{F} \wedge \bar{\star}_4 \bar{F} + 6g^2\,\overline{\mathrm{vol}}_4 \ . \tag{2.35}$$

In remainder of this section, we will construct the reduction ansatz for the class of AdS$_4$ $\mathcal{N} = 2$ solutions with SU(2) structure reviewed in section 2.2 and check that the the eleven-dimensional field equations are indeed satisfied upon imposing the $D = 4$ counterparts. Furthermore, we also show that the supersymmetry variations of the higher-dimensional theory follow from the lower-dimensional ones. This implies that any supersymmetric configuration will uplift to a solution preserving at least as many supercharges. Together with the Sasaki-Einstein case just reviewed, this exhausts all consistent truncations from M-theory down to minimal $D = 4$ $\mathcal{N} = 2$ supergravity.

Minimal supergravity on $\mathcal{N} = 2$ backgrunds

As just recalled, the bosonic sector of pure $D = 4$ $\mathcal{N} = 2$ supergravity [58, 59] includes the metric, $\bar{g}_{\mu\nu}$, $\mu = 0, \dots 3$, and a gauge field $\bar{A}$, the graviphoton, with field strength $\bar{F} = d\bar{A}$. The gauged supergravity has a cosmological constant related to the coupling constant g that couples $\bar{A}$ to the $\mathcal{N} = 2$ gravitini, which can be chosen to be Weyl fermions, $\psi^+_{i\mu}$. Their variation under supersymmetry is

$$\delta\psi^+_{i\mu} = \bar{\nabla}_\mu\bar{\psi}_i + \frac{ig}{2}\epsilon_{ij}\bar{A}_\mu\bar{\psi}_j - \frac{g}{2}\bar{\rho}_\mu(\bar{\psi}_i)^c + \frac{g^2}{32}\bar{F}_{\delta\epsilon}\,\bar{\rho}^{\delta\epsilon}\bar{\rho}_\mu\epsilon_{ij}(\bar{\psi}_j)^c \ , \tag{2.36}$$

for a Weyl spinor parameter $\bar{\psi}_i$ and $\bar{\rho}_\mu$ associated to a local frame for $\bar{g}_4$.

In line with (2.33), we propose the following KK ansatz for the eleven-dimensional fields:

$$g_{11} = e^{2\Delta}(g_4 + \hat{g}_7) \ , \qquad G_{(4)} = m\ \mathrm{vol}_4 + \hat{F}_{(4)} - X \wedge g\bar{F} - Y \wedge g\star_4 \bar{F} \ . \tag{2.37}$$

The metric g_4 is now a general $D = 4$ metric and vol$_4$ its corresponding volume form. Hats over $\hat{g}_7$ and $\hat{F}_{(4)}$ have been employed to signify a shift of the Reeb direction ξ by the $D = 4$ graviphoton $\bar{A}$. This motivates, from (2.15), the definition

$$\hat{E}_1 = \frac{1}{4}\|\xi\|(d\psi + \mathcal{A} - g\bar{A}) \ . \tag{2.38}$$

Accordingly we have, from (2.17) and (2.22),

$$\begin{aligned} \hat{g}_7 &= g_{SU(2)} + \hat{E}_1^2 + E_2^2 + E_3^2 \ , \\ \hat{F}_{(4)} &= \frac{1}{\|\xi\|}\hat{E}_1 \wedge d\left(e^{3\Delta}\sqrt{1-\|\xi\|^2}J_1\right) - m\frac{\sqrt{1-\|\xi\|^2}}{\|\xi\|}J_1 \wedge E_2 \wedge E_3 \ . \end{aligned} \tag{2.39}$$

The graviphoton also enters the KK ansatz (2.37) through its field strength $\bar{F} = d\bar{A}$ and through the Hodge dual of the latter with respect to the four-dimensional metric g_4. The constant g that appears in (2.37) and (2.38) is the gauge coupling of the $D = 4$ supergravity. Finally, X and Y are two-forms on the internal seven-dimensional manifold to be determined.

When g_4 is set equal to the AdS_4 metric and the graviphoton is turned off, $\bar{A} = 0$, $\bar{F} = 0$, the $D = 11$ configuration (2.37) reduces to the $\mathcal{N} = 2$ class of solutions in section 2.2. In this case, the two-forms X, Y drop out from the picture and do not play any rôle in the background geometry. More generally, though, the full configuration (2.37) with general $D = 4$ fields g_4, $\bar{A}$ subject to the field equations of $D = 4$ $\mathcal{N} = 2$ minimal supergravity, (2.34), can still be forced to obey the field equations of $D = 11$ supergravity for suitable X and Y.

The strategy is to substitute (2.37) into the $D = 11$ field equations treating the linear, $\bar{F}$, $\star_4\bar{F}$, and quadratic, $\bar{F} \wedge \bar{F}$, $\star_4\bar{F} \wedge \bar{F}$, combinations of the $D = 4$ graviphoton field strength as independent quantities. Upon imposing the $D = 4$ field equations, a number of differential and algebraic equations for X and Y are produced. Proposing a suitable ansatz for these two-forms in terms of the SU(2)–structure forms and using the torsion conditions (2.19)–(2.21), we can solve this system of equations and, thus, find the explicit consistent KK reduction.

Let us summarise, along these lines, the system of equations that X and Y must obey for the truncation ansatz to be consistent. Further details on the consistency proof are relegated to appendix A.1. In our conventions, the $D = 11$ and $D = 4$ field equations take on the form (I.1) and (2.34). It is convenient to introduce the two-forms $\tilde{X}$, $\tilde{Y}$ containing the contributions to X, Y with no legs along the gauged E_1 direction (see the appendix). Imposing the Bianchi identity for the undeformed four-form in (2.10), and the Bianchi and Maxwell equation for the $D = 4$ graviphoton, the Bianchi identity of the deformed four-form in (2.37) is satisfied provided the unknown forms obey the following constraints:

$$\begin{aligned} \bar{F} \wedge \bar{F}: &\quad i_\xi X = 0\ , & \bar{F}: &\quad \frac{1}{4} i_\xi F_{(4)} + d\tilde{X} = 0\ , \\ \star_4\bar{F} \wedge \bar{F}: &\quad i_\xi Y = 0\ , & \star_4\bar{F}: &\quad d\tilde{Y} = 0\ . \end{aligned} \tag{2.40}$$

These expressions arise in the $D = 11$ five-form $dG_{(4)} = 0$ wedged with the indicated $D = 4$ graviphoton contributions, and must be enforced to vanish separately for arbitrary $\bar{F}$. The constraints coming from the quadratic graviphoton contributions imply $X = \tilde{X}$, $Y = \tilde{Y}$. We will make use of these relations in the sequel to simplify the resulting expressions.

Proceeding similarly, we find the constraints imposed on X and Y by the equation of motion for the $D = 11$ four-form. Assuming, again, that the undeformed four-form (2.10) satisfies the equation of motion and imposing the Bianchi and equation of motion for $\bar{F}$, the equation of motion for $G_{(4)}$ in (2.37) is satisfied provided the following relations hold:

$$\begin{aligned}
\bar{F}\wedge\bar{F}: &\quad \frac{1}{4}e^{3\Delta}i_\xi \star_7 Y + \frac{1}{2}(Y\wedge Y - X\wedge X) = 0\ ,\\
\star_4\bar{F}\wedge\bar{F}: &\quad \frac{1}{4}e^{3\Delta}i_\xi \star_7 X + X\wedge Y = 0\ ,\\
\bar{F}: &\quad \frac{m}{8}\|\xi\| e^{3\Delta} J_3\wedge J_3\wedge E_2\wedge E_3 - \frac{1}{4}e^{3\Delta} d\mathcal{A}\wedge i_\xi \star_7 Y\\
&\qquad + \frac{1}{4}\hat{e}\wedge d\,(e^{3\Delta} i_\xi \star_7 Y) + X\wedge \hat{F}_{(4)} = 0\ ,\\
\star_4\bar{F}: &\quad \frac{1}{4}e^{3\Delta} d\mathcal{A}\wedge i_\xi \star_7 X - \frac{1}{4}\hat{e}\wedge d\,(e^{3\Delta} i_\xi \star_7 X) + Y\wedge \hat{F}_{(4)} = 0\ ,
\end{aligned} \tag{2.41}$$

with $\hat{e}$ defined below (A.2). We have again indicated the linear or quadratic graviphoton combinations with which these expressions appear wedged in the $D = 11$ eight-form equation of motion for $G_{(4)}$.

Finally, we turn to the evaluation of the $D = 11$ Einstein equation on the configuration (2.37). Combining the Ricci tensor (A.8) and the r.h.s. (A.11) of the Einstein equation as given in (I.1), this yields the following three equations,

$$\begin{aligned}
&\mathrm{Ric}_{\alpha\beta} - \frac{g^2}{32}\|\xi\|^2 \bar{F}_{\alpha\gamma}\bar{F}_\beta{}^\gamma - 9(\partial_a\Delta\partial^a\Delta + \nabla_a\nabla^a\Delta)\eta_{\alpha\beta}\\
&\quad = -e^{-6\Delta}\Big\{\frac{1}{3}m^2\eta_{\alpha\beta} - \frac{g^2}{4}(X^2+Y^2)\bar{F}_{\alpha\gamma}\bar{F}_\beta{}^\gamma + \frac{g^2}{24}\eta_{\alpha\beta}\bar{F}^2(X^2+2Y^2)\\
&\qquad + \frac{g^2}{4}\bar{F}_{\gamma(\alpha}\epsilon_{\beta)}{}^{\gamma\mu\nu}\bar{F}_{\mu\nu}X_{cd}Y^{cd} + \frac{g^2}{24}\eta_{\alpha\beta}\epsilon_{\mu\nu\rho\sigma}\bar{F}^{\mu\nu}\bar{F}^{\rho\sigma}X_{cd}Y^{cd}\Big\}\ ,
\end{aligned} \tag{2.42}$$

$$\frac{g}{8}\|\xi\|\delta_{8b}\nabla_\gamma\bar{F}_\alpha{}^\gamma = 0\ , \tag{2.43}$$

$$\begin{aligned}
&\mathrm{Ric}_{ab} + \frac{g^2}{64}\|\xi\|^2\delta_{8a}\delta_{8b}\bar{F}_{\gamma\delta}\bar{F}^{\gamma\delta} + 9[\partial_a\Delta\partial_b\Delta - \nabla_a\nabla_b\Delta - (\partial_c\Delta\partial^c\Delta - \nabla_c\nabla^c\Delta)\\
&\quad = e^{-6\Delta}\Big\{\frac{1}{2}m^2\eta_{ab} + \frac{1}{2}\Big[F_{acde}F_b{}^{cde} - \frac{1}{12}\eta_{ab}F^2\Big]\\
&\qquad + \frac{g^2}{24}\bar{F}^2\left[6(X_{ac}X_b{}^c - Y_{ac}Y_b{}^c) - \eta_{ab}(X^2-Y^2)\right]\\
&\qquad + \frac{g^2}{24}\epsilon_{\mu\nu\rho\sigma}\bar{F}^{\mu\nu}\bar{F}^{\rho\sigma}\Big[3(X_{ac}Y_b{}^c + Y_{ac}X_b{}^c) - \eta_{ab}X_{cd}Y^{cd}\Big]\Big\}\ ,
\end{aligned} \tag{2.44}$$

with $\alpha = 0, \ldots, 3$ and $a = 4, \ldots, 10$ external and internal tangent space indices related to the frame specified in footnote (2). Also, $X^2 = X_{ab}X^{ab}$ and similarly for Y^2, F^2 and $\bar{F}^2$. In (2.42) and (2.44), $\mathrm{Ric}_{\alpha\beta}$ and Ric_{ab} are the Ricci tensors of g_4 and the undeformed g_7 metric. Expectedly, the only non-trivial mixed components, (2.43), of the Einstein equations arise in the direction (the 8-th in the notation of footnote 2) that is gauged. The resulting equation is automatically satisfied on the graviphoton's Maxwell equation (the second equation in (2.34)).

For suitably chosen X and Y in terms of background SU(2)-structure forms, equations (2.40)–(2.44) must be satisfied identically, and equation (2.42) must reduce to the $D = 4$ Einstein equation. As shown in appendix A.1, all these requirements are satisfied by setting

$$X = -\frac{1}{4} e^{3\Delta} \sqrt{1 - \|\xi\|^2} J_1 \ , \qquad Y = -\frac{1}{4} e^{3\Delta} \left(J_3 - \|\xi\| E_2 \wedge E_3 \right) \ . \tag{2.45}$$

The KK ansatz (2.37) is thus consistent, at the level of the bosonic field equations, when the two-form coefficients X, Y are taken as in (2.45).

Furthermore, consistency can be extended to include the fermions, as we now turn to discuss at the level of the supersymmetry variations of the gravitino. See appendix A.2 for further details. First, we decompose the Majorana spinor parameter in $D = 11$ as

$$\epsilon = \sum_i \bar{\psi}_i \otimes e^{\Delta/2} \chi_i + (\bar{\psi}_i)^c \otimes e^{\Delta/2} \chi_i^c \ , \tag{2.46}$$

with $\bar{\psi}_i$ two $D = 4$ Weyl spinors of positive chirality. The difference between (2.46) and (2.13) is that we do not impose the $D = 4$ Killing spinor equation in (2.11) on $\bar{\psi}_i$. Next, we plug the KK ansatz (2.37) with (2.45) into the $D = 11$ gravitino variation (I.3), written in the basis (1.7) for the $D = 11$ Dirac matrices in terms of their four-, ρ_α, and seven-dimensional, γ_a, counterparts. Then, we address the internal and external gravitino variations separately.

A long calculation, summarised in appendix A.2, shows that the internal gravitino variations vanish identically provided the following projections,

$$\left[\|\xi\| (3\gamma^8 + i\gamma^{910}) + \sqrt{1 - \|\xi\|^2} (\gamma^{46} - \gamma^{57}) - i(\gamma^{45} + \gamma^{67}) \right] \chi_i = 0 \ , \tag{2.47}$$

and

$$\begin{gathered} (\gamma^{46} + \gamma^{57}) \chi_i = 0, \qquad (\gamma^{45} - \gamma^{67}) \chi_i = 0 \ , \\ \left[-\sqrt{1 - \|\xi\|^2} \gamma^{46} + i(\gamma^{45} + \|\xi\| \gamma^{910}) \right] \chi_i = 0 \ , \end{gathered} \tag{2.48}$$

are imposed on the internal spinors χ_i. These projections, however, add nothing new: they follow from the undeformed Killing spinor equations (2.14) of the undeformed geometry. This is best seen by sandwiching (2.47), (2.48) with the conjugate spinors $\bar{\chi}_j$: the resulting constraints are identically satisfied by the spinor bilinears that defined the undeformed SU(2)–structure. The internal gravitino variations are thus automatically satisfied for the general class of solutions (2.37), using only the restrictions that characterise the AdS$_4$ solutions (2.10).

The calculation of the external gravitino variations proceeds similarly. Together with (2.47), (2.48), the following projection must be imposed:

$$i\gamma^{45}\chi_i = -\epsilon_{ij}\chi_j^c \; . \tag{2.49}$$

This, like (2.47), (2.48), is still compatible with the original Killing spinor equations (2.14) of the undeformed geometry, as argued in appendix A.2, and does not reduce the amount of supersymmetry or constrain the undeformed geometry further. The calculation allows one to read off the consistent embedding of the $D = 4$ $\mathcal{N} = 2$ gravitini $\psi^+_{i\mu}$, $i = 1, 2$, into its $D = 11$ counterpart Ψ_M, for $M = \mu$:

$$\Psi_\mu = \sum_i \psi^+_{i\mu} \otimes e^{\Delta/2}\chi_i + (\psi^+_{i\mu})^c \otimes e^{\Delta/2}\chi_i^c \; , \tag{2.50}$$

Using (2.50), the external components of the $D = 11$ gravitino variation (I.1) finally reduce to their $D = 4$ $\mathcal{N} = 2$ counterparts, (2.36).

To summarise, any solution of minimal $D = 4$ $\mathcal{N} = 2$ gauged supergravity gives rise to a class of solutions of $D = 11$ supergravity of the form

$$\begin{aligned}
g_{11} &= e^{2\Delta}(g_4 + \hat{g}_7) \; , \\
G_{(4)} &= m \; \mathrm{vol}_4 + \hat{F}_{(4)} \\
&\quad + \frac{g}{4} e^{3\Delta} \sqrt{1 - \|\xi\|^2} J_1 \wedge \bar{F} + \frac{g}{4} e^{3\Delta} \left(J_3 - \|\xi\| E_2 \wedge E_3 \right) \wedge \star_4 \bar{F} \; ,
\end{aligned} \tag{2.51}$$

with $\hat{g}_7$, $\hat{F}_{(4)}$ defined in (2.39), upon uplift on the class of seven-dimensional geometries [17] reviewed in section 2.2. The uplift preserves supersymmetry if originally present in $D = 4$. The general class of solutions (2.51) is completely specified by the $D = 4$ supergravity fields and the same SU(2)-structure that characterises the background AdS_4 class of solutions (2.10) of [17].

It is interesting to determine how our KK truncation ansatz adapts itself to various particular cases of the general geometries of [17]. In the purely magnetic flux case, the geometries [17] reduce, by appropriately taking the $m = 0$ limit, to the $\mathcal{N} = 2$ class of geometries describing M5-branes wrapped on internal SLAG 3-cycles described in [60]. Accordingly, our consistent truncation reduces to the one considered in section 3 of [32].

Let us particularise now our general consistent truncation (2.51) to the class of solutions in (2.28), where the vector ∂_τ along the coordinate τ becomes an isometry of the internal metric g_7. In this case, we find that the

KK ansatz becomes[3]

$$
\begin{aligned}
g_{11} = e^{2\Delta} \Bigg\{ & g_4 + \frac{\alpha f}{4\sqrt{1+(1+r^2)\alpha^2}}\, g_{\mathrm{KE}} + \frac{\alpha^2}{16}\Bigg[dr^2 + \frac{r^2 f^2}{1+r^2}(d\tau + A_{\mathrm{KE}})^2 \\
& + \frac{1+r^2}{1+(1+r^2)\alpha^2}\left(\mathrm{D}\psi - d\tau + \frac{f}{1+r^2}\,(d\tau + A_{\mathrm{KE}})\right)^2\Bigg]\Bigg\} \, , \\
\hat{F}_{(4)} = {} & h_1(r)(\mathrm{D}\psi - d\tau)\wedge dr \wedge \mathbb{J}_1 + h_2(r)(\mathrm{D}\psi - d\tau)\wedge(d\tau + A_{\mathrm{KE}})\wedge \mathbb{J}_2 \\
& + h_3(r)(d\tau + A_{\mathrm{KE}})\wedge dr \wedge \mathbb{J}_1 - X \wedge g\bar{F} - Y \wedge g \star_4 \bar{F} \, ,
\end{aligned}
\tag{2.52}
$$

with the shorthands (2.29). The $D = 11$ metric g_{11} depends on the $D = 4$ metric g_4, explicitly and through the Hodge star operator, and on the $D = 4$ graviphoton $\bar{A}$ through the gauge covariant derivative $\mathrm{D}\psi = d\psi - g\bar{A}$. The latter also enters the $D = 11$ four-form through its field strength $\bar{F}$ and its Hodge dual. These contributions are wedged with internal forms X, Y which now read, from (2.45),

$$
X = -\frac{m^2}{576}\left(\alpha^{-1} e^{-3\Delta} f\right)\mathbb{J}_1 \, , \qquad Y = -\frac{mf}{96}\left[\mathbb{J}_3 - \frac{1}{4} r\,\alpha^2 dr \wedge (d\tau + A_{\mathrm{KE}})\right] .
\tag{2.53}
$$

This can be applied, in particular, for the two solutions corresponding to the IR fixed points of the superpotential deformations of the ABJM SCFT by terms quadratic or cubic in one of the chirals.

As a concluding remark, it is interesting to note that our results bring together in $D = 11$ the separate classification efforts of [61, 62] and [17, 37]. The supersymmetric solutions of $D = 4$ $\mathcal{N} = 2$ minimal gauged supergravity were classified in [61, 62]. By the consistency of our uplift, any such $D = 4$ solution can be fibred over any of the seven-dimensional manifolds of [17] and [37] to produce, via (2.51), a supersymmetric solution of $D = 11$ supergravity.

[3] When the $D = 4$ supergravity fields are turned off, the metric (2.28) agrees, up to a straightforward redefinition of ψ, with (4.13) of [17]. However, the background four-form (2.52) seems to disagree with their (4.14).

Chapter 3

Duality in maximal supergravity truncations

As reviewed in the invitation, duality has been recognised as one of the guiding principles in string theory and supergravity. Its rôle in the introduction of gaugings compatible with $\mathcal{N} = 8$ supersymmetry in $D = 4$ is explored in section 3.1, where the tensor hierarchy and embedding tensor formalisms are presented. Most of the discussion is particularised to the electric SO(8) gauging, and specially to its SU(3)-invariant sector [A], although much of it is only slightly altered for the dyonic ISO(7) and $[\mathrm{SO}(6) \times \mathrm{SO}(1,1)] \ltimes \mathbb{R}^{12}$ gaugings that will be relevant in later chapters.

Section 3.2 follows on discussing how to embed the $D = 4$ SO(8) theory in eleven dimensions while maintaining covariance under the duality group of the lower-dimensional supergravity. This allows one to obtain explicit Kaluza-Klein ansätze for the $D = 11$ fields in terms of the $D = 4$ counterparts, which make it possible to recover at once several previously-known solutions of M-theory and a new consistent truncation to the full minimal $\mathcal{N} = 2$ gauged supergravity. Some of the technical details in this discussion are relegated to appendices B–D.

Some of the materials covered in this chapter, such as gauged supergravity and the embedding tensor formalism, deserve themselves an entire dissertation and some of them in fact enjoy very nice recent reviews (see [63, 64]). For the sake of clarity, we have decided to briefly comment on some of their aspects here in order to fix notation and conventions.

3.1 Maximal supergravity in $D = 4$

3.1.1 Gauged sugra and the Embedding Tensor

In the toroidal reduction in the invitation, we saw how to obtain the field content and Lagrangian of (non-chiral) maximal supergravity in D dimensions out of eleven-dimensional supergravity compactified on T^{11-D} with only the constant modes on the torus kept. For $D = 4$, one can dualise all two-forms into scalars and in (I.17), and the field content from the reduction can be taken to be

$$g_{\mu\nu}(x)\ (\text{metric})\,, \qquad A_\mu^{AB}(x)\ (28\ \text{vectors})\,, \qquad \mathcal{V}_M{}^{\bar{M}}(x)\ (70\ \text{scalars})\,,$$
$$\psi_\mu^i(x)\ (8\ \text{gravitini})\,, \qquad \chi^{ijk}(x)\ (56\ \text{fermions})\,. \tag{3.1}$$

Here, the scalars can be understood as coordinates on the symmetric space

$$\frac{\mathrm{E}_{7(7)}}{\mathrm{SU}(8)}\,, \tag{3.2}$$

with the coset representative $\mathcal{V}$ transforming from the left in the fundamental of $\mathrm{E}_{7(7)}$, with index $M = 1, \ldots, 56$, and in the $\mathbf{28} + \overline{\mathbf{28}}$ of a local SU(8) from the right, whose fundamental is labelled by an index $i = 1, \ldots, 8$. The barred index that the coset representative carries is the shorthand ${}^{\bar{M}} = ({}^{[ij]},\ {}_{[ij]})$, with both sets related by complex conjugation. Similarly, the indices $A, B = 1, \ldots, 8$ that the vectors carry label the fundamental of $\mathrm{SL}(8,\mathbb{R})$, and the gravitini and spin-1/2 fermions live in the $\mathbf{8}$ and $\mathbf{56}$ of SU(8), which is accordingly identified with the R-symmetry of the $\mathcal{N} = 8$ superalgebra in $D = 4$. The dynamics for these fields is dictated by [18, 65]

$$\begin{aligned}\mathcal{L} = {}& R\,\mathrm{vol}_4 - \tfrac{1}{48} d\mathcal{M}_{MN} \wedge * d\mathcal{M}^{MN} \\ & + \tfrac{1}{2}\mathcal{I}_{[AB][CD]}\mathcal{H}_{(2)}^{AB} \wedge *\mathcal{H}_{(2)}^{CD} + \tfrac{1}{2}\mathcal{R}_{[AB][CD]}\mathcal{H}_{(2)}^{AB} \wedge \mathcal{H}_{(2)}^{CD} + (\text{fermions})\,.\end{aligned} \tag{3.3}$$

This is the Lagrangian of $\mathcal{N} = 8$ supergravity in $D = 4$ when the gauge group is $\mathrm{U}(1)^{28}$ and all matter is chargeless with respect to it. All fields appearing, both bosonic and fermionic, are massless. The scalar kinetic term describes a non-linear sigma model with (3.2) as the target space in terms of the symmetric matrix $\mathcal{M}_{MN} = 2\mathcal{V}_{(M}{}^{ij}\mathcal{V}_{N)ij}$. The vector field strengths, $\mathcal{H}_{(2)}^{AB} = dA^{AB}$, are non-minimally coupled to the scalars via the matrices $\mathcal{I}_{[AB][CD]}$ and $\mathcal{R}_{[AB][CD]}$, both symmetric under the exchange of pairs AB and CD, and the first of them positive definite.[1] In terms of these matrices, the field equations for the vectors are

$$d\mathcal{H}^{AB} = 0\,, \qquad d\big(\mathcal{I}_{[AB][CD]} * \mathcal{H}_{(2)}^{CD} + \mathcal{R}_{[AB][CD]}\mathcal{H}_{(2)}^{CD}\big) = 0\,, \tag{3.4}$$

which can be combined as

$$d\mathcal{F}^M = 0 \tag{3.5}$$

in terms of

$$\mathcal{F}^M = \begin{pmatrix} \mathcal{H}^{AB} \\ \tilde{\mathcal{H}}_{AB} \end{pmatrix}\,, \tag{3.6}$$

[1] We employ the "mostly pluses" convention for the signature of spacetime.

with the magnetic field strength

$$\tilde{\mathcal{H}}_{AB} = \mathcal{I}_{[AB][CD]} * \mathcal{H}^{CD}_{(2)} + \mathcal{R}_{[AB][CD]}\mathcal{H}^{CD}_{(2)} \,. \tag{3.7}$$

The definition of $\tilde{\mathcal{H}}_{AB}$ in terms of $\mathcal{H}^{AB}$ implies

$$\begin{aligned} *\mathcal{H} &= -\mathcal{I}^{-1}\mathcal{R}\mathcal{H} + \mathcal{I}^{-1}\tilde{\mathcal{H}}\,, \\ *\tilde{\mathcal{H}} &= -(\mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R})\mathcal{H} + \mathcal{R}\mathcal{I}^{-1}\tilde{\mathcal{H}}\,, \end{aligned} \tag{3.8}$$

which can be phrased as a twisted self-duality condition on (3.6),

$$* \,\mathcal{F}^M = -\Omega^{MN}\hat{\mathcal{M}}_{NP}\mathcal{F}^P \,. \tag{3.9}$$

This condition involves the symplectic form in $\mathbb{R}^{56}$,

$$\Omega^{MN} = \begin{pmatrix} 0 & \delta^{AB}{}_{CD} \\ -\delta_{AB}{}^{CD} & 0 \end{pmatrix} , \tag{3.10}$$

and the matrix

$$\hat{\mathcal{M}} = \begin{pmatrix} -\mathcal{I} - \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & \mathcal{R}\mathcal{I}^{-1} \\ \mathcal{I}^{-1}\mathcal{R} & -\mathcal{I}^{-1} \end{pmatrix} . \tag{3.11}$$

It is easy to check that this matrix is an element of $\mathrm{Sp}(56, \mathbb{R})$, satisfying

$$\hat{\mathcal{M}}_{MP}\,\hat{\mathcal{M}}_{NQ}\,\Omega^{MN} = -\Omega_{PQ}\,, \tag{3.12}$$

with $\Omega^{MN}\Omega_{NP} = -\delta^M{}_P$. Thus, the index $M = 1, \ldots, 56$ can be naturally understood as labelling the fundamental representation of the symplectic group. Conversely, any field strength satisfying (3.9) with (3.10) and (3.11) can be written as (3.6) with (3.7). The self-duality condition is therefore preserved if we transform simultaneously $\mathcal{F}^M$ and $\hat{\mathcal{M}}_{MN}$ under a symplectic rotation $E_M{}^N \in \mathrm{Sp}(56, \mathbb{R})$.

In $\mathcal{N} = 8$ supergravity this is precisely the case, as the symplectic matrix in (3.11) is in fact the coset representative $\mathcal{M}_{MN}$ [18]. Therefore, the full set of equations of motion (including Einstein), is invariant under the global symmetry group $\mathrm{E}_{7(7)} \subset \mathrm{Sp}(56, \mathbb{R})$. For this reason, it is convenient to describe this supergravity in an $\mathrm{E}_{7(7)}$-covariant way, which in the present case just requires supplementing the vector fields A^{AB} with redundant magnetic duals $\tilde{A}_{AB}$ into $\mathcal{A}^M$such that (3.5) is (locally) integrated by $\mathcal{F}^M = d\mathcal{A}^M$.

Contrary to what happens in higher dimensions, the theory (3.3) is not the unique $\mathcal{N} = 8$ supergravity in $D = 4$. One can promote a subgroup of $\mathrm{E}_{7(7)}$ [2]

[2] One can in principle consider gaugings inside $\mathrm{E}_{7(7)} \times \mathbb{R}^+$ [66]. However, when the generator of the trombone participates in the gauging, the resulting equations of motion do not admit an action. We will not consider these trombone gaugings in the following.

to a local symmetry by means of the embedding tensor [67–69], $\Theta_M{}^\alpha$,[3] which selects which of the generators of the global symmetry group of the ungauged theory participate in the gauging. The matter fields, $\Phi^\aleph = \{g,\ \mathcal{V},\ \psi,\ \chi\}$, and gauge bosons then transform under this gauge group as

$$\begin{aligned} \delta_\Lambda \Phi^\aleph &= g\,\Lambda^M\,X_M[\Phi^\aleph]\,, \\ \delta_\Lambda \mathcal{A}^M &= D\Lambda^M = d\Lambda^M + g\,X_{NP}{}^M \mathcal{A}^N \Lambda^P\,, \end{aligned} \tag{3.13}$$

with g the gauge group coupling constant, Λ a spacetime-dependent parameter, and $X_M = \Theta_M{}^\alpha\,t_\alpha$ the generators of the gauge algebra, for t_α the $\mathrm{E}_{7(7)}$ generators in the appropriate representation. In particular, in the fundamental representation $X_{MN}{}^P = \Theta_M{}^\alpha (t_\alpha)_N{}^P$. Covariance under (3.13) requires the introduction of covariant derivatives for the matter fields

$$d \;\;\to\;\; D = d - g\,\mathcal{A}^M\,X_M\,. \tag{3.14}$$

Consistency of the gauging requires it to be compatible with the field content (in particular, with supersymmetry) and that the embedding tensor is invariant under gauge transformations. These requirements are respectively known as the linear and quadratic constraints on the embedding tensor. The former demands

$$X_{(MNP)} = 0\,, \tag{3.15}$$

which implies that all the representations in the product

$$(\mathbf{56}\otimes\mathbf{56}\otimes\mathbf{56})_{\text{sym}} = \mathbf{56}\oplus\mathbf{6480}\oplus\mathbf{24320} \tag{3.16}$$

are absent from the embedding tensor, who could in principle live in

$$\mathbf{56}\otimes\mathbf{133} = \mathbf{56}\oplus\mathbf{912}\oplus\mathbf{6480}\,. \tag{3.17}$$

Therefore, (3.15) implies that $\Theta_M{}^\alpha$ only has components in **912**. On the other hand, the quadratic constraint can be stated as

$$[X_M, X_N] = -X_{MN}{}^P X_P\,, \tag{3.18}$$

which, in combination with (3.15), implies

$$\Omega^{MN}\Theta_M{}^\alpha\Theta_N{}^\beta = 0\,. \tag{3.19}$$

This condition can be understood as saying that there always exists a symplectic rotation such that $\Theta_M{}^\alpha = (\Theta_{AB}{}^\alpha, \Theta^{AB\alpha})$ can be transformed into $\hat{\Theta}_M{}^\alpha = (\hat{\Theta}_{AB}{}^\alpha, 0)$. This is known as the locality constraint, and guarantees that the dimension of the gauge group, equal to the rank of the embedding

[3] The adjoint $\mathrm{E}_{7(7)}$ index, $\alpha = 1,\ldots,133$, should not be confused with the flat indices in the $D=4$ vierbein $e_\mu{}^\alpha$.

tensor, is not bigger than the number of (electric) vectors appearing in the original Lagrangian. Let us note en passant that for non-maximal theories this constraint is not automatic, but an independent requirement [64].

The physical meaning of (3.18) is more profound than it seems at first sight. Obviously, it implies that the gauge algebra generators close into a subalgebra of Lie($E_{7(7)}$). However, the structure constants $X_{MN}{}^P$ are not necessarily antisymmetric in MN, but can be decomposed as

$$X_{MN}{}^P = X_{[MN]}{}^P + Z_{MN}{}^P\,, \tag{3.20}$$

with $Z_{MN}{}^P \equiv X_{(MN)}{}^P$. This symmetric component is generically non-zero, and therefore (3.18) also imposes $Z_{MN}{}^P X_P = 0$. The presence of this symmetric part and the fact that the X's play both the rôle of algebra generators and structure constants implies that the Jacobi identity is also satisfied only up to terms that vanish when contracted with the embedding tensor. Namely, from (3.18) with the generators in the fundamental representation of $E_{7(7)}$, it is easy to obtain

$$X_{[MN]}{}^P X_{[QP]}{}^R + X_{[QM]}{}^P X_{[NP]}{}^R + X_{[NQ]}{}^P X_{[MP]}{}^R = -Z_{P[Q}{}^R X_{MN]}{}^P\,. \tag{3.21}$$

To define a non-abelian gauge theory, apart from defining covariant derivatives we also need to find an appropriate generalisation for field strength of $\mathcal{A}^M$. The natural guess,

$$\mathcal{F}^M_{\text{guess}} = d\mathcal{A}^M + \tfrac{g}{2} X_{NP}{}^M \mathcal{A}^N \wedge \mathcal{A}^P\,, \tag{3.22}$$

turns out not to be a good one precisely because of the failure of the Jacobi identity in (3.21). Under (3.13), $\mathcal{F}^M_{\text{guess}}$ transforms as

$$\delta_\Lambda \mathcal{F}^M_{\text{guess}} = -g\Lambda^P X_{PN}{}^M \mathcal{F}^N_{\text{guess}} + 2g Z_{PQ}{}^M \left(\Lambda^P \mathcal{F}^Q_{\text{guess}} - \mathcal{A}^P \wedge \delta_\Lambda \mathcal{A}^Q\right), \tag{3.23}$$

which is non-covariant given the term proportional to $Z_{PQ}{}^M$. This non-covariance can be subsided by introducing a two-form $\mathcal{B}^{NP} = \mathcal{B}^{(NP)}$ with a Stückelberg coupling

$$\mathcal{F}^M = d\mathcal{A}^M + \tfrac{g}{2} X_{NP}{}^M \mathcal{A}^N \wedge \mathcal{A}^P + g\, Z_{NP}{}^M \mathcal{B}^{NP}\,. \tag{3.24}$$

This augmented field strength is covariant under the following gauge transformations of $\mathcal{A}^M$ and $\mathcal{B}^{MN}$,

$$\begin{aligned} \delta_{\Lambda,\Xi}\mathcal{A}^M &= D\Lambda^M - g\, Z_{NP}{}^M\, \Xi^{NP}\,, \\ \delta_{\Lambda,\Xi}\mathcal{B}^{MN} &= D\Xi^{NP} - \Lambda^{(M}\mathcal{F}^{N)} + \mathcal{A}^{(M} \wedge \delta_{\Lambda,\Xi}\mathcal{A}^{N)}\,, \end{aligned} \tag{3.25}$$

with Ξ a spacetime dependent one-form gauge parameter. Notice that for δ_Λ gauge transformations, (3.25) is tailor-suited to cancel the second term in

(3.23). These two-form potentials do not introduce new degrees of freedom, but are dual (in $D = 4$) to scalar fields. In fact, out of the a priori

$$(\mathbf{56} \otimes \mathbf{56})_{\mathrm{sym}} = \mathbf{133} \oplus \mathbf{1463}\,, \tag{3.26}$$

the linear constraint (3.15) implies that they are only valued in the adjoint of $\mathrm{E}_{7(7)}$ and can be taken as $\mathcal{B}^\alpha$, as expected for the scalar Noether currents. This is a remarkable interplay between duality groups and spacetime dimension that persists in other maximal supergravitites [70].

The addition of the two-forms in (3.24) modifies the Bianchi identity for $\mathcal{F}^M$ by a term depending on the field strength for $\mathcal{B}^\alpha$. This field strength is again not only the naïve expectation, but depends on the vectors and a three-form potential via another Stückelberg coupling to ensure gauge covariance. This three-form in turn needs a four-form potential to enjoy a gauge covariant field strength, and this exhausts the elements of the tensor hierarchy in $D = 4$. The complete $D = 4$ tensor hierarchy is thus comprised by vectors in the **56** of $\mathrm{E}_{7(7)}$, two-forms in the **133**, three-forms in the **912**, and four-forms in the $\mathbf{133} \oplus \mathbf{8645}$ [70, 71]. This is conveniently summarised (up to the four-forms, which will not play a major rôle in the following) in (3.47) below. The higher-rank forms are not dual to any dynamical degrees of freedom, but carry information about the gauging. In particular, the three-forms are dual to the embedding tensor and the four form to the constraints it satisfies, as suggested by the representations that they furnish. The field strengths for these forms are (3.24) and [72]

$$\begin{aligned}
\mathcal{H}^\alpha_{(3)} &= D\mathcal{B}^\alpha + (t^\alpha)_{MN}\, \mathcal{A}^M \wedge \left[d\mathcal{A}^N + \tfrac{1}{3} X_{PQ}{}^N \mathcal{A}^P \wedge \mathcal{A}^Q\right] + Y^\alpha{}_M{}^\beta \mathcal{C}_\beta{}^M\,,\\
\mathcal{H}_{(4)\alpha}{}^M &= D\mathcal{C}_\alpha{}^M + \mathcal{F}^M \wedge \mathcal{B}_\alpha - \tfrac{1}{2} Z_{NP}{}^M \mathcal{B}^{NP} \wedge \mathcal{B}_\alpha\\
&\quad + \tfrac{1}{3}(t_\alpha)_{NP}\mathcal{A}^M \wedge \mathcal{A}^N \wedge \left[\mathcal{F}^P - \tfrac{1}{4} X_{QR}{}^P \mathcal{A}^Q \wedge \mathcal{A}^R - Z_{QR}{}^P B^{QR}\right]\\
&\quad + W_\alpha{}^{M\beta\gamma} D_{\beta\gamma} + W_{\alpha NPQ}{}^M D^{NPQ} + W_{\alpha NP}{}^{M\beta} D_\beta{}^{NP}\,,
\end{aligned} \tag{3.27}$$

where the invariant $\mathrm{E}_{7(7)}$ tensors involved are

$$\begin{aligned}
Y_{\alpha M}{}^\beta &= \Theta_M{}^\gamma f_{\alpha\gamma}{}^\beta - (t_\alpha)_M{}^N \Theta_N{}^\beta\,,\\
W_\alpha{}^{M\beta\gamma} &= \tfrac{1}{2}\Theta^{M[\beta}\delta^{\gamma]}{}_\alpha\,, \qquad W_{\alpha NPQ}{}^M = (t_\alpha)_{(NP}\delta_{Q)}{}^M\,,\\
W_{\alpha NP}{}^{M\beta} &= \Theta_N{}^\gamma f_{\alpha\gamma}{}^\beta \delta_P{}^M + X_{NP}{}^M \delta^\beta{}_\alpha - Y_{\alpha P}{}^\beta \delta_N{}^M\,.
\end{aligned} \tag{3.28}$$

with $f_{\alpha\gamma}{}^\beta$ the $\mathrm{E}_{7(7)}$ structure constants. The corresponding Bianchi identities are, by construction,

$$D\mathcal{F}^M = Z_{NP}{}^M \mathcal{H}^{NP}_{(3)}\,, \qquad D\mathcal{H}^\alpha_{(3)} = Y^\alpha{}_M{}^\beta \mathcal{H}_{(4)\beta}{}^M + (t_\alpha)_{MN}\mathcal{F}^M \wedge \mathcal{F}^N\,, \tag{3.29}$$

plus the trivial $d\mathcal{H}_{(4)} = 0$.

At this point, we have all the necessary ingredients to build an action for our gauged supergravity. Apart from the obvious covariantisation of derivatives and field strengths, gauging additionally introduces a few important changes in (3.3). Firstly, the further insertions of gauge fields in the Lagrangian require the presence of a topological term involving the vectors and two-form potentials to ensure gauge covariance. They also demand mass terms for the fermions for the theory to be supersymmetric at first order in g. These mass matrices are constructed out of contractions of the embedding tensor and scalar representatives,

$$T_{\bar{M}\bar{N}}{}^{\bar{P}} = X_{MN}{}^{P}(\mathcal{V}^{-1})^{M}{}_{\bar{M}}(\mathcal{V}^{-1})^{N}{}_{\bar{N}}\,\mathcal{V}_{P}{}^{\bar{P}}\,, \tag{3.30}$$

and take the form

$$A_{1ij} = \tfrac{4}{21}T_{ikjl}kl\,, \qquad A_{2l}{}^{ijk} = 2\,T_{lm}{}^{imjk}\,, \tag{3.31}$$

and

$$A_{3}{}^{ijk\,lmn} = \tfrac{\sqrt{2}}{144}\,\epsilon^{ijkpqr[lm}A_{2}{}^{n]}{}_{pqr}\,, \tag{3.32}$$

which can be identified as the irreducible SU(8) modules within (3.30), as they satisfy

$$A_{1ij} = A_{1(ij)}\,, \qquad A_{2l}{}^{ijk} = A_{2l}{}^{[ijk]}\,, \qquad A_{2k}{}^{ijk} = 0\,. \tag{3.33}$$

Finally, to preserve supersymmetry at order g^2, a potential for the scalar fields must be introduced. This potential, quadratic in the embedding tensor, reads

$$V = \frac{g^2}{168}X_{MP}{}^{R}X_{NQ}{}^{S}\mathcal{M}^{MN}\big(\mathcal{M}^{PQ}\mathcal{M}_{RS} + 7\delta_S^P\delta_R^Q\big)\,. \tag{3.34}$$

Thereby, the generalisation of (3.3) when a non-trivial gauging is active can be given as

$$\begin{aligned}\mathcal{L} = R\,\mathrm{vol}_4 - \tfrac{1}{48}D\mathcal{M}_{MN}\wedge *D\mathcal{M}^{MN} - V\,\mathrm{vol}_4 \\ + \tfrac{1}{2}\mathcal{M}_{MN}\mathcal{F}^{M}\wedge *\mathcal{F}^{N} + \mathcal{L}_{\text{top}} + (\text{fermions})\,,\end{aligned} \tag{3.35}$$

or, integrating the two-forms from the topological terms to get back to the original fields,

$$\begin{aligned}\mathcal{L} = R\,\mathrm{vol}_4 - \tfrac{1}{48}D\mathcal{M}_{MN}\wedge *D\mathcal{M}^{MN} - V\,\mathrm{vol}_4 \\ + \tfrac{1}{2}\mathcal{I}_{[IJ][KL]}\mathcal{H}^{IJ}_{(2)}\wedge *\mathcal{H}^{KL}_{(2)} + \tfrac{1}{2}\mathcal{R}_{[IJ][KL]}\mathcal{H}^{IJ}_{(2)}\wedge \mathcal{H}^{KL}_{(2)} + (\text{fermions})\,.\end{aligned} \tag{3.36}$$

Let us observe that (a subset of) the equations of motion following from (3.36) can be encoded in the Bianchi identities for the tensor hierarchy fields [72]. The duality relations that recover these equations are (3.6) and

$$\mathcal{H}^{\alpha}_{(3)} = \tfrac{1}{2} * j^{\alpha}\,, \qquad \mathcal{H}_{(4)\alpha}{}^{M} = \tfrac{1}{2} * \Big(\frac{\partial V}{\partial \Theta_{M}{}^{\alpha}}\Big)\,, \tag{3.37}$$

with j^{α} the scalar Noether currents.

Gaugings inside $\mathrm{SL}(8,\mathbb{R})$ and restricted hierarchies

Any embedding tensor satisfying the linear and quadratic constraints (3.15) and (3.18) defines a consistently gauged maximal supergravity. In the following, we shall focus on three choices that enjoy a higher-dimensional interpretation, all of them subgroups of $\mathrm{SL}(8,\mathbb{R}) \subset \mathrm{E}_{7(7)}$.

The relevant representations discussed above decompose as follows

$$\begin{array}{ccc} \mathrm{E}_{7(7)} & \supset & \mathrm{SL}(8,\mathbb{R}) \\ \mathbf{56} & \rightarrow & \mathbf{28} \oplus \mathbf{28'}\ , \\ \mathbf{133} & \rightarrow & \mathbf{63} \oplus \mathbf{70}\ , \\ \mathbf{912} & \rightarrow & \mathbf{36} \oplus \mathbf{36'} \oplus \mathbf{420} \oplus \mathbf{420'}\ . \end{array} \tag{3.38}$$

Accordingly, the embedding tensor can be decomposed as

$$\Theta_M{}^\alpha = \left(\Theta_{AB}{}^C{}_D,\, \Theta_{AB}{}^{CDEF};\, \Theta^{ABC}{}_D,\, \Theta^{ABCDEF}\right), \tag{3.39}$$

with the different components given in terms of θ_{AB}, ξ^{AB}, $\zeta^A{}_{BCD}$ and $\tilde{\zeta}_A{}^{BCD}$ in the **36**, **36′**, **420** and **420′**,

$$\begin{aligned} \Theta_{AB}{}^C{}_D &= 2\delta^C{}_{[A}\theta_{B]D} + \zeta^C{}_{ABD}\,, & \Theta_{AB}{}^{CDEF} &= a\,\tilde{\zeta}_{[A}{}^{GHI}\delta_{B]GHI}{}^{CDEF}\,, \\ \Theta^{ABC}{}_D &= 2\delta_D{}^{[A}\xi^{B]C} + \tilde{\zeta}_D{}^{ABC}\,, & \Theta^{ABCDEF} &= b\,\zeta^{[A}{}_{GHI}\epsilon^{B]CDEFGHI}\,, \end{aligned} \tag{3.40}$$

for a and b constants. Therefore, to have a gauge group which only involves the $t_C{}^D$ generators of $\mathrm{SL}(8,\mathbb{R}) \subset \mathrm{E}_{7(7)}$, we are forced to restricting to the **36** and **36′** components of the embedding tensor and setting $\zeta^A{}_{BCD}$ and $\tilde{\zeta}_A{}^{BCD}$ to zero. Doing so, the quadratic constraint (3.18) requires

$$(\theta\xi)_A{}^C\delta_B{}^D - (\theta\xi)_B{}^D\delta_A{}^C = 0\,, \tag{3.41}$$

which implies

$$(\theta\xi)_A{}^B = \tfrac{1}{8}\mathrm{tr}(\theta\xi)\delta_A{}^B \quad \Leftrightarrow \quad \begin{cases} (\theta\xi)_A{}^B = 0 \\ \xi = c\theta^{-1} \end{cases} . \tag{3.42}$$

We will restrict ourselves to the case in which θ and ξ are singular. Then, by means of an $\mathrm{SL}(8,\mathbb{R})$ conjugation, the most general choice is given by

$$\begin{aligned} \theta_{AB} &= \mathrm{diag}(+,\ldots,+,-,\ldots,-,\ 0\,,\ldots,0\,,\ 0\,,\ldots,0\,,0,\ldots,0)\,, \\ \xi^{AB} &= \mathrm{diag}(\underbrace{0,\ldots,0}_{p},\ \underbrace{0,\ldots,0}_{q},\ \underbrace{+,\ldots,+}_{p'},\underbrace{-,\ldots,-}_{q'},\underbrace{0,\ldots,0}_{r})\,, \end{aligned} \tag{3.43}$$

where $\pm$ denotes ± 1 and $p+q+p'+q'+r=8$. In the following, we will take $r=0$, leading to $[\mathrm{SO}(p,q)\times\mathrm{SO}(p',q')] \ltimes N$ gaugings with N abelian. In particular, we will consider the electric SO(8) gauging [73], with

$$\theta_{AB} = g\,\delta_{AB}\,, \qquad \xi^{AB} = 0\,; \tag{3.44}$$

$$\begin{array}{ccc} \mathrm{SO}(8) & \hookrightarrow & \text{M-theory on } \mathrm{AdS}_4 \times S^7 \\ \mathrm{ISO}(7) & \hookrightarrow & \text{mIIA on } \mathrm{AdS}_4 \times S^6 \\ \big[\mathrm{SO}(6) \times \mathrm{SO}(1,1)\big] \ltimes \mathbb{R}^{12} & \hookrightarrow & \text{IIB on } \mathrm{AdS}_4 \times S^1 \times S^5 \text{ S-fold} \end{array}$$

Figure 3.1: Uplift of gauged maximal supergravity into string theory on spheres.

the dyonic $\mathrm{ISO}(7) \simeq \mathrm{SO}(7) \ltimes \mathbb{R}^7$ gauging [74], with

$$\theta_{AB} = g\,\mathrm{diag}(1_{\times 7}, 0)\,, \qquad \xi^{AB} = \mathrm{diag}(0_{\times 7}, m)\,; \tag{3.45}$$

and the dyonic $[\mathrm{SO}(6) \times \mathrm{SO}(1,1)] \ltimes \mathbb{R}^{12}$ gauging [75], with

$$\theta_{AB} = g\,\mathrm{diag}(1_{\times 6}, 0, 0)\,, \qquad \xi^{AB} = m\,\mathrm{diag}(0_{\times 6}, 1, -1)\,, \tag{3.46}$$

where g and m are respectively the electric and magnetic coupling constants, which can be set to one without loss of generality [76]. These three gaugings stand out for being the only known cases to oxidise respectively into M-theory, massive type IIA and type IIB supergravity on spheres [25, 26, 28]. See figure 3.1 for a schematic description of these embeddings. In the remainder of this chapter, we will study these string theory uplifts, focusing mainly on the electric SO(8) case. In Part II, the three instances will be treated in more equal footing and the Kaluza-Klein spectra around their solutions analysed.

To describe the embeddings of these gaugings into the corresponding ten- or eleven-dimensional supergravities, not all the tensor hierarchy fields are necessary. For the gaugings in $\mathrm{SL}(8,\mathbb{R})$, we can restrict ourselves to the $\mathrm{SL}(8,\mathbb{R})$-covariant tensor hierarchy that only retains

$\mathrm{GL}(4,\mathbb{R})$	$\mathrm{E}_{7(7)}$	$\mathrm{SL}(8,\mathbb{R})$
metric	$\mathbf{1}$ ds_4^2	$\mathbf{1}$ ds_4^2
scalars	$\mathbf{56}$ $\mathcal{V}_M{}^{\bar{M}}$	$\mathbf{28} + \mathbf{28'}$ $\mathcal{V}_{AB}{}^{ij}$, $\mathcal{V}^{AB\,ij}$
vectors	$\mathbf{56}$ $\mathcal{A}^M$	$\mathbf{28} + \mathbf{28'}$ $\mathcal{A}^{AB}$, $\tilde{\mathcal{A}}_{AB}$
two-forms	$\mathbf{133}$ $\mathcal{B}^\alpha$	$\mathbf{63}$ $\mathcal{B}_A{}^B$
three-forms	$\mathbf{912}$ $\mathcal{C}^M{}_\alpha$	$\mathbf{36'}$ $\mathcal{C}^{AB}$

(3.47)

and all the fermionic fields. This subset of fields can be shown to close under the Bianchi identities, duality relations and supersymmetry variations. The field strengths and duality relations depend on the actual embedding tensor considered, via the factors of Z, Y and W in (3.24) and (3.27). For the electric SO(8) gauging (3.44), they are

$$\begin{aligned}
\mathcal{H}_{(2)}^{AB} &= d\mathcal{A}^{AB} - g\,\delta_{CD}\mathcal{A}^{AC}\wedge\mathcal{A}^{DB}\ ,\\
\tilde{\mathcal{H}}_{(2)AB} &= d\tilde{\mathcal{A}}_{AB} + g\,\delta_{C[A}\mathcal{A}^{CD}\wedge\tilde{\mathcal{A}}_{B]D} + 2g\,\delta_{C[A}\mathcal{B}_{B]}{}^{C}\ ,\\
\tilde{\mathcal{H}}_{(3)A}{}^{B} &= D\mathcal{B}_{A}{}^{B} - 2g\,\delta_{AC}\mathcal{C}^{BC} + \tfrac{1}{2}\mathcal{A}^{BC}\wedge d\tilde{\mathcal{A}}_{AC} + \tfrac{1}{2}\tilde{\mathcal{A}}_{AC}\wedge d\mathcal{A}^{BC}\\
&\quad - \tfrac{g}{2}\,\delta_{CD}\mathcal{A}^{BC}\wedge\mathcal{A}^{DE}\wedge\tilde{\mathcal{A}}_{AE} + \tfrac{g}{6}\,\delta_{AC}\mathcal{A}^{BD}\wedge\mathcal{A}^{CE}\wedge\tilde{\mathcal{A}}_{AE} - \tfrac{1}{8}\delta_{A}^{B}(\text{trace})\ ,\\
\mathcal{H}_{(4)}^{AB} &= D\mathcal{C}^{AB} + \tfrac{1}{6}\big[\mathcal{A}^{AC}\wedge\mathcal{A}^{BD}\wedge d\tilde{\mathcal{A}}_{CD} - \mathcal{A}^{C(A}\wedge\tilde{\mathcal{A}}_{CD}\wedge d\mathcal{A}^{B)D}\\
&\quad - g\delta_{CD}\mathcal{A}^{C(A}\wedge\mathcal{A}^{B)E}\wedge\mathcal{A}^{DF}\wedge\tilde{\mathcal{A}}_{EF}\big] - \mathcal{H}_{(2)}^{C(A}\wedge\mathcal{B}_{C}{}^{B)}\,.
\end{aligned}\tag{3.48}$$

with the covariant derivative in (3.14) reducing to

$$D = d - g\,\mathcal{A}^{AB}\,t_{[A}{}^{C}\,\delta_{B]C}\ .\tag{3.49}$$

These fields strengths satisfy the Bianchi identities that follow from (3.29)

$$\begin{aligned}
&D\mathcal{H}_{(2)}^{AB} = 0\ , \qquad D\tilde{\mathcal{H}}_{(2)AB} = -2g\,\mathcal{H}_{(3)[A}{}^{C}\delta_{B]C}\ ,\\
&D\mathcal{H}_{(3)A}{}^{B} = \tilde{\mathcal{H}}_{(2)AC}\wedge\mathcal{H}_{(2)}^{BC} - 2g\,\delta_{AC}\mathcal{H}_{(4)}^{BC} - \tfrac{1}{8}\delta_{A}^{B}(\text{trace})\ , \qquad d\mathcal{H}_{(4)}^{AB} = 0\ .
\end{aligned}\tag{3.50}$$

Finally, the $E_{7(7)}$ duality hierarchy in (3.37) reduces to

$$\begin{aligned}
\tilde{\mathcal{H}}_{(2)AB} &= \mathcal{I}_{[AB][CD]} * \mathcal{H}_{(2)}^{CD} + \mathcal{R}_{[AB][CD]}\mathcal{H}_{(2)}^{CD}\ ,\\
\mathcal{H}_{(3)A}{}^{B} &= \tfrac{1}{12}(t_{A}{}^{B})_{M}{}^{P}\mathcal{M}_{NP} * D\mathcal{M}^{MN}\ ,\\
\mathcal{H}_{4}^{AB} &= \tfrac{1}{84}X_{NQ}{}^{S}(t_{C}{}^{(A|})_{P}{}^{R}\mathcal{M}^{|B)C\,M}\big(\mathcal{M}^{PQ}\mathcal{M}_{RS} + 7\delta_{S}^{P}\delta_{R}^{Q}\big)\,\mathrm{vol}_4\ ,
\end{aligned}\tag{3.51}$$

which make manifest that this restriction is sufficient for the SO(8) gauging. Similar formulae in the case of the ISO(7) gauging can be found in [74]. When written in terms of the original $\mathcal{N} = 8$ supergravity fields (3.1) via (3.51), equations (3.48) and (3.50) recover the equations of motion that follow from (3.36).

3.1.2 SU(3)-invariant truncation of SO(8)-gauging

For most practical applications, the field content in (3.1) is unnecessary and difficult to manage. Often, it proves useful to set many of them to zero by means of a consistent truncation. In the remainder of this section, we study the SO(8) gauging and its associated tensor hierarchy keeping only the fields which are singlets under the unique SU(3) $\subset$ SO(8). This SU(3)–invariant theory corresponds to an $\mathcal{N} = 2$ supergravity coupled to a vector and a hypermultiplet, and, as shown in [77], its potential retains a rich structure.

In addition to the fields entering these $\mathcal{N} = 2$ multiplets, we wish to consider the SU(3)–singlets in the restricted tensor hierarchy in (3.47). The relevant bosonic matter content thus includes

$$
\begin{array}{rcl}
\text{the metric} & : & ds_4^2\,, \\
\text{6 scalars} & : & \varphi\,,\ \chi\,,\ \phi\,,\ a\,,\ \zeta\,,\ \tilde{\zeta}\,, \\
\text{4 vectors} & : & A^0\,,\ A^1\,,\ \tilde{A}_0\,,\ \tilde{A}_1\,, \\
\text{5 two-form} & : & B^0\,,\ B^2\,,\ B^{ab} = B^{(ab)}\,, \\
\text{4 three-form} & : & C^1\,,\ C^{ab} = C^{(ab)}\,,
\end{array}
\tag{3.52}
$$

all of them real. The superscripts on B^0, B^2 and C^1 are just labels without further meaning. The electric and magnetic vectors can be collectively denoted A^Λ and $\tilde{A}_\Lambda$, with the index $\Lambda = 0, 1$ formally labelling "half" the fundamental representation of Sp$(4, \mathbb{R})$. The indices on B^{ab} and C^{ab} take on two values which, for convenience, are labelled $a = 7, 8$. The index a formally labels a doublet of SL(2), but we do not attach any significance to its position as it can be raised and lowered with δ_{ab}. See appendix B for the embedding of the SU(3)–invariant fields (3.52) into their parent $\mathcal{N} = 8$ counterparts in (3.47).

Only the metric, the scalars and the vector fields enter the conventional Lagrangian. The fields φ, ϕ and a are proper scalars, while χ, ζ and $\tilde{\zeta}$ are pseudoscalars. All of these parametrise a submanifold

$$
\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SU}(2,1)}{\mathrm{SU}(2) \times \mathrm{U}(1)}
\tag{3.53}
$$

of $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$ in (3.2), where each factor respectively contains the vector, (φ, χ), and the hypermultiplet, $q^u \equiv (\phi, a, \zeta, \tilde{\zeta})$, (pseudo)scalars.[4] The vectors gauge (electrically, in the usual symplectic frame), the $\mathrm{U}(1)^2$, compact Cartan subgroup of the hypermultiplet isotropy group. In the Iwasawa parametrisation of the scalar manifold (3.53), the bosonic Lagrangian reads

$$
\begin{aligned}
\mathcal{L} = {} & R\,\mathrm{vol}_4 + \tfrac{3}{2}(d\varphi)^2 + \tfrac{3}{2}e^{2\varphi}\,(d\chi)^2 \\
& + 2(D\phi)^2 + \tfrac{1}{2}\,e^{4\phi}\left(Da + \tfrac{1}{2}(\zeta D\tilde{\zeta} - \tilde{\zeta} D\zeta)\right)^2 + \tfrac{1}{2}\,e^{2\phi}\,(D\zeta)^2 + \tfrac{1}{2}\,e^{2\phi}\,(D\tilde{\zeta})^2 \\
& + \tfrac{1}{2}\,\mathcal{I}_{\Lambda\Sigma}\,H^\Lambda_{(2)} \wedge * H^\Sigma_{(2)} + \tfrac{1}{2}\,\mathcal{R}_{\Lambda\Sigma}\,H^\Lambda_{(2)} \wedge H^\Sigma_{(2)} - V\,\mathrm{vol}_4\ ,
\end{aligned}
\tag{3.54}
$$

[4] We will rarely need indices to label the scalars but, when needed, the local indices will be denoted $m = 1, \ldots, 6$, on the entire manifold (3.53), $\alpha = 1, 2$ on the first factor, and $u = 1, \ldots, 4$ on the second.

with $(d\varphi)^2 \equiv d\varphi \wedge *d\varphi$, etc. The covariant derivatives of the hyperscalars take on the form

$$
\begin{aligned}
D\phi = d\phi - gA^0\, a\,, \qquad\qquad Da &= da + gA^0\big(1 + e^{-4\phi}(Z^2 - Y^2)\big)\,,\\
D\zeta &= d\zeta + gA^0\, e^{-2\phi}\big(\zeta\, Z - \tilde{\zeta}\, Y\big) - 3gA^1\, \tilde{\zeta}\,,\\
D\tilde{\zeta} &= d\tilde{\zeta} + gA^0\, e^{-2\phi}\big(\tilde{\zeta}\, Z + \zeta\, Y\big) + 3gA^1\, \zeta\,.
\end{aligned}
\tag{3.55}
$$

Following [74], here and in the following we employ the shorthand definitions

$$
X \equiv 1 + e^{2\varphi}\chi^2\,, \qquad Y \equiv 1 + \tfrac{1}{4}\, e^{2\phi}\,(\zeta^2 + \tilde{\zeta}^2)\,, \qquad Z \equiv e^{2\phi}\, a\,. \tag{3.56}
$$

The covariant derivatives (3.55) correspond to an electric gauging of the $\mathrm{U}(1)^2$ Cartan subgroup of $\mathrm{SU}(2) \times \mathrm{U}(1) \subset \mathrm{SU}(2,1)$ generated by

$$
k_0 = \tfrac{1}{\sqrt{2}}\,(k[E_2] - k[F_2])\,, \qquad\qquad k_1 = -k[H_2]\,, \tag{3.57}
$$

where $k[E_2]$, etc., are $\mathrm{SU}(2,1)$ Killing vectors: see (B.18) and (B.19) for the explicit expressions for the Killing vectors of the scalar manifold (3.53) in our parametrisation.

The scalar potential V in (3.54) reads

$$
\begin{aligned}
g^{-2}V = {}& 6e^{-2\phi-\varphi}(Y-1)\big(e^{4\phi} + Y^2 + Z^2\big)X^2 - 12e^{\varphi}\\
& - 6e^{-2\phi-\varphi}XY\big(e^{4\phi} + Y^2 + Z^2\big) - 12e^{\varphi}(Y-1)\big(1 + Y - \tfrac{3}{2}XY\big)\\
& + e^{-3\varphi}\Big[\tfrac{1}{2}e^{-4\phi} + a^2 - 1 + \tfrac{1}{2}e^{4\phi}(1+a^2)^2 + e^{-4\phi}(Y-1)\big(Z^2 - e^{4\phi}\big)\\
& + \tfrac{1}{2}e^{-4\phi}(Y-1)\big(1 + Y(1 + 2e^{4\phi} + 2Z^2) + Y^2 + Y^3\big)\Big]X^3\,,
\end{aligned}
\tag{3.58}
$$

and derives from the following real superpotential (squared)

$$
\begin{aligned}
W^2 = \tfrac{1}{32}\, g^2\, X \Bigg[& 12e^{-\varphi-2\phi}(X-2)(Y-2)\Big(Y^2 + Z^2 + e^{4\phi}\Big) + 36e^{\varphi}Y^2\\
& + e^{-3\varphi-4\phi}X^2\Big(Y^2 + Z^2 + e^{4\phi}\Big)^2 - 16e^{-3\varphi}X^2(Y-1)\\
& - 48e^{-\varphi-2\phi}\sqrt{(X-1)(Y-1)\Big[\big(e^{4\phi} - Y^2 + Z^2\big)^2 + 4Y^2Z^2\Big]}\;\Bigg]\,,
\end{aligned}
\tag{3.59}
$$

through the usual formula

$$
\tfrac{1}{4}V = 2G^{mn}\partial_m W \partial_n W - 3W^2\,. \tag{3.60}
$$

Here, G_{mn}, $m = 1, \ldots, 6$, denotes the nonlinear sigma model metric on (3.53), and G^{mn} its inverse, which can be read off from the scalar kinetic terms in the Lagrangian (3.54). The superpotential (3.59) corresponds to one of the eigenvalues of the $\mathcal{N} = 8$ gravitino mass matrix restricted to the SU(3)–singlet space. See [78] for the $\mathcal{N} = 2$ special geometry of the model, in unitary gauge for the scalar coset. Superpotentials have previously appeared, also in unitary gauge, in [79, 80].

Finally, the gauge kinetic matrix is

$$\mathcal{N}_{\Lambda\Sigma} = \mathcal{R}_{\Lambda\Sigma} + i\,\mathcal{I}_{\Lambda\Sigma} = \frac{1}{(2\,e^{\varphi}\,\chi + i)} \begin{pmatrix} -\dfrac{e^{3\varphi}}{(e^{\varphi}\,\chi - i)^2} & \dfrac{3\,e^{2\varphi}\,\chi}{(e^{\varphi}\,\chi - i)} \\ \dfrac{3\,e^{2\varphi}\,\chi}{(e^{\varphi}\,\chi - i)} & 3\,(e^{\varphi}\,\chi^2 + e^{-\varphi}) \end{pmatrix}, \tag{3.61}$$

and the (electric) two-form field strengths that appear in (3.54) are simply

$$H^{\Lambda}_{(2)} = dA^{\Lambda}\,, \qquad \Lambda = 0, 1. \tag{3.62}$$

Among the forms in the SL$(8,\mathbb{R})$ restricted tensor hierarchy discussed in the previous section, the ones that survive in this truncation are those that transform as singlets under SU(3) $\subset$ SL$(8,\mathbb{R})$. The complete list is given in (3.52). See appendix B for further details. The field strengths of these SU(3)–invariant tensor hierarchy fields can be similarly obtained by suitably particularising the $\mathcal{N} = 8$ expressions in (3.48). The electric vector field strengths have already been given in (3.62), while the magnetic field strengths are

$$\tilde{H}_{(2)0} = d\tilde{A}_0 + gB^0\,, \qquad \tilde{H}_{(2)1} = d\tilde{A}_1 - 2gB^2\,. \tag{3.63}$$

The three-form field strengths read, in turn,

$$\begin{aligned} H^0_{(3)} &= dB^0\,, \qquad H^2_{(3)} = dB^2\,, \\ H^{ab}_{(3)} &= DB^{ab} + \tfrac{1}{4}\big(3A^0 \wedge d\tilde{A}_0 + 3\tilde{A}_0 \wedge dA^0 - A^1 \wedge d\tilde{A}_1 - \tilde{A}_1 \wedge dA^1\big)\,\delta^{ab} \\ &\qquad + 3g\,C^1\,\delta^{ab} - 4g\,C^{ab} + \tfrac{1}{2}g\,C^c{}_c\,\delta^{ab}\,, \end{aligned} \tag{3.64}$$

with $DB^{ab} = dB^{ab} + 2g\,\epsilon^{c(a}A^0 \wedge B^{b)}{}_c$. Finally, the four-form field strengths in (3.48) reduce to

$$\begin{aligned} H^1_{(4)} &= dC^1 - \tfrac{1}{3}\,H^1_{(2)} \wedge B^2\,, \\ H^{ab}_{(4)} &= DC^{ab} + \tfrac{1}{2}\,H^0_{(2)} \wedge \big(\epsilon^{(a}{}_c\,B^{b)c} + B^0\,\delta^{ab}\big)\,, \end{aligned} \tag{3.65}$$

with $DC^{ab} = dC^{ab} + 2g\epsilon^{c(a}A^0 \wedge C^{b)}{}_c$.

The field strengths (3.62)-(3.65) are subject to the Bianchi identities

$$dH^0_{(2)} = 0\,, \qquad dH^1_{(2)} = 0\,, \qquad d\tilde{H}_{(2)0} = gH_{(3)0}\,, \qquad d\tilde{H}_{(2)1} = -2gH_{(3)2}\,,$$

$$DH^{ab}_{(3)} = \Big(\tfrac{3}{2}H^0_{(2)} \wedge \tilde{H}_{(2)0} - \tfrac{1}{2}H^1_{(2)} \wedge \tilde{H}_{(2)1} + 3gH^1_{(4)} + \tfrac{1}{2}gH_{(4)c}{}^{c}\Big)\delta^{ab} - 4gH^{ab}_{(4)}\,,$$

$$dH^0_{(3)} = 0\,, \qquad dH^2_{(3)} = 0\,, \qquad dH^1_{(4)} = 0\,, \qquad dH^{ab}_{(4)} = 0\,, \quad (3.66)$$

where we have used $DH^{ab}_{(3)} = dH^{ab}_{(3)} - 2g\epsilon^{(a}{}_{c}\,A^0 \wedge H^{b)c}_{(3)}$. These expressions particularise (3.50) to the present case.

As in the $\mathcal{N}=8$ setting, all of the fields in (3.52) carry degrees of freedom, although not independent ones. The magnetic two-form field strengths can be written as scalar-dependent combinations of the electric gauge field strengths and their Hodge duals:

$$\begin{aligned}
\tilde{H}_{(2)0} = \frac{1}{X^2(4X-3)}\,&\big[-e^{3\varphi}(3X-2)*H^0_{(2)} + 3e^{\varphi}X(X-1)*H^1_{(2)} \\
&-2e^{6\varphi}\chi^3\,H^0_{(2)} + 3\chi\,e^{2\varphi}X(2X-1)\,H^1_{(2)}\big]\,, \\
\tilde{H}_{(2)1} = \frac{1}{X(4X-3)}\,&\big[3e^{\varphi}(X-1)*H^0_{(2)} - 3e^{-\varphi}X^2*H^1_{(2)} \\
&+3\chi e^{2\varphi}(2X-1)\,H^0_{(2)} + 6\chi X^2\,H^1_{(2)}\big]\,.
\end{aligned} \quad (3.67)$$

The three-form field strengths are dual to scalar-dependent combinations of derivatives of scalars:

$$\begin{aligned}
H^0_{(3)} &= -*\Big[\big(Y^2-2Y+Z^2+e^{4\phi}\big)\big(Da+\tfrac{1}{2}(\zeta D\tilde{\zeta}-\tilde{\zeta}D\zeta)\big)+Y\big(\zeta D\tilde{\zeta}-\tilde{\zeta}D\zeta\big) \\
&\qquad +2aDY-4aYD\phi\Big]\,, \\
H^2_{(3)} &= 3\,e^{2\phi}*\Big[(Y-1)\big(Da+\tfrac{1}{2}(\zeta D\tilde{\zeta}-\tilde{\zeta}D\zeta)\big)+\tfrac{1}{2}\big(\zeta D\tilde{\zeta}-\tilde{\zeta}D\zeta\big)\Big]\,, \\
H^{77}_{(3)} &= *\Big[Ze^{2\phi}\big(2Da+\zeta D\tilde{\zeta}-\tilde{\zeta}D\zeta\big)+2DY-4YD\phi+3\big(d\varphi-e^{2\varphi}\chi d\chi\big)\Big]\,, \\
H^{78}_{(3)} &= *\Big[\big(Y^2-2Y+Z^2-e^{4\phi}\big)\big(Da+\tfrac{1}{2}(\zeta D\tilde{\zeta}-\tilde{\zeta}D\zeta)\big)+Y\big(\zeta D\tilde{\zeta}-\tilde{\zeta}D\zeta\big) \\
&\qquad +2aDY-4aYD\phi\Big]\,, \\
H^{88}_{(3)} &= -*\Big[Ze^{2\phi}\big(2Da+\zeta D\tilde{\zeta}-\tilde{\zeta}D\zeta\big)+2DY-4YD\phi-3\big(d\varphi-e^{2\varphi}\chi d\chi\big)\Big]\,.
\end{aligned} \quad (3.68)$$

Finally, the four-form field strengths correspond to the following scalar-dependent top forms on four-dimensional spacetime:

$$\begin{aligned}
H^1_{(4)} = g\Big[&2e^{\varphi}Y\big(3X+2Y-3XY\big) \\
&+e^{-\varphi-2\phi}X\big(X+Y-XY\big)\big(Y^2+Z^2+e^{4\phi}\big)\Big]\mathrm{vol}_4\,,
\end{aligned}$$

$$
\begin{aligned}
H^{77}_{(4)} &= -gX\left[e^{-3\varphi}X^2\big(Y^2-2Y+Z^2+e^{4\phi}\big)+6e^{-\varphi+2\phi}\big(XY-X-Y\big)\right]\mathrm{vol}_4\,,\\
H^{78}_{(4)} &= -gXZ\left[e^{-3\varphi-2\phi}X^2\big(Y^2+Z^2+e^{4\phi}\big)+6e^{-\varphi}(XY-X-Y)\right]\mathrm{vol}_4\,,\\
H^{88}_{(4)} &= -gX\left[e^{-3\varphi}X^2\left(Y^2-2Y+Z^2\right)+6e^{-\varphi-2\phi}(XY-X-Y)\left(Y^2+Z^2\right)\right.\\
&\qquad\left.+e^{-3\varphi-4\phi}X^2\big(Y^2+Z^2\big)^2\right]\mathrm{vol}_4\,. \qquad (3.69)
\end{aligned}
$$

The dualisations (3.67)-(3.69) particularise the $\mathrm{SL}(8,\mathbb{R})$ duality hierarchy (3.51) to the SU(3)–invariant case.

It can be checked that the scalar potential (3.58) can be recovered from the dualised four-forms (3.69) via

$$
g\left(6H^{1}_{(4)}+H^{77}_{(4)}+H^{88}_{(4)}\right)=-2\,V\,\mathrm{vol}_4\,. \qquad (3.70)
$$

Likewise, the Bianchi identities (3.66) combined with the dualisation conditions (3.67)-(3.69) partially reproduce the equations of motion that derive from the Lagrangian (3.54). The list of identities needed to verify this includes the action of the $\mathrm{SL}(2,\mathbb{R})$ Killing vector $k[H_0]$ in (B.18) on the gauge kinetic matrix (3.61),

$$
(\partial_\varphi-\chi\,\partial_\chi)\mathcal{N}_{00}=3\,\mathcal{N}_{00}\,,\quad(\partial_\varphi-\chi\,\partial_\chi)\mathcal{N}_{11}=-\mathcal{N}_{11}\,,\quad(\partial_\varphi-\chi\,\partial_\chi)\mathcal{N}_{01}=\mathcal{N}_{01}\,, \qquad (3.71)
$$

and the following identities that can be checked to hold for the dualised three-form field strengths (3.68),

$$
\begin{aligned}
H^{77}_{(3)}-H^{88}_{(3)} &= -4\,h_{uv}\,k^u[H_1]\,*Dq^v\,,\\
H^{78}_{(3)} &= -\sqrt{2}\,h_{uv}\left(k^u[E_2]+k^u[F_2]\right)\,*Dq^v\,, \qquad (3.72)\\
H^{0}_{(3)} &= -2\,h_{uv}\,k^u_0\,*Dq^v\,,\qquad H^{2}_{(3)}=h_{uv}\,k^u_1\,*Dq^v\,,
\end{aligned}
$$

and four-form field strengths (3.69) and the potential (3.58),

$$
\begin{aligned}
3g\big(2H^{1}_{(4)}-H^{77}_{(4)}-H^{88}_{(4)}\big) &= -k^\alpha[H_0]\,\partial_\alpha V\mathrm{vol}_4\,,\\
2g\big(H^{77}_{(4)}-H^{88}_{(4)}\big) &= -\,k^u[H_1]\,\partial_u V\,\mathrm{vol}_4\,,\\
4\sqrt{2}\,gH^{78}_{(4)} &= -\big(k^u[E_2]+k^u[F_2]\big)\,\partial_u V\,\mathrm{vol}_4\,, \qquad (3.73)\\
k^u_0\,\partial_u V=0\,,&\qquad k^u_1\,\partial_u V=0\,.
\end{aligned}
$$

In (3.72) and (3.73), Dq^u, $u=1,\ldots,4$, collectively denote the hypermultiplet covariant derivatives (3.55); k_0 and k_1 are the hypermultiplet Killing vectors (3.57) along which the gauging is turned on; $k[H_0]$ and $k[H_1]$ are other Killing vectors (see (B.18), (B.19)) on each factor of the scalar manifold (3.53); and

Sector	scalars	pseudoscalars	vectors	2-forms	3-forms
SU(3)	3	3	4	5	4
SU(3)×U(1)^2	1	1	4	1	2
SU(3)×U(1)$_v$	3	1	4	4	4
SU(3)×U(1)$_c$	1	3	4	2	2
SU(3)×U(1)$_s$	1	1	4	1	2
SO(6)$_v$	3	0	2	4	4
SU(4)$_c$	0	3	2	1	1
SU(4)$_s$	0	0	2	1	1
SO(7)$_v$	1	0	0	1	2
SO(7)$_c$	0	1	0	0	1
SO(7)$_s$	0	0	0	0	1
G$_2$	1	1	0	1	2

Table 3.1: Number of bosonic tensor hierarchy fields in each subsector.

h_{uv} is the metric that can be read off from the hypermultiplet kinetic terms in the Lagrangian (3.54).

The last two identities in (3.73) reflect the invariance of the potential (3.58) under the gauged hypermultiplet isometries (3.57). These are the only symmetries of the SU(3)–invariant potential (3.58). The symmetry is enhanced in the subsectors that we now turn to discuss.

Subsectors within the SU(3)-invariant truncation

It is interesting to consider further subsectors contained in the SU(3)–invariant sector in the notation that we are using. A natural way to obtain those is to impose invariance under a subgroup G of SO(8) that contains SU(3). The relevant tensor hierarchy field strengths and their dualisation conditions are obtained by bringing the G–invariant restrictions specified on a case-by-case basis below to (3.62)-(3.65) and (3.67)-(3.69). The field content in each of these subsectors is summarised for convenience in table 3.1.

An obvious yet still interesting sector is attained by requiring an additional invariance under the U(1)^2 with which SU(3) commutes inside SO(8). The resulting SU(3) × U(1)^2–invariant sector throws out the hypermultiplet and sets identifications on the restricted tensor hierarchy,[5]

$$\begin{aligned} \mathrm{SU}(3)\times\mathrm{U}(1)^2 \quad &: \phi = a = \zeta = \tilde{\zeta} = 0\,, \\ & B^0 = B^2 = B^{78} = 0\,, \quad B^{77} = B^{88}\,, \\ & C^{78} = 0\,, \quad C^{77} = C^{88}\,. \end{aligned} \tag{3.74}$$

[5] Curiously, B^0 and B^2 are allowed by group theory to be non-vanishing, but are set to $B^0 = B^2 = 0$ by the duality relations (3.68) evaluated with the scalar restrictions (3.74). Similar comments apply to the condition $B^2 = 0$ in (3.76) and $B^0 = -\frac{2}{3}B^2$ in (3.77).

This sector thus reduces to $\mathcal{N} = 2$ supergravity coupled to a vector multiplet with a Fayet-Iliopoulos gauging, namely, to the $\mathrm{U}(1)^4$–invariant sector (*i.e.*, the gauged STU model) with all three vector multiplets identified, along with the relevant tensor hierarchy fields. Inserting (3.74) in (3.54), the Lagrangian indeed reduces to *e.g.* (6.28), (6.29) of [81] with the fields and coupling constants here and there identified as

$$\begin{aligned} e^{\varphi_{\text{there}}} &= e^{-\varphi_{\text{here}}}(1 + e^{2\varphi_{\text{here}}}\chi^2_{\text{here}}) \,, \qquad \chi_{\text{there}}\, e^{\varphi_{\text{there}}} = \chi_{\text{here}}\, e^{\varphi_{\text{here}}} \,, \\ \tilde{A}_{(1)\text{there}} &= -A^0_{\text{here}} \,, \qquad A_{(1)\text{there}} = A^1_{\text{here}} \,, \qquad g_{\text{there}} = -g_{\text{here}} \,. \end{aligned} \tag{3.75}$$

The potential of the $\mathrm{SU}(3) \times \mathrm{U}(1)^2$–invariant sector, (3.58) with (3.74), acquires a symmetry under the compact generator, $k[E_0] - k[F_0]$ in the notation of (B.18), of the vector multiplet scalar manifold. The field redefinition in the first line of (3.75) is a $\mathrm{U}(1) \subset \mathrm{SL}(2, \mathbb{R})$ transformation generated by this Killing vector, followed by a change of sign of χ.

One may also consider $\mathrm{SU}(3)\times\mathrm{U}(1)$–invariant sectors, with $\mathrm{U}(1)$ chosen to be one of the three triality–inequivalent[6] $\mathrm{U}(1)_v$, $\mathrm{U}(1)_s$ or $\mathrm{U}(1)_c$, factors with which $\mathrm{SU}(3)$ commutes inside $\mathrm{SO}(8)$. These invariant sectors are attained by setting

$$\mathrm{SU}(3) \times \mathrm{U}(1)_v \quad : \quad \zeta = \tilde{\zeta} = 0 \,, \qquad B^2 = 0 \,, \tag{3.76}$$

$$\begin{aligned} \mathrm{SU}(3) \times \mathrm{U}(1)_c \quad : \quad & e^{-2\phi} = 1 - \tfrac{1}{4}(\zeta^2 + \tilde{\zeta}^2) \,, \quad a = 0 \,, \\ & B^0 = -\tfrac{2}{3}B^2 \,, \; B^{78} = 0 \,, \quad B^{77} = B^{88} \,, \\ & C^{78} = 0 \,, \quad C^{77} = C^{88} \,, \end{aligned} \tag{3.77}$$

$$\begin{aligned} \mathrm{SU}(3) \times \mathrm{U}(1)_s \quad : \quad & \phi = a = \zeta = \tilde{\zeta} = 0 \,, \\ & B^0 = B^2 = B^{78} = 0 \,, \quad B^{77} = B^{88} \,, \\ & C^{78} = 0 \,, \quad C^{77} = C^{88} \,, \end{aligned} \tag{3.78}$$

while retaining both vectors and their magnetic duals. Only the $\mathrm{SU}(3) \times \mathrm{U}(1)_s$–invariant subtruncation is supersymmetric, and coincides with the $\mathrm{SU}(3) \times \mathrm{U}(1)^2$ sector discussed above –in other words, invariance under $\mathrm{U}(1)_s$ cannot be enforced on top of $\mathrm{SU}(3)$ without also imposing $\mathrm{U}(1)_c$ invariance, but not the other way around. The other two subtruncations retain the would-be vector multiplet and 'half' a hypermultiplet: either the scalars ϕ, a in the $\mathrm{SU}(3) \times \mathrm{U}(1)_v$ sector, or the pseudoscalars ζ, $\tilde{\zeta}$ in the $\mathrm{SU}(3) \times \mathrm{U}(1)_c$ sector, with ϕ a function of the pseudoscalars in the latter case. The covariant derivatives (3.55) simplify accordingly. In the $\mathrm{SU}(3) \times \mathrm{U}(1)_v$

[6] Under triality, the $\mathrm{SO}(7)$ subgroups of $\mathrm{SO}(8)$ are identified by the splitting $\mathbf{8}_v \to \mathbf{1} \oplus \mathbf{7}$ for $\mathrm{SO}(7)_v$, and similarly for $\mathrm{SO}(7)_s$ and $\mathrm{SO}(7)_c$. We follow the spectrum conventions of *e.g.* [82] whereby, at the $\mathrm{SO}(8)$ vacuum, the (graviton, gravitini, vectors, spinors, scalars, pseudoscalars) of $\mathcal{N} = 8$ supergravity lie in the $(\mathbf{1}, \mathbf{8}_s, \mathbf{28}, \mathbf{56}_s, \mathbf{35}_v, \mathbf{35}_c)$ of $\mathrm{SO}(8)$.

sector, ϕ, a remain charged under A^0 and no field is charged under A^1. In the $\mathrm{SU}(3) \times \mathrm{U}(1)_c$ sector the covariant derivatives reduce to

$$D\zeta = d\zeta - g(A^0 + 3A^1)\,\tilde{\zeta}\,, \qquad D\tilde{\zeta} = d\tilde{\zeta} + g(A^0 + 3A^1)\,\zeta\,, \tag{3.79}$$

showing that ζ, $\tilde{\zeta}$ become a doublet charged only under the combined gauge field $A^0 + 3A^1$.

It is possible to further truncate the $\mathrm{SU}(3) \times \mathrm{U}(1)_c$ sector to a two-scalar model retaining (φ, ζ) along with $B^{77} = B^{88}$ and C^1, $C^{77} = C^{88}$ by imposing (3.77)) together with $\chi = 0$, $\tilde{\zeta} = \zeta$, $A^0 = A^1 = 0$ and $B^0 = -\frac{2}{3}B^2 = 0$. The Lagrangian is (3.54) with these identifications and the superpotential reduces, from (3.59), to

$$W = \tfrac{1}{2\sqrt{2}}\, g\, e^{-\frac{3}{2}\varphi}\big(e^{2\phi} - 3e^{2\phi + 2\varphi} - 2\big)\,, \tag{3.80}$$

where $e^{2\phi}$ is shorthand for the expression in terms of $\zeta = \tilde{\zeta}$ that appears in (3.77). This is the model considered in [31]. The identifications

$$e^{-\varphi_{\text{here}}} = \rho^4_{\text{there}}\,, \qquad \zeta^2_{\text{here}} = \tilde{\zeta}^2_{\text{here}} = 2\tanh^2\chi_{\text{there}} \tag{3.81}$$

(the second equation implies $e^{2\phi_{\text{here}}} = \cosh^2\chi_{\text{there}}$ on (3.77)) indeed bring the superpotential (3.80) to (3.9) of [31], up to normalisation.

The $\mathrm{SU}(3) \times \mathrm{U}(1)$–invariant sectors can be further reduced by imposing a larger $\mathrm{SO}(6) \sim \mathrm{SU}(4)$ symmetry. The corresponding sectors are obtained by letting

$$\mathrm{SO}(6)_v \;:\; \zeta = \tilde{\zeta} = \chi = 0\,, \qquad A^1 = \tilde{A}_1 = 0\,, \qquad B^2 = 0\,, \tag{3.82}$$

$$\begin{aligned}
\mathrm{SU}(4)_c \;:\;& e^{-2\phi} = 1 - \tfrac{1}{4}(\zeta^2 + \tilde{\zeta}^2)\,, \quad a = 0\,, \quad e^{-2\varphi} = 1 - \chi^2\,, \\
& A^1 = A^0 \equiv A\,, \quad \tilde{A}_1 = 3\tilde{A}_0\,, \\
& B^0 = -\tfrac{2}{3}B^2\,,\ B^{ab} = 0\,,\ C^1 = C^{77} = C^{88}\,,\ C^{78} = 0\,,
\end{aligned} \tag{3.83}$$

$$\begin{aligned}
\mathrm{SU}(4)_s \;:\;& \phi = a = \zeta = \tilde{\zeta} = \varphi = \chi = 0\,, \\
& A^1 = -A^0\,, \quad \tilde{A}_1 = -3\tilde{A}_0\,, \\
& B^0 = \tfrac{2}{3}B^2\,,\ B^{ab} = 0\,,\ C^1 = C^{77} = C^{88}\,,\ C^{78} = 0\,.
\end{aligned} \tag{3.84}$$

Again, only the $\mathrm{SU}(4)_s$–invariant sector is supersymmetric: it truncates out the vector multiplet of the $\mathrm{SU}(3) \times \mathrm{U}(1)_c$ sector, leading to minimal $\mathcal{N} = 2$ gauged supergravity. Setting all scalars to zero as in (3.84), further setting consistently $B^0 = \frac{2}{3}B^2 = 0$, and rescaling for convenience the metric and the graviphoton as

$$g_{\mu\nu} \equiv \tfrac{1}{4}\,\bar{g}_{\mu\nu}\,, \qquad A^1 = -A^0 \equiv \tfrac{1}{4}\bar{A}\,, \tag{3.85}$$

equation (3.54) reduces to the bosonic Lagrangian of pure $\mathcal{N} = 2$ gauged supergravity (2.35) introduced in the previous chapter. For later reference, we note that the only tensor hierarchy field strengths that are active in the $\mathrm{SU}(4)_s$ sector are

$$
\begin{aligned}
H^1_{(2)} = -H^0_{(2)} \equiv \tfrac{1}{4}\bar{F}\,, \qquad \tilde{H}_{(2)0} = -\tfrac{1}{3}\tilde{H}_{(2)1} = \tfrac{1}{4}{*}\bar{F}\,, \\
H^1_{(4)} = H^{77}_{(4)} = H^{88}_{(4)} = \tfrac{3}{8} g\,\overline{\mathrm{vol}}_4\,,
\end{aligned}
\tag{3.86}
$$

where the bars refer to the rescaled quantities (3.85). The other two truncations (3.82), (3.83) are manifestly non-supersymmetric. Imposing invariance under $\mathrm{SO}(6)_v$ selects the proper scalars φ, ϕ, a along with the gauge field A^0, while invariance under $\mathrm{SU}(4)_c$ retains the pseudoscalars χ, ζ, $\tilde{\zeta}$ along with $A^0 + A^1$. In the latter case, the scalars become functions of the pseudoscalars as indicated in (3.83).

It was noted in [78] that the $\mathrm{SU}(4)_c$–invariant sector coincides with a subtruncation, considered in [83], of the $D = 4$ $\mathcal{N} = 2$ gauged supergravity obtained upon consistent truncation of M-theory on any (skew-whiffed) Sasaki-Einstein seven-manifold [84]. Indeed, using (3.83) and further identifying the pseudoscalars and vectors here and in [83] as

$$
\begin{aligned}
\chi_{\text{here}} = h_{\text{there}}\,, \qquad \zeta_{\text{here}} = -\sqrt{3}\,\mathrm{Im}\,\chi_{\text{there}}\,, \qquad \tilde{\zeta}_{\text{here}} = -\sqrt{3}\,\mathrm{Re}\,\chi_{\text{there}}\,, \\
A^0_{\text{here}} = A^1_{\text{here}} = -A_{1\,\text{there}}\,, \qquad g_{\text{here}} = -\,(2L)^{-1}_{\text{there}}
\end{aligned}
\tag{3.87}
$$

(which further imply $\varphi_{\text{here}} = -2U_{\text{there}} - V_{\text{there}}$ and $\phi_{\text{here}} = -3U_{\text{there}}$, with φ, ϕ here subject to (3.83)) and U, V there subject to their (4.1)), the Lagrangian (3.54) here reproduces (4.3) of [83]. Neither the $\mathrm{SO}(6)_v$ nor the $\mathrm{SU}(4)_c$ sectors admit a further truncation to the Einstein-Maxwell, bosonic Lagrangian (2.35) of minimal $\mathcal{N} = 2$ supergravity.

It is possible to enlarge the symmetry to the three different SO(7) subgroups of SO(8) by further imposing

$$
\begin{aligned}
\mathrm{SO}(7)_v \quad : \quad & \zeta = \tilde{\zeta} = \chi = 0\,, \quad \varphi = \phi\,, \quad a = 0\,, \\
& A^0 = A^1 = \tilde{A}_0 = \tilde{A}_1 = 0\,, \\
& B^0 = B^2 = B^{78} = 0\,,\ B^{88} = -7B^{77}\,, \qquad C^1 = C^{77}\,,\ C^{78} = 0\,,
\end{aligned}
\tag{3.88}
$$

$$
\begin{aligned}
\mathrm{SO}(7)_c \quad : \quad & e^{-2\phi} = 1 - \tfrac{1}{4}(\zeta^2 + \tilde{\zeta}^2) = 1 - \chi^2 = e^{-2\varphi}\,, \quad a = 0\,, \\
& A^0 = A^1 = \tilde{A}_0 = \tilde{A}_1 = 0\,, \\
& B^0 = B^2 = 0\,,\ B^{ab} = 0\,, \qquad C^1 = C^{77} = C^{88}\,,\ C^{78} = 0\,,
\end{aligned}
\tag{3.89}
$$

$$
\begin{aligned}
\mathrm{SO}(7)_s \quad : \quad & \phi = a = \zeta = \tilde{\zeta} = \varphi = \chi = 0\,, \\
& A^0 = -A^1 = 0\,, \\
& B^0 = B^2 = 0\,, \quad B^{ab} = 0\,, \qquad C^1 = C^{77} = C^{88}\,, \quad C^{78} = 0\,.
\end{aligned}
\tag{3.90}
$$

The $\mathrm{SO}(7)_s$ truncation gives minimal $\mathcal{N}=1$ gauged supergravity while the $\mathrm{SO}(7)_v$ and the $\mathrm{SO}(7)_c$ sectors are non-supersymmetric. They respectively retain one dilaton ($\varphi = \phi$) and one axion (χ, together with the identifications (3.89)), along with the relevant tensors in the hierarchy.

All three SO(7) sectors are contained within the G_2–invariant sector. This corresponds to $\mathcal{N}=1$ supergravity coupled to a chiral multiplet with a scalar manifold SL(2)/SO(2) which is diagonally embedded in (3.53) via

$$\begin{aligned}
\mathrm{G}_2 \quad : \quad & \phi = \varphi\,, \quad \tilde{\zeta} = -2\chi\,, \quad a = \zeta = 0\,, \\
& A^0 = A^1 = \tilde{A}_0 = \tilde{A}_1 = 0\,, \\
& B^0 = B^2 = B^{78} = 0\,, \quad B^{88} = -7B^{77}\,, \qquad C^1 = C^{77}\,, \quad C^{78} = 0\,.
\end{aligned} \tag{3.91}$$

The Lagrangian in this sector is (3.54) with the identifications (3.91). It can be cast in canonical $\mathcal{N}=1$ form, in the conventions of *e.g.* section 4.2 of [74], in terms of the following Kähler potential and holomorphic superpotential

$$K = -7\log(-i(z-\bar{z}))\,, \qquad \mathcal{W} = 2g\,(7z^3 + z^7)\,, \tag{3.92}$$

with $z = -\chi + ie^{-\varphi}$. On the identifications (3.91) that define the G_2–invariant sector, the real superpotential (3.59) becomes related to (3.92) via $W^2 = e^K\,\overline{\mathcal{W}}\,\mathcal{W}$.

All of the above truncations arise from symmetry principles, by retaining the fields that are neutral under the relevant invariance groups. For this reason, the above truncations can be directly implemented at the level of the Lagrangian (3.54). In particular, a consistent truncation to minimal $\mathcal{N}=2$ supergravity is obtained by retaining singlets under $\mathrm{SU}(4)_s$, as noted above. We conclude this section by noting an alternate truncation of the SU(3) sector to minimal $\mathcal{N}=2$ supergravity that is inequivalent to the $\mathrm{SU}(4)_s$–invariant truncation. In fact, this alternative minimal truncation is not driven by symmetry principles in any obvious way, so we have verified its consistency at the level of the field equations. Firstly, freeze the scalars to their vacuum expectation values (vevs) at the $\mathrm{SU}(3)\times\mathrm{U}(1)_c$–invariant vacuum (see table 3.2),

$$e^{-2\varphi} = 3\,, \qquad \chi = 0\,, \qquad e^{-2\phi} = 1 - \tfrac{1}{4}(\zeta^2 + \tilde{\zeta}^2) = \tfrac{2}{3}\,, \qquad a = 0\,. \tag{3.93}$$

Secondly, identify the electric and magnetic vectors as

$$A^0 = -3A^1 \equiv \tfrac{1}{2}\,\bar{A}\,, \qquad \tilde{A}_0 = -\tfrac{1}{9}\tilde{A}_1 \equiv \tfrac{1}{6\sqrt{3}}\,\tilde{\bar{A}}\,, \tag{3.94}$$

turn off all the two-forms, and retain one of the three-forms as

$$B^0 = -\tfrac{2}{3}B^2 = B^{ab} = 0\,, \qquad C^{78} = 0\,, \quad C^1 = C^{77} = C^{88}\,. \tag{3.95}$$

Finally, rescale the metric for convenience:

$$g_{\mu\nu} \equiv \tfrac{1}{3\sqrt{3}}\,\bar{g}_{\mu\nu}\,. \tag{3.96}$$

We have verified at the level of the bosonic field equations, including Einstein, that these identifications define a consistent truncation of the theory (3.54) to minimal $\mathcal{N} = 2$ gauged supergravity (2.35).

The identification of the electric vectors in (3.94) retains the SU(3)×U(1)$_c$–invariant vector (see (B.20) with (B.15)) that remains massless (see (3.79)) at the $\mathcal{N} = 2$ vacuum (3.93). For future reference, it is also interesting to keep track of the field strengths for this truncation. On (3.94), (3.95), the two-form potential contributions to the magnetic vector two-form field strengths (3.63) drop out, and the vector field strengths become

$$H^0 = -3H^1 \equiv \tfrac{1}{2}\,\bar{F}\,, \qquad \tilde{H}_0 = -\tfrac{1}{9}\tilde{H}_1 \equiv \tfrac{1}{6\sqrt{3}}\,\tilde{\bar{F}} = -\tfrac{1}{6\sqrt{3}}\,{*\bar{F}}\,, \tag{3.97}$$

with $\bar{F} \equiv d\bar{A}$. The relations here for the magnetic field strengths are compatible with the vector duality relations (3.67) evaluated on the scalar vevs (3.93), and the last equality for the magnetic graviphoton field strength $\tilde{\bar{F}}$ is fixed by $\tilde{\bar{F}} = \partial\mathcal{L}/\partial\bar{F}$, with $\mathcal{L}$ as in (2.35). Moving on to the three-form field strengths, we find that all of them are zero by bringing (3.94), (3.95) to their definitions (3.64) in terms of potentials. This was expected, as the three-form form field strengths are dual to combinations (3.68) of (Hodge duals of) derivatives of scalars, and these have been frozen to their vevs (3.93). Finally, for the four-form field strengths we obtain, from (3.65) with (3.95), $H^{78}_{(4)} = 0$, $H^{1}_{(4)} = H^{77}_{(4)} = H^{88}_{(4)} = dC^1$, expressions which are again compatible with the dualisation conditions (3.69). Rescaling the volume form using (3.96), we find

$$H^{1}_{(4)} = H^{77}_{(4)} = H^{88}_{(4)} = \tfrac{1}{2\sqrt{3}}\,g\,\overline{\mathrm{vol}}_4\,. \tag{3.98}$$

Vacuum structure

The list of vacua of $D = 4$ $\mathcal{N} = 8$ supergravity with an electric SO(8) gauging [73] that preserve at least a subgroup SU(3) of SO(8) was elucidated in [77]. All of them are AdS. These vacua arise as extrema of the scalar potential (3.58), in our conventions, and for convenience we have summarised them in table 3.2. The table includes the residual supersymmetry $\mathcal{N}$ and bosonic symmetry G_0 for each vacuum, as well as its location in the scalar space (3.53) in the parametrisation that we are using. The corresponding cosmological constant, given by (3.58), and the scalar mass spectrum within the SU(3)–invariant sector is also given. See [78] for the bosonic spectra within the full $\mathcal{N} = 8$ supergravity. All three supersymmetric points are also extrema of the superpotential (3.59). On the SO(8) and the G_2 points, the F-terms that derive from the holomorphic superpotential 3.92 also vanish.

It was argued in [30] that some combinations of the four-form field strengths of the duality hierarchy ought to vanish at critical points of the scalar potential, thus yielding necessary conditions for critical points. In our

$\mathcal{N}$	G_0	χ	$e^{-\varphi}$	$e^{-\phi}$	a	ζ	$\tilde{\zeta}$
8	SO(8)	0	1	1	0	0	0
2	$\mathrm{U}(3)_c$	0	$\sqrt{3}$	$\sqrt{\frac{2}{3}}$	0	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{2}{3}}$
1	G_2	$12^{-1/4}$	$\left(\frac{25}{12}\right)^{1/4}$	$\left(\frac{25}{12}\right)^{1/4}$	0	0	$-2\cdot 12^{-1/4}$
0	$\mathrm{SO}(7)_v$	0	$5^{1/4}$	$5^{1/4}$	0	0	0
0	$\mathrm{SO}(7)_c$	$\frac{1}{\sqrt{5}}$	$\frac{2}{\sqrt{5}}$	$\frac{2}{\sqrt{5}}$	0	0	$-\frac{2}{\sqrt{5}}$
0	$\mathrm{SU}(4)_c$	0	1	$\frac{1}{\sqrt{2}}$	0	1	1

$\mathcal{N}$	G_0	$g^{-2}V_0$	L^2M^2
8	SO(8)	-24	$(-2,\,-2,\,-2,\,-2,\,-2,\,-2)$
2	$\mathrm{U}(3)_c$	$-18\sqrt{3}$	$\left(3\pm\sqrt{17},\,2,\,2,\,2,\,0\right)$
1	G_2	$-\frac{2^{11/2}\,3^{13/4}}{5^{5/2}}$	$\left(4\pm\sqrt{6},\,\frac{-11\pm\sqrt{6}}{6},\,0,\,0\right)$
0	$\mathrm{SO}(7)_v$	$-8\times 5^{3/4}$	$\left(6,\,-\frac{12}{5}-\frac{6}{5},\,-\frac{6}{5},\,-\frac{6}{5},\,0\right)$
0	$\mathrm{SO}(7)_c$	$-\frac{25\sqrt{5}}{2}$	$\left(6,\,-\frac{12}{5}-\frac{6}{5},\,-\frac{6}{5},\,-\frac{6}{5},\,0\right)$
0	$\mathrm{SU}(4)_c$	-32	$\left(6,\,6-\frac{3}{4},\,-\frac{3}{4},\,0,\,0\right)$

Table 3.2: All critical points of $D=4$ $\mathcal{N}=8$ supergravity with electric SO(8) gauging with at least SU(3) invariance, reproducing the results of [77] in our parametrisation. For each point we give the residual supersymmetry $\mathcal{N}$ and bosonic symmetry G_0 within the full $\mathcal{N}=8$ theory, their location in the parametrisation that we are using (upper table), the cosmological constant V_0 and the scalar mass spectrum within the SU(3)–invariant sector (lower table). The masses are given in units of the AdS radius, $L^2=-6/V_0$. We have abbreviated $\mathrm{U}(3)_c\equiv\mathrm{SU}(3)\times\mathrm{U}(1)_c$.

SU(3)–invariant case, these conditions read

$$8H^1_{(4)}-\left(6H^1_{(4)}+\delta_{cd}H^{cd}_{(4)}\right)=0\,,\qquad 8H^{ab}_{(4)}-\left(6H^1_{(4)}+\delta_{cd}H^{cd}_{(4)}\right)\delta^{ab}=0\,. \quad (3.99)$$

Using the dualisation conditions (3.69), it can be checked that the relations (3.99) do indeed hold at the critical points summarised in table 3.2.

3.2 The explicit uplift on S^7

As discussed in previous chapters, the existence of a consistent truncation of string theory on a non-trivial compact manifold is a rare phenomenon. Beyond its intrinsic mathematical interest, this problem is also relevant for detailed computations in top-down holographic applications, as the needed computations can be effectively treated as lower-dimensional.

The tensor hierarchy was introduced in the previous section to maintain covariance of the field content under $E_{7(7)}$. This addition of redundant fields was observed in [26, 30, 85] to be very well-adapted to show the consistency of the uplifts of maximal supergravities into string theory. In particular, it was used to show that massive type IIA on $AdS_4 \times S^6$ reduces to the dyonically-gauged ISO(7) supergravity in [26, 85], and to provide explicit details to the proof of consistency of the electric SO(8) gauging uplift to M-theory on $AdS_4 \times S^7$ in [30]. In this section we review the main results of the later reference and employ them to discuss the uplifts of the SU(3)-invariant sector [A] of maximal supergravity on the S^7

3.2.1 $D = 11$ supergravity in $4 + 7$ split and the S^7 truncation

Consider the following ansatz for the M-theory bosons

$$\begin{aligned} d\hat{s}_{11}^2 &= \Delta^{-1} ds_4^2 + g_{mn}(dy^m + B^m)(dy^n + B^n) \, , \\ \hat{A}_{(3)} &= \tfrac{1}{6} A_{\mu\nu\rho} dx^\mu \wedge dx^\nu \wedge dx^\rho + \tfrac{1}{2} A_{\mu\nu m} dx^\mu \wedge dx^\nu \wedge (dy^m + B^m) \\ &\quad + \tfrac{1}{2} A_{\mu mn} dx^\mu \wedge (dy^m + B^m) \wedge (dy^n + B^n) \\ &\quad + \tfrac{1}{6} A_{mnp} (dy^m + B^m) \wedge (dy^n + B^n) \wedge (dy^p + B^p) \, , \end{aligned} \tag{3.100}$$

with the warp factor

$$\Delta^2 = \frac{\det g_{mn}}{\det \mathring{g}_{mn}} \tag{3.101}$$

for an arbitrary background metric $\mathring{g}_{mn}$ homeomorphic to g_{mn}. The presence of this warping guarantees that the reduced theory is in the Einstein frame, in analogy to the choice of coefficients in (I.12). This splitting breaks the eleven-dimensional diffeomorphisms as

$$\begin{array}{ccc} \mathrm{GL}(11,\mathbb{R}) & \to & \mathrm{GL}(4,\mathbb{R}) \times \mathrm{SL}(7,\mathbb{R}) \\ \{x^M\} & \to & \{x^\mu, \ y^m\} \end{array} \ . \tag{3.102}$$

Similarly, the eleven-dimensional gravitino can be split into ψ_μ^I and ψ_m^I, with, following (1.9), I an index in the **8** of an SU(8) local in $D = 11$ coordinates. Using the seven-dimensional Clifford algebra, we can rearrange the latter fermions as a tri-spinor $\chi^{IJK} = \frac{3i}{\sqrt{2}} (\Gamma^a C^{-1})^{[IJ} \psi_a^{K]}$, in the **56** of SU(8) [18, 86].

From a lower dimensional perspective, it proves sometimes useful to dualise the two-form contributions in (3.100) into scalars. In the context of the T^7 reduction, this is what we saw to be needed to show $E_{7(7)}$-covariance of the equations of motion. However, in the spirit of the tensor hierarchy, we will keep the higher forms and only dualise them at the very end. On the other hand, to make contact with the $D = 4$ expressions, we consider the field redefinitions

$$C_\mu{}^{m8} = B_\mu^m \,, \qquad \tilde{C}_{\mu\, mn} = A_{\mu mn} \,, \qquad C_{\mu\nu\, m}{}^8 = -A_{\mu\nu m} + B_{[\mu}{}^n \; A_{\nu]nm} \,,$$
$$C_{\mu\nu\rho}{}^{88} = A_{\mu\nu\rho} - B_{[\mu}{}^m B_\nu{}^n A_{\rho]mn} \,, \tag{3.103}$$

and the generalised vielbeine [86, 87]

$$\begin{aligned} V^{m8}{}_{IJ} &= \tfrac{1}{4}\Delta^{-\frac{1}{2}} e_a{}^m (C\Gamma^a)_{IJ} \,, \\ \tilde{V}_{mn\, IJ} &= \tfrac{1}{4}\Delta^{-\frac{1}{2}} \left[e_m{}^a e_n{}^b (C\Gamma_{ab})_{IJ} + e_a{}^p (C\Gamma^a)_{IJ} A_{pmn} \right] . \end{aligned} \tag{3.104}$$

In these expressions, arbitrary local SU(8) rotations $\Phi^I{}_J(x, y)$ should be included to allow the generalised vielbeine to be complex-valued, as SU(8) covariance demands [86, 88]. However, their presence will be understood but not explicitly included in the following. As usual, complex conjugation is denoted by a change in the position of the SU(8) indices, *e.g.* $(V_{IJ})^* = V^{IJ}$.

The $\mathrm{GL}(4,\mathbb{R}) \times \mathrm{SL}(7,\mathbb{R})$ representations of these fields are

$\mathrm{GL}(4,\mathbb{R})$	$\mathrm{SL}(7,\mathbb{R})$
generalised vielbeine	$\mathbf{7'} + \mathbf{21}$
	$V^{m8}{}_{IJ}$, $V_{mn\, IJ}$
vectors	$\mathbf{7'} + \mathbf{21}$
	$C_\mu{}^{m8}$, $\tilde{C}_{\mu mn}$
two-forms	$\mathbf{7}$
	$C_{\mu\nu m}{}^8$
three-forms	$\mathbf{1}$
	$C_{\mu\nu\rho}{}^{88}$

(3.105)

and it must be emphasised at this point that their full coordinate dependence on both x^μ and y^m is kept.

The $\mathrm{SL}(7,\mathbb{R})$ representations in (3.105) must be confronted with the $\mathrm{SL}(8,\mathbb{R})$ representations in (3.47). The 8 indices in (3.103) and (3.104) are written in hindsight to signal the $\mathrm{SL}(8,\mathbb{R}) \supset \mathrm{SL}(7,\mathbb{R})$ breaking, which suggests how to relate both sets of fields.

In the previous chapter, the bridge between the fields in four and eleven dimensions were the structure group invariant tensors corresponding to the specific supersymmetric background. To relate the fields in (3.105) and (3.47) we must proceed similarly. In the present context, the background will be the round S^7. This sphere can be defined in terms of the $\mathrm{SL}(8,\mathbb{R})$ coordinates that embed it in $\mathbb{R}^8$ as

$$\delta_{AB}\, \mu^A \mu^B = 1 \,. \tag{3.106}$$

The round metric can also be given in terms of these coordinates as the pullback of the flat euclidean metric on $\mathbb{R}^8$ to the (3.106) hypersurface

$$\mathring{g}_{mn} = g^{-2}\delta_{AB}\,\partial_m\mu^A\partial_n\mu^B\,, \tag{3.107}$$

which is invariant under the action of the SO(8) Killing vectors K^m_{AB}. In terms of the μ^A coordinates, these and their derivatives read

$$\begin{aligned} K^{mAB} &= 2g^{-2}\mathring{g}^{mn}\mu^{[A}\partial_n\mu^{B]}\,,\\ K_{mn}{}^{AB} &= 4g^{-2}\partial_{[m}\mu^{[A}\partial_{n]}\mu^{B]}\,. \end{aligned} \tag{3.108}$$

Notice that we have normalised the sphere to have radius g^{-2}.

In terms of these objects, we can perform the reduction of M-theory on S^7 by restricting the y dependence of the fields as [30]

$$\begin{aligned} ds_4^2(x,y) &= ds_4^2(x)\,,\\ C_\mu{}^{m8}(x,y) &= \tfrac{g}{2}K^m{}_{AB}(y)\,\mathcal{A}_\mu^{AB}(x)\,,\\ \tilde{C}_{\mu\,mn}(x,y) &= \tfrac{g}{4}K_{mn}{}^{AB}(y)\,\tilde{\mathcal{A}}_{\mu AB}(x)\,,\\ C_{\mu\nu\,m}{}^{8}(x,y) &= -g^{-1}(\mu_A\partial_m\mu^B)(y)\,\mathcal{B}_{\mu\nu\,B}{}^{A}(x)\,,\\ C_{\mu\nu\rho}{}^{88}(x,y) &= (\mu_A\mu_B)(y)\,\mathcal{C}_{\mu\nu\rho}{}^{AB}(x)\,, \end{aligned} \tag{3.109}$$

and

$$\begin{aligned} V^{m8\,IJ}(x,y) &= \tfrac{g}{2}K^m{}_{AB}(y)\,\eta^I_i(y)\,\eta^J_j(y)\,\mathcal{V}^{AB\,ij}(x)\,,\\ V_{mn}{}^{IJ}(x,y) &= \tfrac{g}{4}K_{mn}{}^{AB}(y)\,\eta^I_i(y)\,\eta^J_j(y)\,\tilde{\mathcal{V}}_{AB}{}^{ij}(x)\,. \end{aligned} \tag{3.110}$$

The objects intertwining the SU(8) indices i and I respectively appearing in the $D=4$ and $D=11$ descriptions are the Killing spinors of the background S^7, satisfying $\eta^I_i\eta^j_I = \delta^j_i$ and $\eta^I_i\eta^i_J = \delta^I_J$. Here, as in (3.104), we have omitted the scalar-dependent SU(8) rotations [86] needed to gauge fix the generalised vielbein, which must also be tuned to appropriately align both SU(8) groups. Explicit expressions can be found in [25, 88].

The history of this truncation ansatz spans more than three decades. In their seminal paper [25], de Wit and Nicolai proved that the factorisation of the metric, scalars and vectors was consistent at the level of the $\mathcal{N}=8$ trasformations in $D=4$, and further checks on the consistency followed in [25, 88]. The factorisation of the two- and three-forms in the tensor hierarchy was put forward in [30], and also verified to be consistent with supersymmetry. This proves instrumental in obtaining explicit expressions for the eleven-dimensional metric and three-form in (3.100).

The inclusion of the $D=4$ forms and metric in the KK ansatz (3.100) can be straightforwardly achieved by plugging (3.109) into (3.103). The relation of the warp factor and purely internal components of the metric and three-form with the $D = 4$ scalars is much more subtle. First one needs to take all possible products between the generalised vielbeine and their conjugates, and trace over the SU(8) indices. Doing so, the dependence on the Killing spinors and SU(8) scalar matrices drops out, and the system reduces to the equations

$$\begin{aligned}
\mathcal{M}^{ABCD}K^{m}{}_{AB}K^{n}{}_{CD} &= 4g^{-2}\Delta^{-1}g^{mn} \; , \\
\mathcal{M}^{AB}{}_{CD}K^{m}{}_{AB}K_{np}{}^{CD} &= 8g^{-1}\Delta^{-1}g^{mq}\,A_{qnp} \; , \\
\mathcal{M}_{ABCD}K_{mn}{}^{AB}K_{pq}{}^{CD} &= 16\Delta^{-1}\big(2g_{m[p}g_{q]n} + g^{rs}A_{rmn}A_{spq}\big) \; ,
\end{aligned} \tag{3.111}$$

in terms of the $\mathrm{SL}(8,\mathbb{R})$-covariant blocks in $\mathcal{M}_{MN}$. It is possible to combine these equations to obtain independent expressions for Δ, g_{mn} and A_{mnp}, and after this is done, the full non-linear embedding of maximal SO(8)-gauged supergravity into M-theory can be given as [30]

$$d\hat{s}_{11}^2 = \Delta^{-1}ds_4^2 + \tfrac{1}{12}g^{-2}\Delta^2\,(t_A{}^B)_M{}^P(t_C{}^D)_N{}^Q\mathcal{M}^{MN}\mathcal{M}_{PQ}\mu_B\mu_D D\mu^A D\mu^C \; , \tag{3.112}$$

and

$$\begin{aligned}
\hat{A}_{(3)} = {}& \mu_A\mu_B\big(\mathcal{C}^{AB} + \tfrac{1}{6}\mathcal{A}^{AC}\wedge\mathcal{A}^{BD}\wedge\tilde{\mathcal{A}}_{CD}\big) \\
&+ g^{-1}\big(\mathcal{B}_B{}^A + \tfrac{1}{2}\mathcal{A}^{AC}\wedge\tilde{\mathcal{A}}_{CB}\big)\wedge\mu_A D\mu^B + \tfrac{1}{2}g^{-2}\tilde{\mathcal{A}}_{AB}\wedge D\mu^A\wedge D\mu^B + A_{(3)} \; ,
\end{aligned} \tag{3.113}$$

with $D\mu^A = d\mu^A - g\,\delta_{BC}\mathcal{A}^{AB}\mu^C$, the internal three-form

$$\begin{aligned}
A_{(3)} = {}& -\tfrac{1}{72}\,g^{-3}\Delta^3\,(t_A{}^B)_P{}^R\,X'_{MQ}{}^S\delta_{NT}\,\Omega^{TU}\,\Theta_U{}^C{}_D \\
&\times\mathcal{M}^{MN}\mathcal{M}^{PQ}\mathcal{M}_{RS}\,\mu^B\,D\mu^A\wedge D\mu_C\wedge D\mu^D \; ,
\end{aligned} \tag{3.114}$$

and the warp factor given by

$$\Delta^{-3} = \tfrac{1}{84}X'_{MP}{}^R\,X'_{NQ}{}^S\,\mathcal{M}^{MN}\big(\mathcal{M}^{PQ}\mathcal{M}_{RS} + 7\delta_S^P\delta_R^Q\big) \; . \tag{3.115}$$

The internal three-form and the warp factor involve the primed embedding tensor $X'_{MN}{}^P = \Theta'_M{}^\alpha(t_\alpha)_N{}^P$. This ancillary object can be decomposed as the actual embedding tensor in (3.40), and is taken to have non-vanishing contributions only in the **36** of $\mathrm{SL}(8,\mathbb{R})$ given by $\theta'_{AB} = \mu_A\mu_B$.

Some comments are in order. First, notice that the relative size of the geometry in (3.112) is controlled by the $D = 4$ coupling constant g. The free limit from the gauged supergravity side corresponds to the decompactification limit from the higher dimensional perspective. In this limit, we can replace a large S^7 by a large T^7, and the invariance under the full $\mathrm{E}_{7(7)}$ is recovered. This is a general rule that prevents the uplift of gaugings with gauge groups that are not contained in the ungauged duality group [89].

On the other hand, at the level of (3.113), it is non-trivial how to express the truncation in terms of the $\mathcal{N}=8$ fields in (3.1), as the tensor hierarchy potentials enter the expressions directly. This can however be achieved at the level of the four-form field strength, which can be written as

$$\hat{F}_{(4)} = \mu_I \mu_J \, \mathcal{H}_{(4)}{}^{IJ} + g^{-1} \mathcal{H}_{(3)J}{}^{I} \wedge \mu_I \, D\mu^J + \tfrac{1}{2} g^{-2} \tilde{\mathcal{H}}_{(2)IJ} \wedge D\mu^I \wedge D\mu^J + dA_{(3)} . \tag{3.116}$$

Employing the dualities in (3.51), the four-form can thus be unambiguously expressed in terms of the fields appearing in the Lagrangian (3.36). Let us remark the highly non-trivial fact that, upon taking the exterior derivative of (3.113), all the derivatives and products of tensor hierarchy forms nicely combine into the covariant field strengths in (3.48).

3.2.2 SU(3)-invariant truncation

We now want to particularise the uplift of the entire $\mathcal{N}=8$ gauged supergravity in (3.112)-(3.116) to the SU(3)-invariant subsector in section 3.1.2. This will allow us to make contact with previous literature, thus checking the uplifting formulae, and to obtain a new consistent truncation of $D=11$ supergravity to minimal $D=4$ $\mathcal{N}=2$ gauged supergravity which does not follow from any obvious group theory argument.

In the present case, the embedding coordinates μ^A, $A=1,\ldots,8$, which define the S^7 as the locus (3.106) in $\mathbb{R}^8$ can be suitably branched into representations of SU(3). Under SU(3), the $\mathbf{8}_v$ of SO(8) breaks down as $\mathbf{8}_v \to \mathbf{3} + \bar{\mathbf{3}} + \mathbf{1} + \mathbf{1}$. In maintaining an explicitly real notation, it is thus convenient to split $\mathbb{R}^8 = \mathbb{R}^6 \times \mathbb{R}^2$, and the indices as $A=(i,a)$, with $i=1,\ldots,6$ and $a=7,8$ respectively labelling the first and second factors.[7] The $D=11$ uplift of the SU(3)–invariant sector utilises the tensors δ_{ij}, $J^{(6)}_{ij}$ (real) and $\Omega^{(6)}_{ijk}$ (complex) that define the natural Calabi-Yau structure of $\mathbb{R}^6$. See (B.9) for our conventions. Inside $\mathbb{R}^8$, these tensors are respectively invariant under $\mathrm{SO}(6)_v \times \mathrm{SO}(2)$, $\mathrm{SU}(3)\times \mathrm{U}(1)^2$ and $\mathrm{SU}(3)\times \mathrm{U}(1)_c$, where SO(2) rotates the $\mathbb{R}^2$ factor in $\mathbb{R}^8 = \mathbb{R}^6 \times \mathbb{R}^2$. Indices on $\mathbb{R}^6$ and $\mathbb{R}^2$ are raised and lowered with δ_{ij} and δ_{ab}, respectively.

As before, only the $D=4$ metric, the scalars, and the electric gauge fields in the SU(3)–invariant restricted duality hierarchy (3.52) enter the $D=11$ metric $d\hat{s}^2_{11}$. In order to express the result, it is useful to introduce a symmetric matrix h_{ab} of $D=4$ scalars and its inverse as[8]

$$h = \begin{pmatrix} e^{2\phi} & Z \\ Z & e^{-2\phi}\left(Y^2+Z^2\right) \end{pmatrix} , \qquad h^{-1} = Y^{-2} \begin{pmatrix} e^{-2\phi}\left(Y^2+Z^2\right) & -Z \\ -Z & e^{2\phi} \end{pmatrix} , \tag{3.117}$$

[7] The index i in the $\mathbf{3}\oplus\bar{\mathbf{3}}$ of SU(3) should not be confused with the fundamental index of the local SU(8) index in $D=4$.

[8] This matrix h_{ab} should not be confused with the metric h_{uv} on the hypermultiplet scalar manifold.

and the following combination of $D = 4$ scalars and coordinates μ^i, μ^a,

$$\Delta_1 = e^{2\varphi}\, Y \mu_i \mu^i + X\, h_{ab}\, \mu^a \mu^b \ . \tag{3.118}$$

With these definitions, the embedding into the $D = 11$ metric in (3.112) reads

$$\begin{aligned} d\hat{s}_{11}^2 = e^{-\varphi}\, X^{1/3}\, \Delta_1^{2/3} \Big[ds_4^2 + g^{-2}\, e^{\varphi}\, \Delta_1^{-1} \Big(D\mu_i D\mu^i + e^{2\varphi} \tfrac{Y}{X} (h^{-1})_{ab} D\mu^a D\mu^b \Big) \\ + g^{-2} e^{3\varphi} X^{-1} Y^{-1} (Y - X)\, \Delta_1^{-2} \Big(Y\, J^{(6)}_{ij}\, \mu^i D\mu^j + h_{ab} \epsilon^{bc} \mu^a D\mu_c \Big)^2 \Big] \,, \end{aligned} \tag{3.119}$$

where ϵ^{ab} is the totally antisymmetric symbol with two indices, and the covariant derivatives are defined as

$$D\mu^i = d\mu^i - g\, A^1 J^{(6)ij} \mu_j \,, \qquad D\mu^a = d\mu^a - g\, A^0 \epsilon^{ab} \mu_b \ . \tag{3.120}$$

For generic values of the $D = 4$ scalars, the metric (3.119) enjoys an $\mathrm{SU}(3) \times \mathrm{U}(1)_v$ isometry.

Moving on to the $D = 11$ three-form $\hat{A}_{(3)}$, all the $D = 4$ fields in the tensor hierarchy (3.52), except for the metric, enter its expression. A long calculation shows that (3.113) becomes

$$\begin{aligned} \hat{A}_{(3)} = {} & C^1 \mu_i \mu^i + C_{ab}\, \mu^a \mu^b - \tfrac{1}{12} g^{-1} \big[\big(B_a{}^a + 2A^1 \wedge \tilde{A}_1 \big)\, \delta_{ij} + 4B^2 J^{(6)}_{ij} \big] \wedge \mu^i D\mu^j \\ & + \tfrac{1}{2} g^{-1} \big[B_{ab} - A^0 \wedge \tilde{A}_0\, \delta_{ab} + B^0\, \epsilon_{ab} \big] \wedge \mu^a D\mu^b \\ & + \tfrac{1}{6}\, g^{-2}\, \tilde{A}_1 \wedge J^{(6)}_{ij} D\mu^i \wedge D\mu^j + \tfrac{1}{2}\, g^{-2} \tilde{A}_0 \wedge \epsilon_{ab} D\mu^a \wedge D\mu^b + A \,, \end{aligned} \tag{3.121}$$

where the internal three-form A reducing from (3.114) reads

$$\begin{aligned} A = -g^{-3}\, \Delta_1^{-1} \Big[& \tfrac{1}{2}\, e^{4\varphi}\, \chi\, X^{-1} Y\, J^{(6)}_{ij} \mu^i D\mu^j \wedge \epsilon_{ab}\, D\mu^a \wedge D\mu^b \\ & + \tfrac{1}{2}\, \chi\, e^{2\varphi} \big(Y J^{(6)}_{ij} \mu^i D\mu^j + h_{ab} \epsilon^{bc} \mu^a D\mu_c \big) \wedge J^{(6)}_{kl} D\mu^k \wedge D\mu^l \\ & - \tfrac{1}{4}\, e^{2\varphi} \big(V_1 \,\mathrm{Re}\, \Omega^{(6)}_{ijk} + V_2 \,\mathrm{Im}\, \Omega^{(6)}_{ijk} \big) \wedge \mu^i D\mu^j \wedge D\mu^k \\ & + \tfrac{1}{12}\, e^{2\phi}\, X \big(v_1 \,\mathrm{Re}\, \Omega^{(6)}_{ijk} + v_2 \,\mathrm{Im}\, \Omega^{(6)}_{ijk} \big) D\mu^i \wedge D\mu^j \wedge D\mu^k \Big] \ . \end{aligned} \tag{3.122}$$

Here, we have defined the shorthand functions

$$v_1 = \mu_7 \zeta + \mu_8 e^{-2\phi} (\zeta Z + \tilde{\zeta} Y) \,, \qquad v_2 = \mu_7 \tilde{\zeta} - \mu_8 e^{-2\phi} (\zeta Y - \tilde{\zeta} Z) \,, \tag{3.123}$$

and one-forms

$$V_1 = (\zeta Y - \tilde{\zeta} Z) D\mu^7 + e^{2\phi} \tilde{\zeta}\, D\mu^8 \,, \qquad V_2 = (\zeta Z + \tilde{\zeta} Y) D\mu^7 - e^{2\phi} \zeta\, D\mu^8 \ . \tag{3.124}$$

The field strength four-form $\hat{F}_{(4)} = d\hat{A}_{(3)}$ is computed to be

$$\begin{aligned}
\hat{F}_{(4)} &= H^1_{(4)}\,\mu_i\mu^i + H^{ab}_{(4)}\,\mu_a\mu_b - \tfrac{1}{12}g^{-1}\big[H_{(3)a}{}^a\,\delta_{ij} + 4H^2_{(3)}\,J^{(6)}_{ij}\big]\wedge\mu^i D\mu^j \\
&\quad + \tfrac{1}{2}g^{-1}\big[H^{ab}_{(3)} + H^0_{(3)}\,\epsilon^{ab}\big]\wedge\mu_a D\mu_b + \tfrac{1}{6}g^{-2}\tilde{H}_{(2)1}\wedge J^{(6)}_{ij}D\mu^i\wedge D\mu^j \\
&\quad + \tfrac{1}{2}g^{-2}\tilde{H}_{(2)0}\wedge\epsilon_{ab}D\mu^a\wedge D\mu^b \\
&\quad + \tfrac{1}{4}\,g^{-2}\,e^{2\varphi}\Delta_1^{-1}\Big[4\chi e^{2\varphi}\,X^{-1}Y\,J^{(6)}_{ij}\mu^i D\mu^j\wedge\mu_k D\mu^k \\
&\qquad\qquad + e^{2\phi}\Big(v_2\,\mathrm{Re}\,\Omega^{(6)}_{ijk} - v_1\,\mathrm{Im}\,\Omega^{(6)}_{ijk}\Big)\,\mu^i D\mu^j\wedge D\mu^k\Big]\wedge H^0_{(2)} \\
&\quad - \tfrac{1}{4}g^{-2}\,\Delta_1^{-1}\Big[2\chi e^{2\varphi}\,X^{-1}Y\mu_k\mu^k\big(XJ^{(6)}_{ij}D\mu^i\wedge D\mu^j + e^{2\varphi}\epsilon_{ab}\,D\mu^a\wedge D\mu^b\big) \\
&\qquad\qquad - 4\chi\,e^{2\varphi}\,\mu_k D\mu^k\wedge\big(Y J^{(6)}_{ij}\mu^i D\mu^j + h^{ac}\,\epsilon_{cb}\,\mu_a D\mu^b\big) \\
&\qquad\qquad + e^{2\phi}X\Big(v_2\,\mathrm{Re}\,\Omega^{(6)}_{ijk} - v_1\,\mathrm{Im}\,\Omega^{(6)}_{ijk}\Big)\,\mu^i D\mu^j\wedge D\mu^k\Big]\wedge H^1_{(2)} \\
&\quad + dA_{\text{scalars}}\,.
\end{aligned}\tag{3.125}$$

In this expression, $H^1_{(4)}$, $H^{ab}_{(4)}$, etc., turn out to reproduce the $D = 4$ four-, three- and magnetic two-form field strengths (3.63)-(3.64) of the restricted tensor hierarchy (3.52). This provides a $D = 11$ crosscheck of the $D = 4$ calculation of section 3.1.2. The terms that contain the electric two-form field strengths $H^0_{(2)}$, $H^1_{(2)}$, come from the vector contributions in the covariant derivatives $D\mu^i$ and $D\mu^a$ in (3.122). Finally, dA_{scalars} contains two types of terms. The first type includes contributions of covariant derivatives of $D = 4$ scalars, wedged with three-forms on the internal S^7. The second type includes internal four-forms with coefficients that depend on the $D = 4$ scalars algebraically only. The presence in $\hat{A}_{(3)}$ of $J^{(6)}_{ij}$, $\Omega^{(6)}_{ijk}$ and h_{ab} breaks the symmetry of the full $D = 11$ configuration to SU(3), in agreement with the symmetry of the $D = 4$ model.

The above expressions provide the complete embedding of the SU(3)–invariant, restricted tensor hierarchy (3.52) into $D = 11$ supergravity. As such, these expressions contain redundant $D = 4$ degrees of freedom. As argued below (3.116), these redundancies can be eliminated at the level of the $D = 11$ four-form field strength by making use of the $D = 4$ duality relations. Indeed, regarding the tensor field strengths in (3.125) as shorthand for the dualisation conditions (3.67)-(3.69), equations (3.119), (3.125) then express the embedding into $D = 11$ supergravity exclusively in terms of the dynamically independent (metric, electric-vector and scalar) degrees of freedom that enter the $D = 4$ Lagrangian (3.54).

In particular, the Freund-Rubin term (the first two contributions on the r.h.s. of (3.125)), can be simplified by using the identities (3.70), (3.73) that relate the dualised four-form field strengths (3.69) to the scalar potential

(3.54) and its derivatives:

$$
\begin{aligned}
H^1_{(4)}\,\mu_i\mu^i + H^{ab}_{(4)}\,\mu_a\mu_b = -\tfrac{1}{4g}\Big[V &+ \tfrac{1}{6}(\mu_i\mu^i - 3\mu_a\mu^a)\,k^\alpha[H_0]\,\partial_\alpha V \\
&+ \big((\mu^7)^2 - (\mu^8)^2\big)\,k^u[H_1]\,\partial_u V \\
&+ \sqrt{2}\,\mu^7\mu^8\big(k^u[E_2] + k^u[F_2]\big)\,\partial_u V\Big]\mathrm{vol}_4\;.
\end{aligned} \tag{3.126}
$$

At a critical point, the terms in derivatives of the potential drop out and the Freund-Rubin term becomes proportional to the AdS_4 cosmological constant, in agreement with the general $\mathcal{N}=8$ discussion of [30]. See also [90] for a related discussion.

The uplifting formulae (3.119)-(3.122) simplify by imposing a symmetry enlargement, carried over to $D=11$ by restricting the $D=4$ fields to the previous subsectors studied in section 3.1.2. We now turn to the description of these subsectors in terms of the intrinsic S^7 angles that are best adapted to making the relevant symmetry apparent in $D=11$. See appendix C for some relevant geometric structures on S^7.

SU(3) × U(1)^2–invariant sector

For the $\mathrm{SU}(3)\times\mathrm{U}(1)^2$–invariant sector (3.74), the embedding formulae for the $D=11$ metric, (3.119), and three-form, (3.121), (3.122), become

$$
\begin{aligned}
d\hat{s}^2_{11} = e^{-\varphi}X^{1/3}\Delta_1^{2/3}ds_4^2 + g^{-2}\Big[&X^{-2/3}\Delta_1^{2/3}d\alpha^2 + X^{1/3}\Delta_1^{-1/3}\cos^2\alpha\,ds^2(\mathbb{CP}^2) \\
&+ e^{2\varphi}X^{-2/3}\Delta_1^{2/3}\Delta_2^{-1}\sin^2\alpha\cos^2\alpha\,(D\tau_- + \sigma)^2 \\
&+ X^{-2/3}\Delta_2\,\Delta_1^{-4/3}\Big(D\psi_- + \Delta_3\Delta_2^{-1}\,\cos^2\alpha\,\big(D\tau_- + \sigma\big)\Big)^2\Big]\,,
\end{aligned} \tag{3.127}
$$

$$
\begin{aligned}
\hat{A}_{(3)} = {}& C_1\cos^2\alpha + C_{77}\sin^2\alpha \\
&+ \tfrac{1}{12}\,g^{-1}\,\sin 2\alpha\,\big(4B_{77} + A^1\wedge\tilde{A}_1 - 3A^0\wedge\tilde{A}_0\big)\wedge d\alpha \\
&- \tfrac{1}{6}\,g^{-2}\sin 2\alpha\,(\tilde{A}_1 + 3\tilde{A}_0)\wedge d\alpha\wedge D\psi_- \\
&+ \tfrac{1}{3}\,g^{-2}\cos\alpha\,\tilde{A}_1\wedge\big[\cos\alpha\,\boldsymbol{J}^{(4)} - \sin\alpha\,d\alpha\wedge(D\tau_- + \sigma)\big] \\
&+ \tfrac{1}{2}\,g^{-3}\chi e^{2\varphi}X^{-1}\sin 2\alpha\,d\alpha\wedge D\psi_-\wedge(D\tau_- + \sigma) \\
&- g^{-3}\chi e^{2\varphi}\Delta_1^{-1}\cos^4\alpha\,(D\tau_- + \sigma)\wedge\boldsymbol{J}^{(4)} \\
&- g^{-3}\chi e^{2\varphi}\Delta_1^{-1}\cos^2\alpha\cos 2\alpha\,D\psi_-\wedge\boldsymbol{J}^{(4)}\,.
\end{aligned} \tag{3.128}
$$

In these expressions, α, τ_-, ψ_- are angles on S^7 whose relation to the constrained coordinates μ^A of $\mathbb{R}^8$ is given in appendix C. The covariant derivatives for the last two are

$$D\psi_- = d\psi_- - gA^0\,, \qquad D\tau_- = d\tau_- + g(A^0 + A^1)\,. \tag{3.129}$$

The line element $ds^2(\mathbb{CP}^2)$ and the two-form $\boldsymbol{J}^{(4)}$ respectively correspond to the Fubini-Study metric, normalised so that its Ricci tensor is six times the metric, and the Kähler form, with potential one-form σ such that $d\sigma = 2\boldsymbol{J}^{(4)}$, on the the complex projective plane. Finally, Δ_1, Δ_2 and Δ_3 are the following functions of the S^7 angle α and the $\mathrm{SU}(3)\times\mathrm{U}(1)^2$–invariant, $D=4$ vector multiplet scalars

$$\begin{aligned}
\Delta_1 &= X\sin^2\alpha + e^{2\varphi}\cos^2\alpha\,,\\
\Delta_2 &= e^{2\varphi}\left[\sin^4\alpha + \left(e^{2\varphi} + 2\chi^2 e^{2\varphi} + e^{-2\varphi}X^2\right)\sin^2\alpha\cos^2\alpha + \cos^4\alpha\right]\,,\\
\Delta_3 &= \left(X^2 + \chi^2 e^{4\varphi}\right)\sin^2\alpha + e^{2\varphi}\cos^2\alpha\,.
\end{aligned} \tag{3.130}$$

with Δ_1 being simply the particularisation of (3.118) to the present case.

The field strength corresponding to (3.128) can be computed to be

$$\begin{aligned}
\hat{F}_{(4)} &= 2g\Big[2\left(e^{\varphi}\cos^2\alpha + e^{-\varphi}X\sin^2\alpha\right) + Xe^{-\varphi}\Big]\,\mathrm{vol}_4\\
&\quad + g^{-1}\,\sin 2\alpha\,(*d\varphi - e^{2\varphi}\chi * d\chi)\wedge d\alpha\\
&\quad - \tfrac{1}{6}\,g^{-2}\Big[\sin 2\alpha\,(\tilde{H}_1 + 3\tilde{H}_0)\wedge d\alpha\wedge D\psi_-\\
&\qquad\qquad - 2\tilde{H}_1\wedge\left(\cos^2\alpha\,\boldsymbol{J}^{(4)} - \sin\alpha\cos\alpha\,d\alpha\wedge(D\tau_- + \sigma)\right)\Big]\\
&\quad + \tfrac{1}{2}\,g^{-2}\chi e^{2\varphi}\Big[X^{-1}\sin 2\alpha\,d\alpha\wedge\left(H^0\wedge(D\tau_- + \sigma) + (H^0 + H^1)\wedge D\psi_-\right)\\
&\qquad\qquad - 2\Delta_1^{-1}\cos^4\alpha\,(H^0 + H^1)\wedge\boldsymbol{J}^{(4)} + 2\Delta_1^{-1}\cos^2\alpha\cos 2\alpha\,H^0\wedge\boldsymbol{J}^{(4)}\Big]\\
&\quad + \frac{1}{2}\,g^{-3}\,e^{2\varphi}\,X^{-2}\sin 2\alpha\Big[2\chi\,d\varphi - (X-2)d\chi\Big]\wedge d\alpha\wedge D\psi_-\wedge(D\tau_- + \sigma)\\
&\quad + g^{-3}\,e^{2\varphi}\Big\{\Delta_1^{-2}\,(D\tau_- + \sigma)\wedge\Big[-2\chi(\Delta_1 + X)\,\sin\alpha\cos^3\alpha\,d\alpha\\
&\qquad\qquad + \cos^4\alpha\Big(2\chi\,\sin^2\alpha\,d\varphi + \left(e^{2\varphi}\cos^2\alpha - (X-2)\sin^2\alpha\right)d\chi\Big)\Big]\\
&\qquad + \Delta_1^{-2}D\psi_-\wedge\Big[-\tfrac{1}{2}\,\chi\,\sin 2\alpha\Big(4e^{2\varphi}\cos^4\alpha + X\left(\sin^2 2\alpha + 2\cos 2\alpha\right)\Big)\,d\alpha\\
&\qquad\qquad + \cos^2\alpha\cos 2\alpha\Big(2\chi\,\sin^2\alpha\,d\varphi + \left(e^{2\varphi}\cos^2\alpha - (X-2)\sin^2\alpha\right)d\chi\Big)\Big]\\
&\qquad + \chi\,X^{-1}\sin 2\alpha\,d\alpha\wedge D\psi_- - 2\chi\,\Delta_1^{-1}\cos^4\alpha\,\boldsymbol{J}^{(4)}\Big\}\wedge\boldsymbol{J}^{(4)}\,.
\end{aligned} \tag{3.131}$$

Here, we have explicitly made use of the dualisation conditions (3.68), (3.69) for the three- and four-form field strengths, particularised to $\mathrm{SU}(3)\times\mathrm{U}(1)^2$–scalars via (3.74). The magnetic two-form field strengths $\tilde{H}_\Lambda$, $\Lambda = 0, 1$, stand for the dualised expressions (3.67).

As noted in section 3.1.2, the $\mathrm{SU}(3)\times\mathrm{U}(1)^2$–invariant sector coincides with the gauged STU model with all three vector multiplets identified. This was embedded in $D=11$ supergravity in [81] (see also [91]), along with the entire STU model. Our uplifting formulae (3.127), (3.131), obtained instead from the $D=11$ embedding of the SU(3) sector, are in perfect agreement with (6.22)-(6.24) of [81]. This can be seen by using the $D=4$ redefinitions (3.75), which also imply $\tilde{H}_{0\,\text{here}}=\tilde{R}_{\text{there}}$ and $\tilde{H}_{1\,\text{here}}=-R_{\text{there}}$, along with the S^7 angle and one-form identifications

$$\begin{aligned}
&\xi_{\text{there}}=\alpha_{\text{here}}+\tfrac{\pi}{2}\,, && \phi_{1\text{there}}=\psi_{-\,\text{here}}\,,\\
&\psi_{\text{there}}=\psi_{-\,\text{here}}+\tau_{-\,\text{here}}\,, && B_{\text{there}}=\sigma_{\text{here}}\,,
\end{aligned}\tag{3.132}$$

or, in terms of the ψ, τ defined in equation (C.1) of appendix C, $\phi_{1\text{there}}=-\psi$, $\psi_{\text{there}}=\tau$.

SU(4)–invariant sectors

While the deformations inflicted on the internal S^7 by the SU(3)–invariant $D=4$ fields are inhomogeneous, enlarging the symmetry to $\mathrm{SU}(4)_c$ and $\mathrm{SU}(4)_s$ results in the deformations becoming homogeneous.

For the $\mathrm{SU}(4)_c$–invariant $D=4$ fields (3.83), the $D=11$ embedding formulae (3.119), (3.121)), (3.122) simplify to

$$d\hat{s}^2_{11}=e^{\frac{4}{3}\phi+\varphi}\,ds^2_4+g^{-2}\Big[e^{-\frac{2}{3}\phi}ds^2(\mathbb{CP}^3_+)+e^{\frac{4}{3}\phi-2\varphi}(\boldsymbol{\eta}^{(7)}_+ +gA)^2\Big]\,,\tag{3.133}$$

$$\begin{aligned}
\hat{A}_{(3)}=&\;C^1+\tfrac{1}{2}\,g^{-1}\,B^0\wedge(\boldsymbol{\eta}^{(7)}_+ +gA)+g^{-2}\tilde{A}_0\wedge\boldsymbol{J}^{(7)}_+\\
&-g^{-3}\Big[\chi\,\boldsymbol{J}^{(7)}_+\wedge(\boldsymbol{\eta}^{(7)}_+ +gA)-\tfrac{1}{2}\,\zeta\,\mathrm{Re}\,\boldsymbol{\Omega}^{(7)}_+ -\tfrac{1}{2}\,\tilde{\zeta}\,\mathrm{Im}\,\boldsymbol{\Omega}^{(7)}_+\Big]\,,
\end{aligned}\tag{3.134}$$

where ϕ, φ stand for the expressions in terms of $\chi,\ \zeta,\ \tilde{\zeta}$ given in (3.83). Here, $ds^2(\mathbb{CP}^3_+)$ is the Fubini-Study metric on $\mathbb{CP}^3$ normalised so that the Ricci tensor is eight times the metric, and $\boldsymbol{\eta}^{(7)}_+$, $\boldsymbol{J}^{(7)}_+$, $\boldsymbol{\Omega}^{(7)}_+$ are the homogeneous Sasaki-Einstein forms on S^7 defined in appendix C. The four-form field strength corresponding to (3.134) reads

$$\begin{aligned}
\hat{F}_{(4)}=&-6g\,e^{4\phi+3\varphi}\Big[-1+\chi^2+\tfrac{1}{3}\big(\zeta^2+\tilde{\zeta}^2\big)\Big]\,\mathrm{vol}_4\\
&+\tfrac{1}{2}\,g^{-1}\,e^{4\phi}*\Big(\tilde{\zeta}D\zeta-\zeta D\tilde{\zeta}\Big)\wedge(\boldsymbol{\eta}^{(7)}_+ +gA)\\
&+\frac{g^{-2}(1-\chi^2)}{1+3\chi^2}\Big[2\chi\,F-\sqrt{1-\chi^2}\,*F\Big]\wedge\boldsymbol{J}^{(7)}_+\\
&-g^{-3}\Big[d\chi\wedge\boldsymbol{J}^{(7)}_+\wedge(\boldsymbol{\eta}^{(7)}_+ +gA)-\tfrac{1}{2}D\zeta\wedge\mathrm{Re}\,\boldsymbol{\Omega}^{(7)}_+ -\tfrac{1}{2}D\tilde{\zeta}\wedge\mathrm{Im}\,\boldsymbol{\Omega}^{(7)}_+\Big]\\
&-2g^{-3}\chi\boldsymbol{J}^{(7)}_+\wedge\boldsymbol{J}^{(7)}_+ -2g^{-3}\big(\tilde{\zeta}\,\mathrm{Re}\,\boldsymbol{\Omega}^{(7)}_+ -\zeta\,\mathrm{Im}\,\boldsymbol{\Omega}^{(7)}_+\big)\wedge(\boldsymbol{\eta}^{(7)}_+ +gA)\,,
\end{aligned}\tag{3.135}$$

with, again, ϕ, φ written in terms of χ, ζ, $\tilde{\zeta}$ as in (3.83). As noted in section 3.1.2 following [78], the $\mathrm{SU}(4)_c$–invariant sector of SO(8) supergravity coincides with the model considered in [83]. Using the redefinitions (3.87) and straightforwardly identifying our Sasaki-Einstein structure with theirs, our uplifting formulae (3.133), (3.135) do indeed match (2.2), (2.3) of [83] when the identifications of their equation (4.1) are taken into account.

The $\mathrm{SU}(4)_s$–sector coincides with minimal $\mathcal{N}=2$ gauged supergravity, (2.35). The $D=11$ uplift of this sector can be achieved by bringing the restrictions (3.84) to the general formulae (3.119)-(3.122) or, equivalently, by further setting $\varphi=\chi=0$, $A^1=-A^0\equiv\frac{1}{4}\bar{A}$, and $\tilde{A}_1=-3\tilde{A}_0$ in the uplifting formulae (3.127)-(3.131) of the $\mathrm{SU}(3)\times\mathrm{U}(1)^2$ sector. Using the rescaled fields (3.85) and the $D=4$ field strengths (3.86), and combining the resulting expressions in terms of the Sasaki-Einstein forms $\boldsymbol{J}_-^{(7)}$, $\boldsymbol{\eta}_-^{(7)}$ specified in appendix C, the $D=11$ uplift of the $\mathrm{SU}(4)_s$–sector can be written as

$$
\begin{aligned}
ds_{11}^2 &= \tfrac{1}{4}\,d\bar{s}_4^2 + g^{-2}\left(ds^2(\mathbb{CP}^3_-) + (\boldsymbol{\eta}_-^{(7)} + \tfrac{1}{4}g\bar{A})^2\right), \\
\hat{F}_{(4)} &= \tfrac{3}{8}\,g\,\overline{\mathrm{vol}}_4 - \tfrac{1}{4}\,g^{-2}\,\bar{*}\bar{F}\wedge\boldsymbol{J}_-^{(7)}\,.
\end{aligned}
\tag{3.136}
$$

This coincides with the consistent truncation of $D=11$ supergravity down to minimal $\mathcal{N}=2$ gauged supergravity obtained in [32], with straightforward identifications. An alternate $D=11$ embedding of minimal $\mathcal{N}=2$ supergravity based on (3.93)-(3.96) will be given in (3.142)-(3.143) whose consistency does not follow from group theory arguments in any manifest way.

G_2–invariant sector

The $D=11$ embedding formulae (3.119)-(3.122) particularised to the G_2–invariant sector (3.91) become, in the relevant set of intrinsic coordinates described in appendix C,

$$
\begin{aligned}
ds_{11}^2 &= e^{-\varphi}X^{1/3}\Delta_1^{2/3}ds_4^2 + g^{-2}X^{1/3}\Delta_1^{-1/3}\Big(e^{2\varphi}X^{-3}\Delta_1 d\beta^2 + \sin^2\beta\, ds^2(S^6)\Big), \\
\hat{A}_{(3)} &= C_1\sin^2\beta + C_{88}\cos^2\beta + 4g^{-1}\sin\beta\cos\beta\, B_{77}\wedge d\beta \\
&\quad + g^{-3}\,\chi\,\Delta_1^{-1}\sin^2\!\beta\Big[e^{2\varphi}X^{-1}\Delta_1\mathcal{J}\wedge d\beta \\
&\qquad + X^2\sin\beta\cos\beta\,\mathrm{Re}\,\Omega + e^{2\varphi}X\,\sin^2\!\beta\,\mathrm{Im}\,\Omega\Big],
\end{aligned}
\tag{3.137}
$$

where β is an angle on S^7, $ds^2(S^6)$ is the round metric on S^6 normalised so that the Ricci tensor equals five times the metric, $\mathcal{J}$ and Ω are the homogeneous nearly-Kähler forms on S^6 and the function Δ_1 is, from (3.118) with (C.22),

$$
\Delta_1 = X\left(e^{-2\varphi}X^2\cos^2\!\beta + e^{2\varphi}\sin^2\!\beta\right). \tag{3.138}
$$

The associated four-form field strength reads

$$
\begin{aligned}
\hat{F}_{(4)} = & -g\, e^{-3\varphi} X^2 \Big[\big[(X-2)X^2 + e^{4\varphi}(7X-12) \big] \sin^2\!\beta \\
& \qquad\qquad + e^{-4\varphi} X^2 \big[X^3 + 7e^{4\varphi}(X-2) \big] \cos^2\!\beta \Big] \mathrm{vol}_4 \\
& - 4g^{-1} \sin\!\beta \cos\beta \left(* \, d\varphi - e^{2\varphi} \chi * d\chi \right) \wedge d\beta \\
& + g^{-3} e^{2\varphi} X^{-2} \sin^2\beta \left(2\chi d\varphi - (X-2) d\chi \right) \wedge \mathcal{J} \wedge d\beta \\
& - 2 g^{-3} e^{2\varphi} \chi X \Delta_1^{-1} \sin^4\beta \, \mathcal{J} \wedge \mathcal{J} \\
& + g^{-3} \, \Delta_1^{-2} \sin^2\beta \Big\{ \chi X^3 \sin^2\!2\beta \, d\varphi \wedge \mathrm{Im}\,\Omega \\
& \quad + \chi X \sin 2\beta \left(\Delta_1 - 2e^{2\varphi} X \sin^2\beta \right) d\varphi \wedge \mathrm{Re}\,\Omega \\
& \quad + X^2 d\chi \wedge \Big[\sin^2\beta \big[e^{4\varphi} \sin^2\beta - X(3X-4)\cos^2\beta \big] \,\mathrm{Im}\,\Omega \\
& \qquad + \tfrac{1}{2} \sin 2\beta \left[e^{2\varphi}(3X-2)\sin^2\beta - e^{-2\varphi} X^2 (X-2) \cos^2\beta \right] \, \mathrm{Re}\,\Omega \Big] \\
& \quad + \chi X \sin^2\beta \, d\beta \wedge \Big[X \sin 2\beta \big[(e^{4\varphi} + X^2)\sin^2\beta + 2X^2 \cos^2\beta \big] \,\mathrm{Im}\,\Omega \\
& \qquad - e^{-2\varphi} \big[e^{4\varphi} \left(3e^{4\varphi} + X^2 \right) \sin^2\beta + X^2 \left(5e^{4\varphi} - X^2 \right) \cos^2\beta \big] \, \mathrm{Re}\,\Omega \Big] \Big\} \, .
\end{aligned}
\tag{3.139}
$$

In order to obtain this expression, we have again made explicit use of the dualisation conditions (3.68), (3.69) for the three- and four-form field strengths, particularised to the G_2–invariant sector (3.91). The $D = 11$ uplift of the various SO(7)–invariant sectors can be straightforwardly obtained by bringing (3.88)-(3.90) to (3.137)-(3.139). See [90] for a previous $D = 11$ uplift of the G_2–invariant sector.

A new embedding of minimal $\mathcal{N} = 2$ gauged supergravity

It was noted below (3.84) that the $\mathrm{SU}(4)_s$ sector coincides with minimal $\mathcal{N} = 2$ gauged supergravity. In (3.136), the corresponding $D = 11$ uplift was obtained and shown to coincide with the consistent embedding of [32]. It was also discussed in that section that the SU(3)–sector admits an alternative truncation to minimal $\mathcal{N} = 2$ supergravity, by fixing the scalars to their vevs (3.93) at the $\mathcal{N} = 2$, $\mathrm{SU}(3) \times \mathrm{U}(1)_c$–invariant point and selecting the $\mathcal{N} = 2$ graviphoton as in (3.94). Bringing these $D = 4$ identifications to the general SU(3)–invariant consistent uplifting formulae, we obtain a new embedding of pure $\mathcal{N} = 2$ gauged supergravity into $D = 11$.

We find it convenient to present the result in local intrinsic S^7 coordinates ψ', τ', α, and in terms of a local five-dimensional Sasaki-Einstein structure $\boldsymbol{\eta}'$, $\boldsymbol{J}'$ and $\boldsymbol{\Omega}'$. The former are locally related to the global coordinates ψ, τ,

α, defined in (C.1), that are adapted to the topological description of S^7 as the join of S^5 and S^1, with α here identified with that in (C.1) and

$$\psi = \psi' \,, \qquad \tau = \tau' - \tfrac{1}{3}\,\psi' \,. \tag{3.140}$$

The local five-dimensional Sasaki-Einstein structure forms $\boldsymbol{\eta}'$, $\boldsymbol{J}'$ and $\boldsymbol{\Omega}'$ are related to their globally defined counterparts $\boldsymbol{\eta}^{(5)}$, $\boldsymbol{J}^{(5)}$ and $\boldsymbol{\Omega}^{(5)}$ discussed in appendix C and the global coordinate ψ via

$$\boldsymbol{\eta}' \equiv d\tau' + \sigma \equiv \boldsymbol{\eta}^{(5)} + \tfrac{1}{3}d\psi \,, \qquad \boldsymbol{J}' \equiv \boldsymbol{J}^{(5)} \,, \qquad \boldsymbol{\Omega}' \equiv e^{i(\psi+\frac{\pi}{4})}\,\boldsymbol{\Omega}^{(5)} \,. \tag{3.141}$$

The real two-form $\boldsymbol{J}'$ coincides with the Kähler form on $\mathbb{CP}^2$, σ is a one-form on the latter such that $d\sigma = 2\boldsymbol{J}'$ (given *e.g.* by (C.11)) and the constant phase $e^{i\frac{\pi}{4}}$ in the complex two-form $\boldsymbol{\Omega}'$ has been chosen for convenience, in order to simplify the resulting expressions. The primed forms defined in (3.141) satisfy the Sasaki-Einstein conditions (C.5) and (C.6).

Bringing all these definitions, along with the $D = 4$ restrictions (3.93)-(3.96), to the uplifting formulae (3.119), (3.121), (3.122), we find a new consistent embedding of minimal $D = 4$ $\mathcal{N} = 2$ gauged supergravity (2.35) into the $D = 11$ metric and three-form:

$$\begin{aligned} d\hat{s}^2_{11} = \frac{1}{3}\cdot 2^{-2/3}\,(1+2\sin^2\alpha)^{2/3}\Bigg[d\bar{s}^2_4 + g^{-2}\Big[\, 2\,d\alpha^2 + \frac{6\cos^2\alpha}{1+2\sin^2\alpha} ds^2(\mathbb{CP}^2) \\ + \frac{18\sin^2\alpha\cos^2\alpha}{1+8\sin^4\alpha}\,\boldsymbol{\eta}'^{\,2} + \frac{1+8\sin^4\alpha}{\big(1+2\sin^2\alpha\big)^2}\Big(D\psi' - \frac{3\cos^2\alpha}{1+8\sin^4\alpha}\boldsymbol{\eta}'\Big)^2\Big]\Bigg] \,, \end{aligned} \tag{3.142}$$

$$\begin{aligned} \hat{A}_{(3)} = C^1 - \frac{1}{2\sqrt{3}}\,g^{-2}\,\cos\alpha\,\tilde{\bar{A}}\wedge\Big[\cos\alpha\;\boldsymbol{J}' - \sin\alpha\,d\alpha\wedge\boldsymbol{\eta}'\Big] \\ + \frac{1}{\sqrt{3}}\,g^{-3}\,\cos^2\alpha\,\Big[d\alpha\wedge\mathrm{Im}\,\boldsymbol{\Omega}' + \frac{\sin\alpha\cos\alpha}{1+2\sin^2\alpha}\Big(2D\psi' - 3\boldsymbol{\eta}'\Big)\wedge\mathrm{Re}\,\boldsymbol{\Omega}'\Big] \,. \end{aligned} \tag{3.143}$$

These expressions depend explicitly on the dynamical $D = 4$ metric $d\bar{s}^2_4$ and graviphoton $\bar{A}$. The former only features in $d\hat{s}^2_{11}$ but not in $\hat{A}_{(3)}$. The latter appears both in $d\hat{s}^2_{11}$ and in $\hat{A}_{(3)}$, but only through the gauge covariant derivative

$$D\psi' = d\psi' + \tfrac{1}{2}g\bar{A} \,. \tag{3.144}$$

This singles out ψ' as the angle on the local $\mathcal{N} = 2$ "Reeb" direction and thus justifies the primed coordinates (3.140) that we chose to present the result. Two other $D = 4$ fields enter the consistent embedding through the three-form (3.143): the magnetic dual, $\tilde{\bar{A}}$, of the $D = 4$ graviphoton, and the auxiliary three-form potential C^1.

The four-form field strength corresponding to $\hat{A}_{(3)}$ in (3.142) can be computed with the help of (the primed version of) the Sasaki-Einstein conditions (C.5), (C.6). We find

$$\hat{F}_{(4)} = \frac{g}{2\sqrt{3}}\overline{\text{vol}}_4 + \frac{g^{-3}}{\sqrt{3}}\Bigg[-\frac{\cos^2\alpha\,(7-10\cos 2\alpha+\cos 4\alpha)}{\big(1+2\sin^2\alpha\big)^2}d\alpha\wedge D\psi'\wedge \text{Re}\,\boldsymbol{\Omega}'$$

$$-\frac{6\cos^4\alpha}{(1+2\sin^2\alpha)^2}d\alpha\wedge\boldsymbol{\eta}'\wedge\text{Re}\,\boldsymbol{\Omega}'+\frac{6\sin\alpha\cos^3\alpha}{1+2\sin^2\alpha}D\psi'\wedge\boldsymbol{\eta}'\wedge\text{Im}\,\boldsymbol{\Omega}'\Bigg]$$

$$+\frac{g^{-2}}{4\sqrt{3}}\left[\frac{4\sin\alpha\cos^3\alpha}{1+2\sin^2\alpha}\,\bar{F}\wedge\text{Re}\,\boldsymbol{\Omega}'+\bar{*}\,\bar{F}\wedge d\big(\cos^2\!\alpha\,\boldsymbol{\eta}'\big)\right]\,. \tag{3.145}$$

Again, we have made use of appropriate dualisation conditions, (3.97), (3.98) in this case, to express the result for the embedding (3.145) into the four-form only in terms of the independent $D=4$ degrees of freedom (the metric $d\bar{s}_4^2$, the graviphoton field strength $\bar{F}=d\bar{A}$ and its Hodge dual), that appear in the Lagrangian (2.35).

The truncation (3.142), (3.145) of $D=11$ supergravity down to pure $D=4$ $\mathcal{N}=2$ gauged supergravity (2.35) is consistent by construction. We have explicitly verified consistency at the level of the Bianchi identities and equations of motion for the $D=11$ four-form: its field equations are indeed satisfied, provided the $D=4$ Bianchi, $d\bar{F}=0$, and equation of motion, $d\,\bar{*}\,\bar{F}=0$, of the $D=4$ graviphoton are imposed. Some details can be found in appendix D.1. Moreover, these local uplifting formulae are still valid if, more generally, $\boldsymbol{\eta}'$, $\boldsymbol{J}'$, $\boldsymbol{\Omega}'$ are taken to be the defining forms of *any* Sasaki-Einstein five-manifold, and $ds^2(\mathbb{CP}^2)$ is replaced with the metric on the corresponding local Kähler-Einstein base. This is in agreement with the G-structures perspective provided in chapter 2, as this truncation can be recovered from that route as (2.52) with (2.30).

Recovering $D=11$ AdS$_4$ solutions

Setting the scalars to the vevs at each critical point with at least SU(3) invariance that were recorded in table 3.2, and turning off the relevant tensor hierarchy fields, the consistent embedding formulae (3.119)-(3.122) produce AdS$_4$ solutions of $D=11$ supergravity. All these $D=11$ solutions are known, so our presentation must necessarily be brief. The main motivation to work out these solutions is rather to test the consistency of the uplifting formulae presented in section 3.2.1 (and their particularisation to an explicit, SU(3)–invariant, subsector). Except for the more involved $D=11$ Einstein equation, the metrics and four-forms that we write below have been verified to solve the eleven-dimensional field equations. Please refer to appendix D.2 for details.

We present the solutions in the appropriate intrinsic S^7 angles defined in appendix C. Also, AdS$_4$ is always taken to be unit radius (so that the Ricci tensor equals -3 times the metric). As a consequence, the metric

$ds^2(\mathrm{AdS}_4)$ that appears in the expressions below is related to the metric ds_4^2 that appears in the $D=4$ Lagrangian (3.54) and $D=11$ embedding (3.119) by a rescaling

$$ds_4^2 = -6\, V_0^{-1}\, ds^2(\mathrm{AdS}_4)\,, \tag{3.146}$$

where V_0 is the cosmological constant at each critical point given in table 3.2. The Freund-Rubin term is rescaled accordingly with respect to (3.126).

Let us first discuss the supersymmetric solutions. The $\mathcal{N}=8$, SO(8) point uplifts to the Freund-Rubin solution [92] for which the internal four-form vanishes and the internal metric is the round, Einstein metric $ds^2(S^7)$, given in *e.g.* (C.3) or (C.17). The $\mathcal{N}=2$, $\mathrm{SU}(3)\times\mathrm{U}(1)_c$ critical point uplifts to the $D=11$ CPW solution [31]. A local form of this solution can be obtained from (3.142)-(3.145) by turning off the $D=4$ graviphoton, $\bar{A}=0$, $\bar{F}=0$, and fixing the metric to $d\bar{s}_4^2 = g^{-2} ds^2(\mathrm{AdS}_4)$. As a check, we have verified that the solution in $\mathbb{R}^8$ embedding coordinates μ^A, directly obtained from (3.119)-(3.122), perfectly agrees with the CPW solution as given in [93]. Finally, the $\mathcal{N}=1$ G_2–invariant solution can be written, using (3.137)-(3.139), as

$$\begin{aligned}
d\hat{s}_{11}^2 &= g^{-2}\left(\tfrac{25}{12}\right)^{\frac{1}{6}}(2+\cos 2\beta)^{\frac{2}{3}}\left[\tfrac{5}{24}ds^2(\mathrm{AdS}_4)+\tfrac{1}{3}d\beta^2+\frac{\sin^2\beta}{2+\cos 2\beta}\,ds^2(S^6)\right]\,,\\
\hat{F}_{(4)} &= \tfrac{1}{8}\left(\tfrac{25}{12}\right)^{\frac{5}{4}}\, g^{-3}\ \mathrm{vol}(\mathrm{AdS}_4) + \frac{\sqrt{2}g^{-3}\sin^2\beta}{3^{1/4}(2+\cos 2\beta)^2}\Big[\sqrt{3}\sin^2\beta\,\mathrm{Re}\,\Omega\wedge d\beta\\
&\quad -\sin\beta\cos\beta\,(5+\cos 2\beta)\,\mathrm{Im}\,\Omega\wedge d\beta - \sin^2\beta(2+\cos 2\beta)\,\mathcal{J}\wedge\mathcal{J}\Big]\,,
\end{aligned} \tag{3.147}$$

with internal three-form potential

$$A = \frac{g^{-3}\sin^2\beta}{3^{3/4}\sqrt{2}(2+\cos 2\beta)}\Big[\tfrac{3}{2}\sin 2\beta\,\mathrm{Re}\,\Omega+\sqrt{3}\sin^2\beta\,\mathrm{Im}\,\Omega+(2+\cos 2\beta)\mathcal{J}\wedge d\beta\Big]\,, \tag{3.148}$$

in terms of the nearly-Kähler structure of the S^6 in S^7. This solution was first obtained by de Wit, Nicolai and Warner [94].

Turning to the non-supersymmetric solutions, the SO(7) critical points can again be uplifted using (3.137)-(3.139). The $\mathrm{SO}(7)_v$ solution uplifts to a solution first written by de Wit and Nicolai [95]. In our conventions, we get

$$\begin{aligned}
d\hat{s}_{11}^2 &= 5^{-\frac{5}{6}}\, g^{-2}\,(3+2\cos 2\beta)^{\frac{2}{3}}\left[\tfrac{3}{4}ds^2(\mathrm{AdS}_4)+d\beta^2+\frac{5\sin^2\beta}{3+2\cos 2\beta}\,ds^2(S^6)\right]\,,\\
\hat{F}_{(4)} &= \tfrac{9}{8}\cdot 5^{-\frac{3}{4}}\, g^{-3}\ \mathrm{vol}(\mathrm{AdS}_4)\,,
\end{aligned} \tag{3.149}$$

while the $\mathrm{SO}(7)_c$ point uplifts to Englert's solution [96]

$$d\hat{s}_{11}^2 = g^{-2}\left(\tfrac{4}{5}\right)^{\frac{1}{3}}\left[\tfrac{3}{10}ds^2(\mathrm{AdS}_4)+ds^2(S^7)\right]\,,$$

$$\hat{F}_{(4)} = g^{-3} \tfrac{18}{25\sqrt{5}} \,\mathrm{vol}(\mathrm{AdS}_4) + \frac{4\sin^4\beta}{\sqrt{5}\, g^3}\Big[\mathrm{Re}\,\Omega\wedge d\beta - \cot\beta\,\mathrm{Im}\,\Omega\wedge d\beta - \tfrac{1}{2}\,\mathcal{J}^2\Big]\,, \tag{3.150}$$

with internal three-form

$$A = \frac{\sin^2\beta}{2\sqrt{5}\, g^3}\Big[2\sin^2\beta\,\mathrm{Im}\,\Omega + 2\mathcal{J}\wedge d\beta + \sin 2\beta\,\mathrm{Re}\,\Omega\Big]\,. \tag{3.151}$$

In the $\mathrm{SO}(7)_c$ solution, $ds^2(S^7)$ is, as always, the round, SO(8)–invariant metric. It should be understood in this context as the sine-cone form (C.23). Since $\mathrm{SO}(7)_c \supset \mathrm{SU}(4)_c$, this solution can also be re-obtained from the $\mathrm{SU}(4)_c$–invariant truncation and written in terms of the homogeneous Sasaki-Einstein structure on S^7. The $D = 11$ metric is the same appearing in (3.150) with $ds^2(S^7)$ now understood as the Hopf fibration (C.17), and the four-form is given by

$$\hat{F}_{(4)} = \frac{18}{25\sqrt{5}\, g^3}\,\mathrm{vol}(\mathrm{AdS}_4) + \frac{2}{\sqrt{5}g^3}\Big[2\,\mathrm{Re}\,\boldsymbol{\Omega}^{(7)}_+\wedge\boldsymbol{\eta}^{(7)}_+ - \boldsymbol{J}^{(7)}_+\wedge\boldsymbol{J}^{(7)}_+\Big]\,, \tag{3.152}$$

with internal three-form

$$A = -\frac{1}{\sqrt{5}\, g^3}\Big[\boldsymbol{J}^{(7)}_+\wedge\boldsymbol{\eta}^{(7)}_+ + \mathrm{Im}\,\boldsymbol{\Omega}^{(7)}_+\Big]\,. \tag{3.153}$$

The metric in (3.150) and four-form (3.152) for the $\mathrm{SO}(7)_c$ solution coincide with (3.11) of [83] upon using the redefinitions (3.87), and making an appropriate choice for the phase of the complex scalar $\chi_{\text{there}} \equiv -\frac{1}{\sqrt{3}}(\tilde{\zeta}_{\text{here}} + i\zeta_{\text{here}})$, which is unfixed at the critical point. We obtain perfect agreement with [83] upon shifting that phase by π.

Finally, the $\mathrm{SU}(4)_c$–invariant point gives rise to the Pope-Warner solution [97] in eleven dimensions. Using (3.133)-(3.135), this solution can also be written in terms of the homogeneous Sasaki-Einstein structure on S^7 as

$$\begin{aligned} d\hat{s}^2_{11} &= \frac{1}{2^{1/3}\, g^2}\left[\tfrac{3}{8}ds^2(\mathrm{AdS}_4) + ds^2(\mathbb{CP}^3_+) + 2\,\boldsymbol{\eta}^{(7)}_+\otimes\boldsymbol{\eta}^{(7)}_+\right]\,, \\ \hat{F}_{(4)} &= \frac{9}{32g^3}\,\mathrm{vol}(\mathrm{AdS}_4) - \frac{2}{g^3}\Big[\mathrm{Re}\,\boldsymbol{\Omega}^{(7)}_+\wedge\boldsymbol{\eta}^{(7)}_+ - \mathrm{Im}\,\boldsymbol{\Omega}^{(7)}_+\wedge\boldsymbol{\eta}^{(7)}_+\Big]\,, \end{aligned} \tag{3.154}$$

where the internal three-form potential is now

$$A = \tfrac{1}{2}\, g^{-3}\left[\mathrm{Re}\,\boldsymbol{\Omega}^{(7)}_+ + \mathrm{Im}\,\boldsymbol{\Omega}^{(7)}_+\right]\,. \tag{3.155}$$

We again find agreement with [83]: 3.154 coincides with (3.8) of that reference when the identifications (3.87) are taken into account and the phase of $\chi_{\text{there}} \equiv -\frac{1}{\sqrt{3}}(\tilde{\zeta}_{\text{here}} + i\zeta_{\text{here}})$, which is again unfixed at the critical point, is shifted by $\frac{\pi}{4}$.

Part II

Kaluza-Klein Spectroscopy

Chapter 4

Introduction: Weighing ripples of the world

On any compactification of string theory there are several competing scales. This can already be seen in the spectrum of states of the bosonic theory compactified on a circle,

$$M^2 = \frac{k^2}{R^2} + \frac{R^2}{\alpha'^2} w^2 + \frac{2}{\alpha'}(N + \tilde{N} - 2)\,, \tag{4.1}$$

where k labels the KK level, as in the invitation, w the winding number of the string around the compact dimension, and N, $\tilde{N}$ the number of string oscillators applied onto the vacuum. For $\alpha' \ll R^2$, *i.e.* when the characteristic string length is much smaller than the size of the internal dimensions, the states with non-zero winding or $N + \tilde{N} - 2 > 0$ are very heavy compared to the characteristic KK mass scale and thus decouple. This idea persists in the more interesting superstring theories, where $\alpha' \ll R^2$ characterises the regime in which the supergravity approximation is valid, and is therefore the limit that we are interested in.

In this limit, we can understand the dynamics of the *full* higher-dimensional theory from a lower dimensional point of view. This tactic is of special relevance in holography, as KK modes on an AdS_{d+1} solution describe the spectrum of single-trace operators of the dual CFT_d at strong coupling and large N through the dictionary [98–103]

$$\begin{aligned}
\text{gravitons/stress-energy} &\quad:\quad \Delta = \tfrac{1}{2}(d + \sqrt{d^2 + 4L^2M^2})\,,\\
p\text{-forms/conserved currents} &\quad:\quad \Delta = \tfrac{1}{2}(d + \sqrt{(d-2p)^2 + 4L^2M^2})\,,\\
\text{scalars} &\quad:\quad \Delta = \tfrac{1}{2}(d \pm \sqrt{d^2 + 4L^2M^2})\,,\\
\text{grivitini/supercurrents} &\quad:\quad \Delta = |LM| + \tfrac{d}{2}\,,\\
\text{spin-}1/2\text{ fields} &\quad:\quad \Delta = |LM| + \tfrac{d}{2}\,,
\end{aligned} \tag{4.2}$$

where Δ is the conformal dimension of the CFT operators and M the mass of the supergravity KK modes.[1] We normalise with respect to the effective AdS length L so that the combination LM is dimensionless. The two possible signs for the scalars [107] correspond to the two roots of $L^2M^2 = \Delta(\Delta-3)$. If $L^2M^2 > -\frac{d^2}{4}+1$, only the larger root leads to admissible boundary conditions. However, in the range $-\frac{d^2}{4} < L^2M^2 < -\frac{d^2}{4}+1$ both boundary conditions are possible. Notice that for scalar masses below the Breitenlohner-Freedman bound [104]

$$L^2M^2 > -\frac{d^2}{4}\,, \tag{4.3}$$

the dimensions obtained from (4.2) are imaginary, and thus incompatible with the representation theory of the conformal group.

Operators in a CFT must arrange themselves in representations of the conformal group $\mathrm{SO}(2,d)$, and the conformal dimensions Δ are one of the fundamental pieces of information describing these multiplets which are usually very difficult to determine in strongly interacting field theories. This is one of the cases mentioned in the invitation where gravity becomes informative about interesting properties of non-trivial quantum field theories.

On top of a given solution in higher dimensions, the KK masses appearing in (4.2) can be computed by shifting the background field configuration by small deformations $\delta\varphi(x,y)$ and linearising the equations of motion. For generic solutions of the higher-dimensional supergravities, this computation is simply out of reach. The main difficulty is that usually these perturbations couple to the metric and background fluxes in a very intricate way, and it is very subtle to find the precise combination of perturbations which carry definite mass. The situation is different if one focuses again on homogeneous solutions, which can be addressed by group theory methods [108, 109]. This approach led to the early computation of the complete spectra of M-theory on the S^7 [110] and of type IIB supergravity on the S^5 [111] maximally supersymmetric solutions, which can be thought in this context as the coset spaces (1.1). More recent illustrations of this method can be found in [112, 113], but in general unless both G and H in G/H are sufficiently big, this computation becomes quickly unfeasible. For this reason, this approach will not be covered here.

A more general case in which the situation simplifies drastically is when one focuses on the set of massive gravitons only. On maximally symmetric backgrounds, these modes decouple from the fluxes [114] and are only sensitive to the internal metric and warping of the higher-dimensional metric. The precise form of these perturbations reads

$$d\hat{s}^2_D = e^{2A(y)}\Big[(\bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x,y))dx^\mu dx^\nu + d\bar{s}^2_d(y)\Big]\,, \tag{4.4}$$

[1] These formulae were in fact known long before, with Δ then interpreted as the scaling dimension of the AdS supergroup so that modes could be arranged in supermultiplets at supersymmetric solutions. See [24, 104–106].

with the coordinates and indices ranging as in (I.8). We take the external metric $\bar{g}_{\mu\nu}\, dx^\mu dx^\nu \equiv ds^2(\mathrm{AdS}_d)$ to be the unit-radius anti-de Sitter metric. The internal $d\bar{s}^2_d(y)$ denotes a background metric on the n-dimensional space, where the warp factor $e^{2A(y)}$ also takes values. The perturbation, $h_{\mu\nu}$, is assumed to take the factorised form

$$h_{\mu\nu}(x,y) = h^{[tt]}_{\mu\nu}(x)\mathcal{Y}(y)\,, \tag{4.5}$$

with $\mathcal{Y}(y)$ a function on the internal space only, and $h^{[tt]}$ transverse ($\bar{\nabla}^\mu h^{[tt]}_{\mu\nu}= 0$) with respect to the Levi-Civita connection corresponding to $\bar{g}_{\mu\nu}$, traceless ($\bar{g}^{\mu\nu}h^{[tt]}_{\mu\nu} = 0$), and subject to the Fierz-Pauli equation

$$\bar{\Box}h^{[tt]}_{\mu\nu} = (M^2L^2 - 2)h^{[tt]}_{\mu\nu}\,. \tag{4.6}$$

The masses and modes are then related by the differential equation [114, 115]

$$\mathcal{L}\mathcal{Y} = -\frac{e^{-(D-2)A}}{\sqrt{\bar{g}}}\partial_m\big(e^{(D-2)A}\sqrt{\bar{g}}\,\bar{g}^{mn}\partial_n\mathcal{Y}\big) = M^2\mathcal{Y}\,. \tag{4.7}$$

The differential operator appearing here is therefore a warped Laplacian completely insensitive to the background fluxes. These approach has been previously used to compute massive spin-2 spectra in a variety of cases with holographic interest [93, 114, 116–122].

Finally, another interesting case is the class of solutions that uplift from the vacua of a lower-dimensional maximal gauged supergravity. As in the study of consistent truncations of chapter 3, duality plays a prominent rôle in this context. A systematic way of taking advantage of the duality groups that appear in the lower-dimensional supergravities are Exceptional Field Theory (ExFT) [123–130] and Generalised Geometry [45–52, 54, 56], which are reformulations of the full higher-dimensional theories in a duality-covariant fashion that resembles the lower-dimensional counterparts (see [131] for a recent review). In this duality-covariant language, the consistent truncations studied in chapter 3 can be expressed as a factorisation of the dependence on external and internal coordinates by means of a Scherk-Schwarz ansatz [55, 132] generalising (3.109)-(3.110). Given that these reformulations encode the full higher-dimensional dynamics, their usefulness does not restrict to consistent truncations, but can also be effectively applied to the study of the KK tower. This route has received much attention in recent times [133–141] given the ability of ExFT to recover the masses of modes of all spins in terms of mass matrices controlled by very limited data.

The rest of this part proceeds as follows. In chapter 5 we employ (4.7) to analyse the KK spectrum of massive gravitons on the $\mathcal{N} = 2$ solutions of M-theory reviewed in chapter 2 which enjoy a $\mathrm{U}(1)^2$ isometry group. For the solution which cannot be obtained within the truncation to maximal supergravity, this analysis reveals interesting structures in the dual theory such as the space invaders scenario for the completion of supermultipets [24] or the apparent protection of operators not saturating the unitarity bound.

In chapter 6 we analyse how, based on $\mathrm{SL}(8,\mathbb{R})$ covariance, the spectrum of KK gravitons can be computed algebraically in terms of the embedding tensor. To address lower-spin fields, the recent toolkit from ExFT must be employed, and we first review the basic notions of $\mathrm{E}_{7(7)}$ ExFT to subsequently introduce the KK mass matrices on top of solutions that uplift from $D = 4$ maximal gauged supergravity. One of the powerful aspects of this new method is that, contrary to all others, the fully-fledged uplifts in higher-dimensions are not needed to compute the spectrum, and in fact many of the solutions here and in the literature are only known at the $D = 4$ level.

Finally, in chapter 7 we apply these tools to compute the KK spectrum on different (families of) solutions in eleven-dimensional and type II supergravity. Out of the obtained spectra, we discuss the global properties of conformal manifolds of the CFTs dual to a class type IIB S-fold solutions. Finally, we also note some intriguing universality properties in the spectrum of massive spin-2 modes. This universality seems a robust phenomenon in all solutions analysed so far, but both gravity and QFT understanding are still pending.

Chapter 5

KK spectrum on the cubic deformation of ABJM

As emphasised in chapter 4, the Kaluza-Klein modes with spin-2 decouple from most of the details about the higher dimensional configuration and obey the simple modified-laplacian equation (4.7). The simplicity of the latter has allowed its solution on a variety of configurations [93, 114, 116–122, 142–144], and the modes thus obtained enjoy a nice holographic interpretation as the duals of heavier cousins of the stress-energy tensor in the corresponding SCFTs.

This chapter explores the dual of a relevant deformation of the Aharony-Bergman-Jafferis-Maldacena (ABJM) [39] SCFT, which describes the worldvolume of a stack of M2 branes at an orbifold singularity. The reformulation of this theory in $\mathcal{N}=2$ language will be recalled in section 5.1, together with its AdS dual, which is a particularisation of the class of solutions discussed in section 2.2. The spectrum of KK excitations will be analysed in section 5.2, addressing first its algebraic structure based on the R-charges induced by the relevant deformation, and then focusing on the spin-2 sector. Section 5.3 concludes with comments about some intriguing patterns that our results exhibit.

5.1 ABJM, its deformations and their duals

The ABJM theory [39] is the superconformal U(N)×U(N) Chern-Simons-matter gauge theory with $\mathcal{N}=6$ supersymmetry describing the worldvolume of a stack of M2-branes on a $\mathbb{C}^4/\mathbb{Z}_k$ orbifold singularity. In $D=3$ $\mathcal{N}=2$ superfield language, its field content comprises gauge and chiral superfields. The gauge superfields $\mathcal{V}^a_b$ and $\hat{\mathcal{V}}^{\hat{a}}_{\hat{b}}$, with $a, \hat{a}$ labelling the fundamental of each U(N) factor in the gauge group, are governed by a Chern-Simons action at levels k and $-k$ respectively. The matter superfields are $(\mathcal{Z}^{\mathsf{A}})^a_{\hat{a}}$ and $(\mathcal{W}_{\mathsf{A}})^{\hat{a}}_a$, with $\mathsf{A}=1,2$, transforming in the $(\mathbf{N},\bar{\mathbf{N}})$ and $(\bar{\mathbf{N}},\mathbf{N})$ of the gauge group as well as in the fundamental of two global SU(2)'s. Apart from the standard kinetic term for the chiral matter, the theory also contains the quartic superpotential

$$W = \frac{2\pi}{k} \epsilon_{\mathsf{AC}} \epsilon^{\mathsf{BD}} \, \mathrm{tr}(\mathcal{Z}^{\mathsf{A}} \mathcal{W}_{\mathsf{B}} \mathcal{Z}^{\mathsf{C}} \mathcal{W}_{\mathsf{D}}) \,. \tag{5.1}$$

The theory is manifestly invariant under $\mathrm{U}(1)_R \times \mathrm{SU}(2) \times \mathrm{SU}(2)$. However, for $k = 1, 2$, supersymmetry is expected to enhance to $\mathcal{N} = 8$, with the global symmetry correspondingly upgrading to a manifest $\mathrm{U}(1)_R \times \mathrm{SU}(4)$. To make the theory manifestly invariant under this larger group, t'Hooft monopole operators [145] must be used, see *e.g.* [146]. These operators, $(\mathcal{M}^q)^{a_1,\ldots,a_q}_{\hat{a}_1,\ldots,\hat{a}_q}$, carry q units of the baryonic $\mathrm{U}(1)_b$ flux, with $\mathrm{U}(1)_b \subset \mathrm{U(N)} \times \mathrm{U(N)}$ being the linear combination of U(1)'s orthogonal to the one corresponding to the centre of mass of the branes. With the help of these monopole operators, a new set of chiral superfields $\mathcal{Z}^I = (\mathcal{Z}^1, \mathcal{Z}^2, \mathcal{Z}^3, \mathcal{Z}^4)$ in the fundamental of SU(4) and in the $(\mathbf{N}, \bar{\mathbf{N}})$ of the gauge group, can be introduced related to the original ABJM ones as

$$(\mathcal{Z}^3)^a_{\hat{a}} = (\mathcal{W}^1)^{\hat{b}}_b (\mathcal{M}^2)^{ab}_{\hat{a}\hat{b}} \,, \qquad (\mathcal{Z}^4)^a_{\hat{a}} = (\mathcal{W}^2)^{\hat{b}}_b (\mathcal{M}^2)^{ab}_{\hat{a}\hat{b}} \,. \tag{5.2}$$

The SU(4)-invariant [147, 148] superpotential can then be written as

$$W = \frac{4\pi}{k} \, (\mathcal{Z}^1)^a_{\hat{a}} (\mathcal{Z}^2)^b_{\hat{b}} (\mathcal{Z}^3)^c_{\hat{c}} (\mathcal{Z}^4)^d_{\hat{d}} \left[(\mathcal{M}^{-2})^{\hat{a}\hat{c}}_{bc} (\mathcal{M}^{-2})^{\hat{b}\hat{d}}_{ad} - (\mathcal{M}^{-2})^{\hat{a}\hat{d}}_{bd} (\mathcal{M}^{-2})^{\hat{b}\hat{c}}_{ac} \right] . \tag{5.3}$$

Although not manifestly, for $k = 1$ the supersymmetry of the model is increased to $\mathcal{N} = 8$ [148]. The supersymmetry superalgebra is therefore OSp(8|4), and the R-symmetry group contained within the superalgebra is accordingly enhanced to SO(8).

5.1.1 Relevant Superpotential Deformations

For $\mathcal{N} = 8$ ABJM, the superpotential (5.3) can be deformed by introducing an operator polynomial in one of the chirals, say $\mathcal{Z}^4$. Schematically,

$$\Delta W = (\mathcal{Z}^4)^p \,. \tag{5.4}$$

This deformation preserves $\mathrm{SU}(3) \times \mathrm{U}(1)_p \subset \mathrm{SO}(8)$, with $\mathrm{SU}(3) \subset \mathrm{SU}(4)$ the flavour group that rotates the remaining $\mathcal{Z}^A$, $A = 1, 2, 3$, and $\mathrm{U}(1)_p$ the R-symmetry associated to the manifestly preserved $\mathcal{N} = 2$ supersymmetry.

For $p = 2$ and $p = 3$, the operator in (5.4) induces a relevant deformation. In the $p = 2$ case, it corresponds to the mass deformation introduced in [147]. Here, we will be more interested in the case in which the deformation is instead cubic in $\mathcal{Z}^4$. In either case, the actual deformations are

$$p = 2 \;: \qquad \Delta W = \alpha (\mathcal{Z}^4)^a_{\hat{a}} (\mathcal{Z}^4)^b_{\hat{b}} (\mathcal{M}^{-2})^{\hat{a}\hat{b}}_{ab} \,, \tag{5.5}$$

and

$$p = 3 \ : \qquad \Delta W = \alpha (\mathcal{Z}^4)^a_{\hat{a}} (\mathcal{Z}^4)^b_{\hat{b}} (\mathcal{Z}^4)^c_{\hat{c}} (\mathcal{M}^{-3})^{\hat{a}\hat{b}\hat{c}}_{abc} \,, \tag{5.6}$$

where α is a coupling constant, and the gauge indices have been contracted using a monopole operator, making equation (5.4) more precise.

The IR R-charges of the chirals under the R-symmetry group $\mathrm{U}(1)_p$ for each case can be computed by requiring that the total superpotential, (5.3) plus (5.6), has R-charge two and that the free energy be extremal [149]. Assuming that the monopole operators are R-neutral, the result for these $\mathrm{U}(1)_2$ and $\mathrm{U}(1)_3$ IR R-charges are, correspondingly [82, 150],

$$p = 2 \ : \qquad R_1 \equiv R(\mathcal{Z}^A) = \tfrac{1}{3} \,, \ A = 1,2,3 \,, \qquad R_2 \equiv R(\mathcal{Z}^4) = 1 \,, \tag{5.7}$$

and

$$p = 3 \ : \qquad R_1 \equiv R(\mathcal{Z}^A) = \tfrac{4}{9} \,, \ A = 1,2,3 \,, \qquad R_2 \equiv R(\mathcal{Z}^4) = \tfrac{2}{3} \,. \tag{5.8}$$

The SU(3) flavour group of both the $p = 2$ and $p = 3$ IR phases is the same subgroup of the SO(8) R-symmetry of the ultraviolet (UV) $\mathcal{N} = 8$ ABJM theory: it is, in fact, the unique $\mathrm{SU}(3) \subset \mathrm{SO}(8)$. However, (5.7) and (5.8) show that the $\mathrm{U}(1)_p$ R-symmetry groups for $p = 2$ and $p = 3$ are different U(1) subgroups of SO(8): they are different U(1) combinations of the $\mathrm{U}(1) \times \mathrm{U}(1)$ that commutes with SU(3) inside SO(8), as explained in appendix E.2. The full (super)symmetry of these IR SCFTs is thus $\mathrm{OSp}(2|4)_p \times \mathrm{SU}(3)$, with $\mathrm{U}(1)_p \subset \mathrm{OSp}(2|4)_p$ having a subscript $p = 2$ or $p = 3$ attached to signify that they are different (super)groups.

It is also useful to look at the deformation at the level of the Lagrangian. In terms of field components, the chiral superfields can be expanded in the usual way as [151]

$$\mathcal{Z}^I = Z^I + \sqrt{2}\theta\chi^I + \theta^2\zeta^I + i(\theta\gamma^\mu\bar{\theta})\partial_\mu Z^I - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\chi^I\gamma^\mu\bar{\theta} - \frac{1}{4}\theta^2\bar{\theta}^2\Box Z^I \,, \tag{5.9}$$

with Z^I and χ^I respectively the dynamical scalars and fermions, and ζ^I non-propagating auxiliary fields. The potential that derives from a superpotential then reads

$$V = -\int d^2\theta \ \mathcal{W}(\mathcal{Z}) + h.c. = -\frac{\partial\mathcal{W}(Z)}{\partial Z^I}\zeta^I + \frac{1}{2}\frac{\partial^2\mathcal{W}(Z)}{\partial Z^I\partial Z^J}\chi^I\chi^J + h.c. \tag{5.10}$$

The auxiliary fields can be integrated out using their equations of motion, which set them to $\zeta^I = -\partial\bar{\mathcal{W}}/\partial\bar{Z}^I$, with the potential becoming

$$V = \frac{\partial\mathcal{W}(Z)}{\partial Z^I}\frac{\partial\bar{\mathcal{W}}(\bar{Z})}{\partial\bar{Z}^I} + \frac{1}{2}\frac{\partial^2\mathcal{W}(Z)}{\partial Z^I\partial Z^J}\chi^I\chi^J + \frac{1}{2}\frac{\partial^2\bar{\mathcal{W}}(\bar{Z})}{\partial\bar{Z}^I\partial\bar{Z}^J}\bar{\chi}^I\bar{\chi}^J \,. \tag{5.11}$$

Therefore, the effect of the deformation (5.6) on top of (5.3) is to augment the ABJM Lagrangian with the following schematic interaction terms:

$$p = 2 \ : \qquad \Delta\mathcal{L} = \tfrac{1}{2}|\alpha|^2 \, Z^4 \, \bar{Z}_4 + \tfrac{1}{2}\alpha \, \chi^4\chi^4 + h.c. \,, \tag{5.12}$$

and

$$p = 3 \; : \qquad \Delta\mathcal{L} = \tfrac{1}{2}|\alpha|^2\,(Z^4)^2(\bar{Z}_4)^2 + \tfrac{1}{2}\alpha\,\chi^4\chi^4 Z^4 + h.c.\,, \tag{5.13}$$

where the contractions occur with monopole operators, which we have suppressed to avoid cluttering.

In real notation, Z^I and χ^I respectively transform in the $\mathbf{8}_v$ and $\mathbf{8}_c$ of the SO(8) R-symmetry group of $\mathcal{N} = 8$ ABJM. Accordingly, the deformations in (5.12) correspond, up to terms in the ABJM analogue of the $\mathcal{N} = 4$ super-Yang-Mills Konishi operator, to mass terms for the scalars and fermions. These mass terms have canonical dimension, $\Delta = 1$ and $\Delta = 2$, and branch from the $\mathbf{35}_v$ and $\mathbf{35}_c$ of SO(8). On the other hand, the operators in (5.13) respectively branch from the $\mathbf{294}_v$ and $\mathbf{224}_{cv}$ representations of SO(8). These operators have relevant dimension $\Delta = 2$ and $\Delta = \frac{5}{2}$, and thus do indeed generate RG flow as expected.

5.1.2 Gravity duals

The relevant operators in (5.12) are dual to scalar and pseudoscalar KK modes that branch from the $\mathbf{35}_v$ and $\mathbf{35}_c$ representations of SO(8), respectively. Both these modes arise at KK level $n = 0$ in the spectrum of the $\mathcal{N} = 8$ $\mathrm{AdS}_4 \times S^7$ Freund-Rubin (FR) solution of $D = 11$ supergravity, dual to $\mathcal{N} = 8$ ABJM: see [24] for a review and table 2 of [82] for a convenient summary. As is well-known, a consistent truncation of $D = 11$ supergravity on S^7 exists [25] that retains all $n = 0$ KK modes and reconstructs their full non-linear interactions. The resulting $D = 4$ supergravity is $\mathcal{N} = 8$ and has gauge group SO(8) [73], and the RG flow triggered by (5.12) can be described at the level of this truncation.

In contrast, the operators in (5.13) that trigger the $p = 3$ RG flow are dual to scalar and pseudoscalar KK modes which arise at KK levels $n = 2$ and $n = 1$. There is no known consistent truncation, maximally supersymmetric or otherwise, that retains these modes,[1] and this prevents a purely four-dimensional description of the flow. For this reason, unlike for $p = 2$, the geometry dual to the $p = 3$ IR SCFT must be engineered directly in $D = 11$. The general class of M-theory solutions involving $\mathcal{N} = 2$ supersymmetry and an AdS_4 factor was analysed in [17] and reviewed in section 2.2. What we are referring to here as the $p = 3$ Gabella-Martelli-Passias-Sparks (GMPS) geometry is a particular solution to their formalism which the authors of [17] discuss in detail. The $p = 2$ CPW geometry [31] can also be recovered [17] as a different solution in the same class. The local form of the family of

[1] Some consistent truncations are known [84, 152, 153] that retain modes up the KK towers, but not the required ones. For example, the $\mathcal{N} = 2$ truncation of [84] keeps $\mathrm{SU}(4)_s$-invariant scalar and pseudoscalar modes from KK level $n = 2$, dual to irrelevant operators. A different consistent truncation retaining massive modes has recently been constructed in [G].

geometries that encompasses both specific solutions takes the form (2.28) in section 2.2. In particular, the seven-dimensional internal metric takes on the local form in (2.28)

$$\begin{aligned} ds_7^2 = & \frac{f \cdot \alpha}{4\sqrt{1+(1+r^2)\alpha^2}}\, ds^2(\mathbb{CP}_2) + \frac{\alpha^2}{16}\Big[dr^2 + \frac{r^2 f^2}{1+r^2}(d\tilde{\tau}+\sigma)^2 \\ & + \frac{1+r^2}{1+(1+r^2)\alpha^2}\big(d\tilde{\psi} + \frac{f}{1+r^2}(d\tilde{\tau}+\sigma)\big)^2\Big]\,, \end{aligned} \tag{5.14}$$

in terms of coordinates r, $\tilde{\psi}$, $\tilde{\tau}$. The Kähler-Einstein base in (2.28) has been chosen as the Fubini-Study metric on the complex projective plane, normalised so that the Ricci tensor equals six times the metric, and σ is a local one-form potential for the Kähler form J on $\mathbb{CP}_2$, normalised as $d\sigma = 2\,J$. Finally, α and f are functions of the coordinate r only, the former simply a rewrite of the warp factor:

$$e^{6\Delta} \equiv \big(\tfrac{m}{6}\big)^2(1+r^2+\alpha^{-2})\,. \tag{5.15}$$

These functions are subject to the following system of non-linear differential equations:

$$\frac{f'}{f} = -\frac{1}{2}r\alpha^2\,, \qquad\qquad \frac{(r\alpha' - r^2\alpha^3)f}{\sqrt{1+(1+r^2)\alpha^2}} = -3\,, \tag{5.16}$$

where a prime denotes derivative with respect to r. The vectors $\partial_{\tilde{\psi}}$ and $\partial_{\tilde{\tau}}$ are Killing, and the isometry of the metric (5.14) is manifestly $\mathrm{SU}(3)\times\mathrm{U}(1)\times\mathrm{U}(1)$. The former vector defines the local $\mathcal{N}=2$ Reeb direction corresponding to the $\mathrm{U}(1)_p$ R-symmetry, and the latter is broken by the internal four-form $F_{(4)}$ in (2.28). The internal symmetry of the full $D=11$ configuration (2.10) is thus $\mathrm{SU}(3)\times\mathrm{U}(1)_p$.

Each solution f and α to the system of ODEs (5.16) gives rise to an $\mathcal{N}=2$ solution to the equations of motion of $D=11$ supergravity of the form (2.28) and (2.29) with (5.15). The two solutions, GMPS and CPW, of interest here correspond to specific choices of f and α subject to the boundary conditions

$$\begin{aligned} & f \xrightarrow[r\to 0]{} \frac{3p}{p-1}\,, \qquad \alpha \xrightarrow[r\to 0]{} w r^{-1+1/p}\,, \qquad \text{with } w>0\,, \\ & f \xrightarrow[r\to r_0]{} \frac{2\sqrt{1+r_0^2}}{r_0}\,(r_0 - r)\,, \qquad \alpha \xrightarrow[r\to r_0]{} \sqrt{\frac{2}{r_0(r_0-r)}}\,, \end{aligned} \tag{5.17}$$

for $p=2$ or $p=3$. For these choices, the local geometry (5.14) extends globally over S^7. The coordinate r is globally defined and ranges in $0 \le r \le r_0$ for a solution-dependent constant r_0. The coordinates $\tilde{\psi}$ and $\tilde{\tau}$ are only

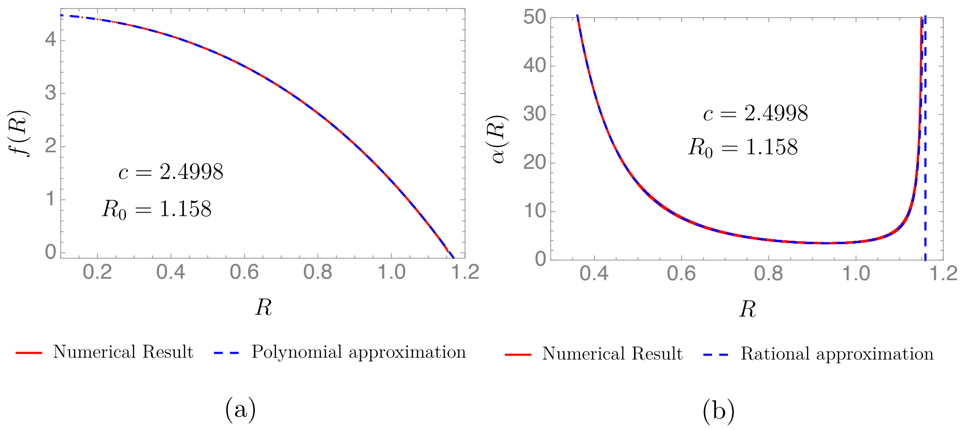


Figure 5.1: Comparison between the numerically obtained functions f and α for the GMPS solution and their respective polynomial, (5.23), and rational, (5.24), approximations.

defined locally, but can be related to globally defined angles ψ and τ of period 2π via the transformation[2]

$$\psi = \tfrac{1}{p}\,\tilde{\psi}\,, \qquad\qquad \tau = \tilde{\tau} + \tfrac{1}{3}\big(1 - \tfrac{1}{p}\big)\tilde{\psi} \tag{5.18}$$

for $p = 2$ or $p = 3$. The global coordinates ψ and τ are the angles on the Hopf fibres of S^7 and on the S^5 inside S^7. In terms of the globally-defined angles, the $\mathcal{N} = 2$ Reeb vector is

$$R = \tfrac{4(p-1)}{3p}\,\partial_\tau + \tfrac{4}{p}\,\partial_\psi \equiv 4\,\partial_{\tilde{\psi}}\,. \tag{5.19}$$

The analytical $p = 2$ CPW solution [31] is recovered for [17]

$$f = 6\Big(1 - \frac{r}{r_0}\Big)\,, \qquad \alpha = \sqrt{\frac{2}{r(r_0 - r)}}\,, \qquad r_0 = 2\sqrt{2}\,. \tag{5.20}$$

The $p = 3$ GMPS solution is only known numerically [17]. We re-derive it here following [17] in order to calibrate our numerics. The equations (5.16) can be combined into a single non-linear ODE for f,

$$\frac{1}{9} f\left(R\ddot{f} - 5\dot{f}\right) + \frac{1}{3} R\dot{f}^2 = \sqrt{-\dot{f}\left(6R^5 f - 4\dot{f}\,(1 + R^6)\right)}\,, \tag{5.21}$$

in terms of a convenient new independent variable

$$R = r^{1/3}\,. \tag{5.22}$$

[2]In the notation of [17], $\tilde{\psi}_{\text{here}} = \psi_{\text{there}}$, $\tilde{\tau}_{\text{here}} = \tau_{\text{there}}$ and $\psi_{\text{here}} = {\varphi_0}_{\text{there}}$ and $\tau_{\text{here}} = \varphi_{\text{there}}$ up to orientation.

In (5.21), a dot denotes derivative with respect to R. An approximate solution to equation (5.21) can be found by expanding in Taylor series about $R=0$:

$$f(R)=\frac{9}{2}-cR^2-\frac{c^2}{9}R^4+\frac{\left(2187-128c^3\right)}{3888}R^6+\frac{\left(19683c-1264c^4\right)}{104976}R^8+\mathcal{O}(R^{10})\,, \tag{5.23}$$

with c an integration constant. Using (5.23), the function α derives from (5.16) as

$$\begin{aligned}\alpha^2(R)\approx\tfrac{4}{3R^4}&\left[177147R^4+26244c\left(3R^6-4\right)-23328c^2R^2-10368c^3R^4-5056c^4R^6\right]\\&\left[1264c^4R^8+3456c^3R^6+11664c^2R^4-6561c\left(3R^6-16\right)R^2-59049\left(R^6+8\right)\right]^{-1}.\end{aligned} \tag{5.24}$$

The approximate analytical solutions (5.23), (5.24) can now be used to kick off a numerical integration of the system of ODEs (5.16). Imposing the right asymptotic behaviour near $R=R_0$, given by (5.17) with $p=3$ through (5.22), the integration constant c and the upper limit r_0 for the variable r become fixed to

$$c\approx 2.4998\,,\qquad R_0\approx 1.1585\quad\Longleftrightarrow\quad r_0\approx 1.555\,. \tag{5.25}$$

Interestingly, the approximate solutions (5.23), (5.24) found close to $R=0$ fit the numerically integrated functions very well across the entire range $0\leq R\leq R_0$ for the value of c in (5.25): see figure 5.1.

5.2 Massive KK modes on the GMPS solution

After reviewing the principal aspects of the configuration to be perturbed, let us analyse its ripples. This problems is technically very complicated, and we will content ourselves with drawing some conclusions from group theory about its structure, as similarly done in [82] for the CPW solution [31] and focusing on the KK graviton towers. The main observation is that the KK spectrum displays a *space invaders scenario* similar to that described in [24] for the KK spectrum on the squashed S^7 solution [154].

5.2.1 Algebraic Structure

As remarked in the previous section, the full (super)symmetry group of both the CPW and GMPS solutions is $\mathrm{OSp}(2|4)\times\mathrm{SU}(3)$, and the KK spectrum must accordingly organise itself in representations of this (super)group. See appendix A of [82] for a convenient summary of $\mathrm{OSp}(2|4)$ multiplets. On the other hand, these configurations are connected through an RG flow to the $\mathcal{N}=8$ $\mathrm{AdS}_4\times S^7$ FR solution, whose spectrum at KK level n is organised in terms of the SO(8) representations

$$\text{graviton}: G_n\equiv[n,0,0,0]\,,$$

$$\begin{aligned}
\text{gravitini} : \mathcal{G}_n &\equiv [n,0,0,1] \oplus [n-1,0,1,0]\,, \\
\text{vectors} : V_n &\equiv [n,1,0,0] \oplus [n-1,0,1,1] \oplus [n-2,1,0,0]\,, \\
\text{fermions} : \mathcal{F}_n &\equiv [n+1,0,1,0] \oplus [n-1,1,1,0] \\
&\quad \oplus [n-2,1,0,1] \oplus [n-2,0,0,1]\,, \\
\text{scalars} : S_n^+ &\equiv [n+2,0,0,0] \oplus [n-2,2,0,0] \oplus [n-2,0,0,0]\,, \\
\text{pseudoscalars} : S_n^- &\equiv [n,0,2,0] \oplus [n-2,0,0,2]\,,
\end{aligned} \tag{5.26}$$

where only representations with non-negative Dynkin labels contribute.

This connection implies that the $\mathrm{SU}(3)\times\mathrm{U}(1)_p$ representations in the deformed solution must be related to the SO(8) representations in the round one by branching them under

$$\mathrm{SO}(8) \supset \mathrm{SU}(3) \times \mathrm{U}(1)_p\,. \tag{5.27}$$

For the gravitons, the result of this branching is

$$[n,0,0,0] \xrightarrow{\mathrm{SU}(3)\times\mathrm{U}(1)_p} \bigoplus_{\ell=0}^{n} \bigoplus_{t=0}^{n-\ell} \bigoplus_{p=0}^{\ell} [p,\ \ell-p]_{-R_1(\ell-2p)+R_2(n-\ell-2t)}\,, \tag{5.28}$$

where R_1 and R_2 are the IR R-charges (5.8) (or (5.7) for CPW) of the coordinates transverse to the M2-branes. The branchings of the lower spins in (5.26) are summarised in appendix E.2.

The next step is to allocate fields of different spin but the same SU(3) charges into OSp(2|4) multiplets. For CPW [31] this exercise was carried out in [82], and crucially relies on the assignment of R-charges (5.7). Under the assumption that the allocation into supermultiplets should take place KK level by KK level, group theory alone was found to narrow down the possible spectrum of (short) multiplets to two possibilities dubbed scenarios I and II in [82]. Both scenarios differ by the embedding of the $\mathrm{U}(1)_2$ IR isometry into SO(8), and are related by a triality rotation [143]. The actual calculation of the KK graviton spectrum [93] confirmed scenario I as the correct choice.

Going through the same exercise for the GMPS solution [17] we find that we need to relax the assumption that the allocation of $\mathrm{SU}(3) \times \mathrm{U}(1)_3$ states into OSp(2|4) multiplets should proceed KK level by KK level. Otherwise, the problem has no solution starting from the R-charge assignment (5.8), and that is not an option. Instead, states entering the same OSp(2|4) multiplet must be retrieved from different SO(8) KK levels n. For example, states from higher KK levels are needed to complete Short Gravitino multiplets in the $[1,0]_{\frac{1}{9}}$ and $[1,0]_{-\frac{1}{9}}$ and a Long Vector in the $[0,0]_0$, whose states come mostly from $n=0$. Table 5.1 shows a possible distribution of the $n=0$ states into OSp(2|4) multiplets that assumes that all needed space invaders descend from KK level $n=1$. Group theory is not enough to determine whether this or another invasion pattern is the correct one, though.

Spin	SO(8)	$\mathrm{SU}(3) \times \mathrm{U}(1)_3$												
2	$\mathbf{1}$	$\mathbf{1}_0$												
$\frac{3}{2}$	$\mathbf{8}_s$	$\mathbf{1}_{+1}$ $\mathbf{1}_{-1}$	$\mathbf{3}_{\frac{1}{9}}$	$\bar{\mathbf{3}}_{-\frac{1}{9}}$										
1	$\mathbf{28}$	$\mathbf{1}_0$	$\mathbf{3}_{-\frac{8}{9}}$ $\mathbf{3}_{\frac{10}{9}}$	$\bar{\mathbf{3}}_{\frac{8}{9}}$ $\bar{\mathbf{3}}_{-\frac{10}{9}}$	$\mathbf{8}_0$	$\mathbf{3}_{-\frac{2}{9}}$	$\bar{\mathbf{3}}_{\frac{2}{9}}$	$\mathbf{1}_0$						
			$\mathbf{3}_{-\frac{8}{9}}$	$\bar{\mathbf{3}}_{\frac{8}{9}}$										
$\frac{1}{2}$	$\mathbf{56}_s$		$\mathbf{3}_{\frac{1}{9}}$ $\mathbf{3}_{\frac{1}{9}}$	$\bar{\mathbf{3}}_{-\frac{1}{9}}$ $\bar{\mathbf{3}}_{-\frac{1}{9}}$	$\mathbf{8}_{+1}$ $\mathbf{8}_{-1}$	$\mathbf{3}_{\frac{7}{9}}$ $\mathbf{3}_{-\frac{11}{9}}$	$\bar{\mathbf{3}}_{-\frac{7}{9}}$ $\bar{\mathbf{3}}_{\frac{11}{9}}$	$\mathbf{1}_{+1}$ $\mathbf{1}_{-1}$	$\mathbf{6}_{-\frac{1}{9}}$	$\bar{\mathbf{6}}_{\frac{1}{9}}$	$\mathbf{1}_{\frac{1}{3}}$	$\mathbf{1}_{-\frac{1}{3}}$	
			$\mathbf{3}_{-\frac{17}{9}}$	$\bar{\mathbf{3}}_{\frac{17}{9}}$		$\mathbf{3}_{\frac{7}{9}}$	$\bar{\mathbf{3}}_{-\frac{7}{9}}$	$\mathbf{1}_{+1}$ $\mathbf{1}_{-1}$					$\mathbf{3}_{\frac{1}{9}}$ $\bar{\mathbf{3}}_{-\frac{1}{9}}$
0	$\mathbf{35}_v$				$\mathbf{8}_0$	$\mathbf{3}_{-\frac{2}{9}}$	$\bar{\mathbf{3}}_{\frac{2}{9}}$	$\mathbf{1}_0$	$\mathbf{6}_{\frac{8}{9}}$	$\bar{\mathbf{6}}_{-\frac{8}{9}}$	$\mathbf{1}_{\frac{4}{3}}$	$\mathbf{1}_{-\frac{4}{3}}$	$\mathbf{3}_{\frac{10}{9}}$ $\bar{\mathbf{3}}_{-\frac{10}{9}}$
	$\mathbf{35}_c$		$\mathbf{3}_{-\frac{8}{9}}$	$\bar{\mathbf{3}}_{\frac{8}{9}}$	$\mathbf{8}_0$	$\mathbf{3}_{-\frac{2}{9}}$	$\bar{\mathbf{3}}_{\frac{2}{9}}$	$\mathbf{1}_0$	$\mathbf{6}_{-\frac{10}{9}}$	$\bar{\mathbf{6}}_{\frac{10}{9}}$	$\mathbf{1}_{-\frac{2}{3}}$	$\mathbf{1}_{\frac{2}{3}}$	
						$\mathbf{3}_{\frac{16}{9}}$	$\bar{\mathbf{3}}_{-\frac{16}{9}}$	$\mathbf{1}_0$ $\mathbf{1}_{+2}$ $\mathbf{1}_{-2}$					$\mathbf{3}_{-\frac{8}{9}}, \bar{\mathbf{3}}_{\frac{8}{9}}$ $\mathbf{3}_{-\frac{8}{9}}, \bar{\mathbf{3}}_{\frac{8}{9}}$ $\mathbf{3}_{-\frac{2}{9}}, \bar{\mathbf{3}}_{\frac{2}{9}}$ $\mathbf{1}_0$
		Massless graviton	Short gravitino	Short gravitino	Massless vector	Short vector	Short vector	Long vector	Massive hyper	Massive hyper	Massive hyper	Massive hyper	Eaten modes

Table 5.1: Possible branching of the $\mathcal{N} = 8$ massless graviton multiplet into $\mathrm{Osp}(2|4) \times \mathrm{SU}(3)$ representations. The symbol denotes space invader states coming from higher KK levels.

5.2.2 Spin-2 sector

The spectrum of massive KK gravitons about the CPW solution [31] was determined analytically in [93]. Here, we pose the analogue boundary value problem for the GMPS solution [17] and turn to solve it numerically. The numerical integration can be systematised using the group theory of section 5.2.1, and the *complete* graviton spectrum can be found. Here we present the numerical outcomes up to KK level $n = 3$, out of which analytic results on the short graviton spectrum and on a specific type of long OSp(2|4) supermultiplets can be found. This results will be used in section 5.3 to show that the GMPS metric does not descend from the flat Euclidean metric on $\mathbb{R}^8$.

Boundary value problem

As reviewed in chapter 4, we consider the line element

$$d\hat{s}_{11}^2 = e^{2A}\Big[\big(\bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x,y)\big)dx^\mu dx^\nu + d\bar{s}_7^2(y)\Big]\,, \tag{5.29}$$

where we have rescaled for convenience the warp factor and internal metric in (5.14) as

$$e^{2A} = \tfrac{1}{4}\, e^{2\Delta}\,, \qquad d\bar{s}_7^2 = 4\, ds_7^2\,, \tag{5.30}$$

with respect to (5.15) and (5.14). We fix the functions f and α appearing in the internal squashed and stretched metric on S^7 and warp factor to those corresponding to the $p = 3$ GMPS solution [17] as reviewed in section 5.1.

Accordingly, using (4.7), the KK graviton mass operator associated to (5.29) reads [114]

$$\begin{aligned}
\mathcal{L} = & -\frac{4}{r\alpha^2 f^3}\partial_r\Big[r f^3 \partial_r\Big] - \frac{\sqrt{1+(1+r^2)\alpha^2}}{f\cdot\alpha}\Box_{S^5} \\
& -\frac{4}{9}\Big(1+\frac{1}{r^2\alpha^2}\Big)\partial_\psi^2 - \frac{8}{3}\Big[\frac{2}{9}\Big(1+\frac{1}{r^2\alpha^2}\Big) - \frac{1}{r^2\alpha^2 f}\Big]\partial_\psi\partial_\tau \\
& -\Big[-\frac{\sqrt{1+(1+r^2)\alpha^2}}{f\cdot\alpha} + \frac{16}{81}\Big(1+\frac{1}{r^2\alpha^2}\Big) + \frac{4(1+r^2)}{r^2\alpha^2 f^2} - \frac{16}{9r^2\alpha^2 f}\Big]\partial_\tau^2\,.
\end{aligned} \tag{5.31}$$

in terms of the global coordinates (5.18) for $p = 3$. Here, $\Box_{S^5}$ is the Laplacian on the round, unit radius S^5. With a graviton perturbation of the form (4.5) subject to the field equation (4.6), the linearised Einstein equation satisfied by (5.29) becomes an eigenvalue problem for the mass operator (5.31):

$$\mathcal{L}\,\mathcal{Y} = L^2 M^2\,\mathcal{Y}\,. \tag{5.32}$$

with the combination L^2M^2 being dimensionless.

At this point, we can exploit the $\mathrm{SU}(3)\times\mathrm{U}(1)_\tau\times\mathrm{U}(1)_\psi$ isometry of the metric (5.14) and expand the $\mathcal{L}$–eigenfunction $\mathcal{Y}$ as

$$\mathcal{Y} = \sum_{\ell,m,j} \xi_{\ell,m,j}(r)\, Y_{\ell,m}(z,\bar{z},\tau)\, e^{ij\psi}\,. \tag{5.33}$$

Here, $\xi_{\ell,m,j}(r)$ is a function of r only and $Y_{\ell,m}(z,\bar{z},\tau)$ are the S^5 spherical harmonics (with definite $\mathrm{U}(1)_\tau$ charge)

$$\Box_{S^5} Y_{\ell,m} = -\ell(\ell+4)\,Y_{\ell,m}\,, \qquad \partial_\tau Y_{\ell,m} = im\,Y_{\ell,m}\,. \tag{5.34}$$

The quantum numbers in (5.33) and (5.34) range as

$$\ell = 0,1,2,\dots\,, \qquad m = -\ell, -\ell+2, \dots, \ell-2, \ell\,, \qquad j = 0,\,\pm1,\,\pm2,\,\dots \tag{5.35}$$

(note that i in (5.33) and (5.34) is the imaginary unit). This quantum numbers follow the structure set by (5.28) under the usual relation $m = 2p-\ell$ and a proper identification of the U(1) charges that will be given in (5.49) below.

$k\backslash j$	0	1	2	3
0	0.00	2.44	5.78	9.99
1	5.92	10.00	14.86	20.54
2	14.94	20.57	26.94	34.11
3	27.03	34.13	42.05	50.71

Table 5.2: KK graviton masses $L^2 M^2_{k,j,\ell=0,m=0}$ on the GMPS background for a few values of the quantum numbers k and j, at $\ell = m = 0$, as obtained from figure 5.2. The KK tower with $k = 0$ corresponds to short gravitons (*c.f.* (5.51) and below).

The partial differential equation (5.32) thus reduces to the following Sturm-Liouville problem in $\xi_{\ell,m,j}(r)$ where, to avoid cluttering, we omit the quantum number subscripts on ξ:

$$\begin{aligned} L^2M^2\xi = &-\frac{4}{r\alpha^2 f^3}\frac{d}{dr}\Big[rf^3\frac{d\xi}{dr}\Big] + \frac{\sqrt{1+(1+r^2)\alpha^2}}{f\cdot\alpha}\ell(\ell+4)\xi \\ &+\frac{4}{9}\Big(1+\frac{1}{r^2\alpha^2}\Big)j^2\xi + \frac{8}{3}\Big[\frac{2}{9}\Big(1+\frac{1}{r^2\alpha^2}\Big) - \frac{1}{r^2\alpha^2 f}\Big]jm\xi \\ &-\Big[\frac{\sqrt{1+(1+r^2)\alpha^2}}{f\cdot\alpha} - \frac{16}{81}\Big(1+\frac{1}{r^2\alpha^2}\Big) - \frac{4(1+r^2)}{r^2\alpha^2 f^2} + \frac{16}{9r^2\alpha^2 f}\Big]m^2\xi\,. \end{aligned} \tag{5.36}$$

The normalisable spin-2 modes correspond to the solutions of this ODE such that [114, 116]

$$\int_0^{r_0} dr\, r\alpha^2 f^3 |\xi|^2 < \infty\,, \tag{5.37}$$

supplemented with the fall-offs (5.17) with $p = 3$ for the metric functions.

Numerics

Solving the ODE (5.36) on the GMPS background entails a non-trivial numerical integration over a numerical background. We have nevertheless managed to address the *complete* graviton spectrum. In this section, we set up our numerics.

We start by conveniently rewriting the ODE (5.36) in terms of the variable R defined in (5.22), whereby it becomes

$$\ddot{\xi} - \Big(\frac{9}{2}R^5\alpha^2 - R^{-1}\Big)\dot{\xi} + \Big(\frac{9}{4}L^2M^2R^4\alpha^2 + Aj^2 + B\ell(\ell+4) + Cm^2 + Djm\Big)\xi = 0\,. \tag{5.38}$$

Here we have defined

$$A \equiv -\big(R^4\alpha^2 + R^{-2}\big)\,,$$

$$B \equiv -\frac{9}{4}R^4\alpha f^{-1}\sqrt{1+(1+R^6)\alpha^2}\,,$$

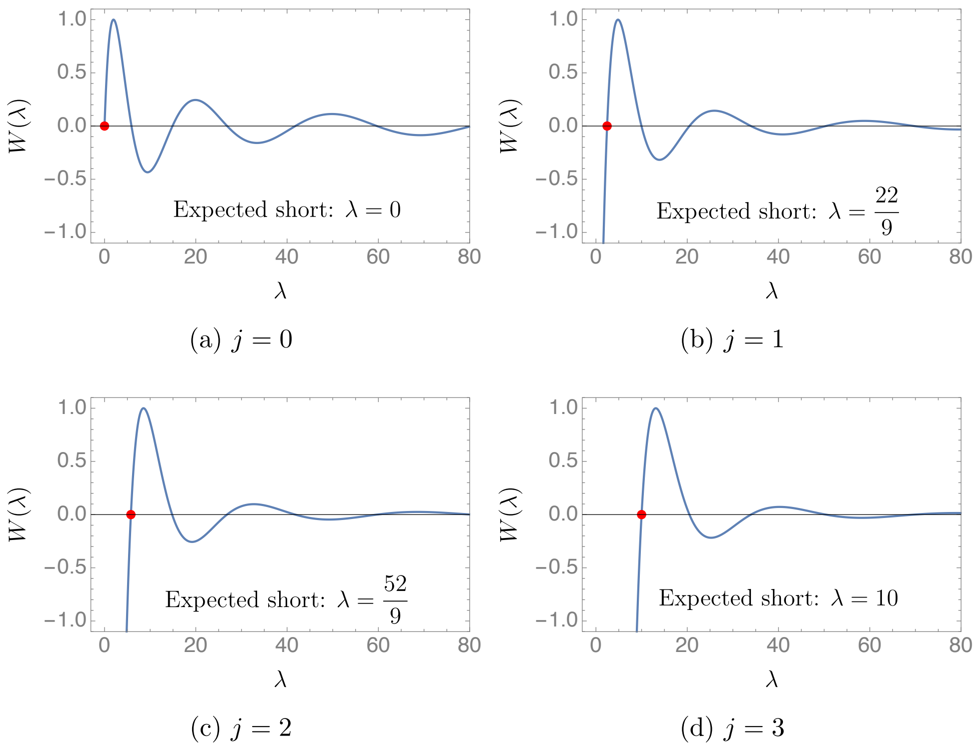


Figure 5.2: Wronskian W in (5.48) of the numerical functions $\xi^L_\lambda(R)$ and $\xi^R_\lambda(R)$ at $R = R_0/2$ for $\ell = m = 0$ and various values of j. The masses in table 5.2 correspond to the zeroes of W. The masses lying in short multiplets are marked with red dots (*c.f.* (5.51) and below).

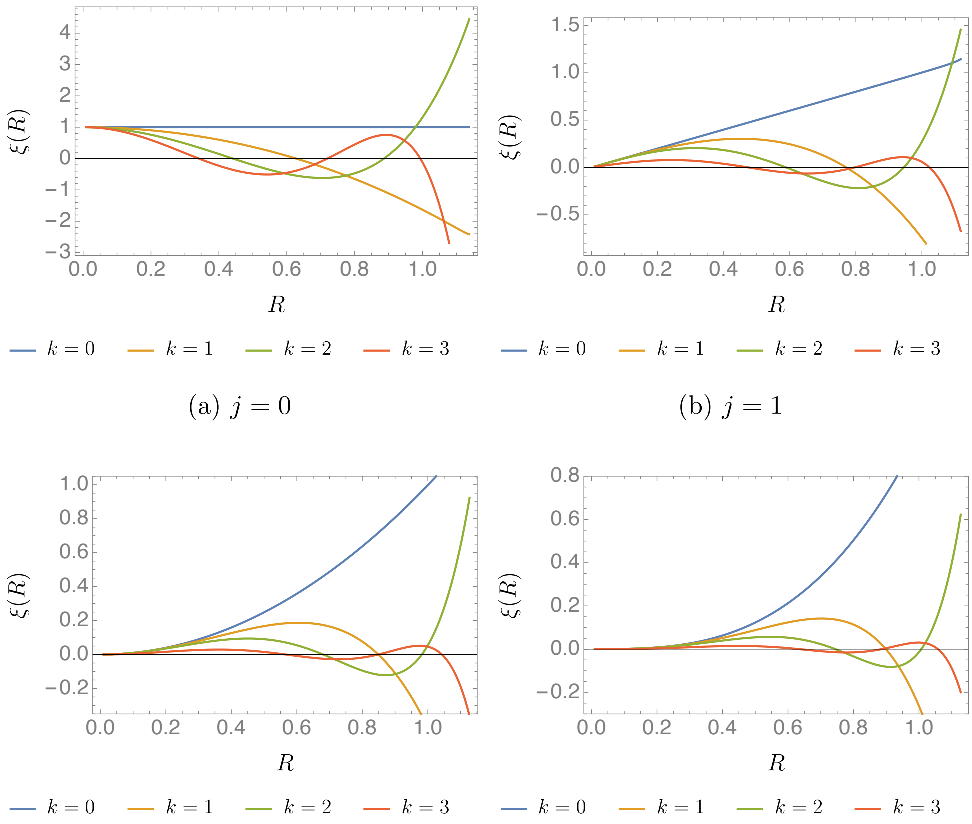
$$
\begin{aligned}
C &\equiv \frac{9}{4}R^4\alpha f^{-1}\sqrt{1+(1+R^6)\alpha^2} \\
&\qquad - \frac{4}{9}\big(R^4\alpha^2 + R^{-2}\big) - 9R^{-2}(1+R^6)f^{-2} + 4R^{-2}f^{-1}\,, \\
D &\equiv -\frac{4}{3}\big(R^4\alpha^2 + R^{-2}\big) + 6R^{-2}f^{-1}\,.
\end{aligned}
\tag{5.39}
$$

Next, we obtain asymptotic forms of the normalisable solution to (5.38) close to each endpoint, $R = 0$ and $R = R_0$, of the domain of R. Near $R = 0$, the asymptotic form of (5.38) implied by (5.23) and (5.24) depends on whether the quantum number j is zero or not. For $j \neq 0$, the ODE (5.38) close to $R = 0$ takes on the form

$$
\ddot{\xi} + \frac{1}{R}\dot{\xi} - \frac{j^2}{R^2}\xi = 0\,, \tag{5.40}
$$

where the term in the eigenvalue L^2M^2 drops out as it is subleading. The ODE (5.40) has solutions

$$
\xi = aR^j + bR^{-j}\,, \tag{5.41}
$$



(a) $j=0$ (b) $j=1$

(c) $j=2$ (d) $j=3$

Figure 5.3: Numerical eigenfunctions for the modes with masses in table 5.2.

with a, b constants. Compatibility with the normalisability condition (5.37) requires $a=0$ for $j<0$ and $b=0$ for $j>0$. When $j=0$, (5.38) close to $R=0$ reduces instead to

$$\ddot{\xi}+\frac{1}{R}\dot{\xi}+\Big(\frac{2c}{3}L^2M^2-\frac{4c}{27}\ell(\ell+4)+\frac{4c}{243}m^2\Big)\xi=0\,, \tag{5.42}$$

with the constant c given in (5.25). The solutions of (5.42) are now

$$\begin{aligned}\xi=&\,a\,J_0\Big(\sqrt{\tfrac{2c}{3}L^2M^2-\tfrac{4c}{27}\ell(\ell+4)+\tfrac{4c}{243}m^2}R\Big)\\&+b\,Y_0\Big(\sqrt{\tfrac{2c}{3}L^2M^2-\tfrac{4c}{27}\ell(\ell+4)+\tfrac{4c}{243}m^2}R\Big)\,,\end{aligned} \tag{5.43}$$

with a, b again integration constants and J_0 and Y_0 Bessel functions. In this case, normalisability, (5.37), requires $b=0$.

Near $R=R_0$, with R_0 specified in (5.25), the asymptotic form turns out to depend on the quantum number ℓ. For $\ell=0$, (5.38) close to $R=R_0$

becomes

$$\ddot{\xi} - \frac{3}{R_0 - R}\dot{\xi} + \frac{1}{R_0(R_0 - R)}\left(\frac{3}{2}L^2M^2 - \frac{2}{3}j^2\right)\xi = 0\,. \tag{5.44}$$

This has solutions

$$\begin{aligned}\xi = &\frac{u}{R_0 - R} I_2\left(\sqrt{\frac{2}{3}\frac{(4j^2 - 9L^2M^2)(R_0 - R)}{R_0}}\right)\\ &+ \frac{v}{R_0 - R} K_2\left(\sqrt{\frac{2}{3}\frac{(4j^2 - 9L^2M^2)(R_0 - R)}{R_0}}\right)\,,\end{aligned} \tag{5.45}$$

where u, v are constants and I_2 and K_2 modified Bessel functions. If $\ell \neq 0$, then (5.38) close to $R = R_0$ can be approximated to

$$\ddot{\xi} - \frac{3}{R_0 - R}\dot{\xi} - \frac{1}{4(R_0 - R)^2}\ell(\ell + 4)\xi = 0\,, \tag{5.46}$$

which has solutions

$$\xi = u(R_0 - R)^{\ell/2} + v(R_0 - R)^{-(\ell+4)/2}\,. \tag{5.47}$$

In this case, normalisability requires $v = 0$ in both (5.45) and (5.47).

Now, the above asymptotic functions near $R = 0$ and $R = R_0$ can be used as seeds for the numerical integration of the ODE (5.38). Following [116], we have performed the integration starting from both ends of the R interval, in terms of a parameter λ that labels the possible dimensionless squared masses. Denoting the functions obtained, for each λ, by integrating from the left and from the right as $\xi^L_\lambda(R)$ and $\xi^R_\lambda(R)$, the valid solutions to (5.38) can only arise for the specific values of λ for which both $\xi^L_\lambda(R)$ and $\xi^R_\lambda(R)$ are linearly dependent. This requires that the Wronskian,

$$W(\lambda, R) = \xi^L_\lambda(R)\,\dot{\xi}^R_\lambda(R) - \xi^R_\lambda(R)\,\dot{\xi}^L_\lambda(R)\,, \tag{5.48}$$

vanishes for all R in its range. We choose, without loss of generality, to evaluate (5.48) at the midpoint of the interval in order to minimise the accumulated numerical error of each solution, $\xi^L_\lambda(R)$ and $\xi^R_\lambda(R)$. Plotting $W(\lambda, \frac{R_0}{2})$ as a function of λ at fixed value of the quantum numbers j, ℓ and m, the physical masses occur at the zeros of this function: see for example figure 5.2 for the $\ell = m = 0$ case. The zeroes turn out to form an infinite discrete set, which we label by a non-negative integer $k = 0, 1, 2, \ldots$ (the first zero corresponding to $k = 0$). We have tabulated a few results in table 5.2. Finally, the eigenfunctions can be plotted numerically: see figure 5.3 for a few examples.

n	$[p,\ell-p]_{\frac{4}{9}(2p-\ell)+\frac{2}{3}(n-\ell-2t)}$	$d_{p,\,\ell-p}$	$L^2M^2_{n,\ell,t,p}$	$\Delta_{n,\ell,t,p}$	Dual operator	Short?
0	$[0,0]_0$	1	0	3	$\mathcal{T}^{(0)}_{\alpha\beta}\|_{s=2}$	✓
1	$[0,0]_{\pm\frac{2}{3}}$	1	$\frac{22}{9}$	$\frac{11}{3}$	$\mathcal{T}^{(0)}_{\alpha\beta}\mathcal{Z}^4\|_{s=2}$, c.c.	✓
	$[1,0]_{\frac{4}{9}}, [0,1]_{-\frac{4}{9}}$	3	1.76	3.50	$\mathcal{T}^{(0)}_{\alpha\beta}\mathcal{Z}^a\|_{s=2}$, c.c.	
2	$[0,0]_{\pm\frac{4}{3}}$	1	$\frac{52}{9}$	$\frac{13}{3}$	$\mathcal{T}^{(0)}_{\alpha\beta}(\mathcal{Z}^4)^2\|_{s=2}$, c.c.	✓
	$[1,0]_{-\frac{2}{9}}, [0,1]_{\frac{2}{9}}$	3	4.68	4.13	$\mathcal{T}^{(0)}_{\alpha\beta}\mathcal{Z}^a\bar{\mathcal{Z}}_4\|_{s=2}$, c.c.	
	$[2,0]_{\frac{8}{9}}, [0,2]_{-\frac{8}{9}}$	6	3.88	3.97	$\mathcal{T}^{(0)}_{\alpha\beta}\mathcal{Z}^{(a}\mathcal{Z}^{b)}\|_{s=2}$, c.c.	
	$[1,0]_{\frac{10}{9}}, [0,1]_{-\frac{10}{9}}$	3	5.07	4.21	$\mathcal{T}^{(0)}_{\alpha\beta}\mathcal{Z}^a\mathcal{Z}^4\|_{s=2}$, c.c.	
	$[0,0]_0$	1	5.92	4.36	$\mathcal{T}^{(0)}_{\alpha\beta}(1-4a^2\mathcal{Z}^4\bar{\mathcal{Z}}_4+b\mathcal{Z}^a\bar{\mathcal{Z}}_a)\|_{s=2}$	
	$[1,1]_0$	8	4	4	$\mathcal{T}^{(0)}_{\alpha\beta}(\mathcal{Z}^a\bar{\mathcal{Z}}_b-\frac{1}{3}\delta^a_b\mathcal{Z}^c\bar{\mathcal{Z}}_c)\|_{s=2}$	
3	$[0,0]_{\pm 2}$	1	10	5	$\mathcal{T}^{(0)}_{\alpha\beta}(\mathcal{Z}^4)^3\|_{s=2}$, c.c.	✓
	$[1,0]_{-\frac{8}{9}}, [0,1]_{\frac{8}{9}}$	3	8.48	4.77	$\mathcal{T}^{(0)}_{\alpha\beta}\mathcal{Z}^a(\bar{\mathcal{Z}}_4)^2\|_{s=2}$, c.c	
	$[2,0]_{\frac{2}{9}}, [0,2]_{-\frac{2}{9}}$	6	7.27	4.59	$\mathcal{T}^{(0)}_{\alpha\beta}\mathcal{Z}^{(a}\mathcal{Z}^{b)}(\bar{\mathcal{Z}}_4)\|_{s=2}$, c.c.	
	$[3,0]_{\frac{4}{3}}, [0,3]_{-\frac{4}{3}}$	10	6.36	4.43	$\mathcal{T}^{(0)}_{\alpha\beta}\mathcal{Z}^{(a}\mathcal{Z}^b\mathcal{Z}^{c)}\|_{s=2}$, c.c.	
	$[0,0]_{\pm\frac{2}{3}}$	1	10.00	5.00	$\mathcal{T}^{(0)}_{\alpha\beta}(2-5a^2\mathcal{Z}^4\bar{\mathcal{Z}}_4+b\mathcal{Z}^c\bar{\mathcal{Z}}_c)\mathcal{Z}^4\|_{s=2}$, c.c.	
	$[1,0]_{\frac{16}{9}}, [0,1]_{-\frac{16}{6}}$	3	9.28	4.90	$\mathcal{T}^{(0)}_{\alpha\beta}\mathcal{Z}^a(\mathcal{Z}^4)^2\|_{s=2}$, c.c.	
	$[1,0]_{\frac{4}{9}}, [0,1]_{-\frac{4}{9}}$	3	9.08	4.87	$\mathcal{T}^{(0)}_{\alpha\beta}\mathcal{Z}^a(1-5a^2\mathcal{Z}^4\bar{\mathcal{Z}}_4+b\mathcal{Z}^c\bar{\mathcal{Z}}_c)\|_{s=2}$, c.c.	
	$[1,1]_{\pm\frac{2}{3}}$	8	$\frac{70}{9}$	$\frac{14}{3}$	$\mathcal{T}^{(0)}_{\alpha\beta}(\mathcal{Z}^a\bar{\mathcal{Z}}_b-\frac{1}{3}\delta^a_b\mathcal{Z}^c\bar{\mathcal{Z}}_c)\mathcal{Z}^4\|_{s=2}$, c.c.	
	$[2,0]_{\frac{14}{9}}, [0,2]_{-\frac{14}{9}}$	6	8.02	4.70	$\mathcal{T}^{(0)}_{\alpha\beta}\mathcal{Z}^{(a}\mathcal{Z}^{b)}\mathcal{Z}^4\|_{s=2}$, c.c	
	$[2,1]_{\frac{4}{9}}, [1,2]_{-\frac{4}{9}}$	15	6.60	4.48	$\mathcal{T}^{(0)}_{\alpha\beta}(\mathcal{Z}^{(a}\mathcal{Z}^{b)}\bar{\mathcal{Z}}_c-\delta^{(a}_c\mathcal{Z}^{b)}\mathcal{Z}^d\bar{\mathcal{Z}}_d)\|_{s=2}$, c.c.	

Table 5.3: The complete KK graviton spectrum on the GMPS solution up to KK level $n=3$. For each state, the $\mathrm{SU}(3)\times\mathrm{U}(1)_3$ representation where it belongs is shown, along with its degeneracy $d_{p,\,\ell-p}$, mass $L^2M^2_{n,\ell,t,p}$, and conformal dimension $\Delta_{n,\ell,t,p}$. The schematic form of the dual operator is shown, with $\mathcal{T}^{(0)}_{\alpha\beta}$ denoting the IR SCFT stress-energy operator. Masses that correspond to short multiplets (ticked in the last column) and *shadow* long multiplets have been given analytically in (5.51) and (5.62), respectively.

Repeating this process for other values of the quantum numbers j, ℓ and m, we are guaranteed to sweep over the complete KK graviton spectrum by the arguments in section 5.2.1. Group theory arguments also allow us to translate between the set of quantum numbers (k,j,ℓ,m), with the quantum numbers (n,ℓ,p,t) adapted to the branching (5.28):

$$n=2k+|j|+\ell\,,\qquad m=2p-\ell\,,\qquad j=n-\ell-2t\,,\tag{5.49}$$

Finally, it can be checked that the quantum numbers (n,ℓ,p,t) that characterise the KK graviton spectrum range as

$$n=0,1,\dots,\qquad \ell=0,1,\dots,n\,,\qquad t=0,1,\dots,n-\ell\,,\qquad p=0,1,\dots,\ell\,,\tag{5.50}$$

in agreement with the branching (5.28). The eigenfunctions, and thus the schematic form of the dual operators, can be similarly inferred from the branching (5.28). Table 5.3 summarises our results up to SO(8) KK level $n=3$.

Analytic results: short and shadow gravitons

In the previous section, we arranged the GMPS graviton spectrum in representations of the $\mathrm{SU}(3)\times\mathrm{U}(1)_3$ residual bosonic symmetry of the background. This geometry also preserves $\mathcal{N}=2$ supersymmetry, so the graviton spectrum must organise itself into representations of the full (super)symmetry group $\mathrm{OSp}(2|4)\times\mathrm{SU}(3)$ (with the $\mathrm{U}(1)_3$ R-symmetry contained in the $\mathrm{OSp}(2|4)$ factor). Recall that there are three types of $\mathrm{OSp}(2|4)$ multiplets that contain states up to spin $s=2$: massless, short and long. See *e.g.* tables 8, 9 and 10 of [82] for a summary of their state contents.

From table 5.3 we see that we obtain, as expected, a massless graviton which is an $\mathrm{SU}(3)\times\mathrm{U}(1)_3$ singlet. In addition to the $D=4$ metric and gravitini, the $\mathcal{N}=2$ massless graviton multiplet contains a vector. A fully non-linear consistent truncation on GMPS [B] (and on CPW [A, B]) beyond the linearised analysis presented here exists to this $D=4$ field content. This is in agreement with the general statements of [32, 44].

Inspection of our numerical results also allows us to detect analytically a tower of short gravitons. We indeed observe that, for every n, our numerical eigenvalues for the states with $\mathrm{SU}(3)\times\mathrm{U}(1)_3$ quantum numbers $[0,0]_{\pm R_2 n}$, with R_2 given by the R-charge of $\mathcal{Z}^4$ in (5.8), are very well approximated by the analytic expression

$$L^2M_n^2 = R_2 n\left(R_2 n+3\right). \tag{5.51}$$

These states are thus short, since their conformal dimensions

$$\Delta_n = R_2 n + 3\,, \tag{5.52}$$

which arise from (5.51) as the larger solution to the equation

$$\Delta(\Delta-3) = M^2L^2\,, \tag{5.53}$$

are locked in terms of their R-symmetry charges

$$R_n = \pm R_2 n \tag{5.54}$$

through the relation

$$\Delta_n = |R_n| + 3\,. \tag{5.55}$$

For these states, the numerically obtained value of the masses has been replaced in table 5.3 with the analytic value (5.51).

From the branching (5.28), the short graviton multiplets can be seen to correspond to bound states of the energy-momentum superfield and the operator $\mathcal{Z}^4$ that is integrated out in the IR. Schematically,

$$\mathcal{T}^{(0)}_{\alpha\beta}\left(\mathcal{Z}^4\right)^n\,, \quad n=0,1,2,\dots\,, \tag{5.56}$$

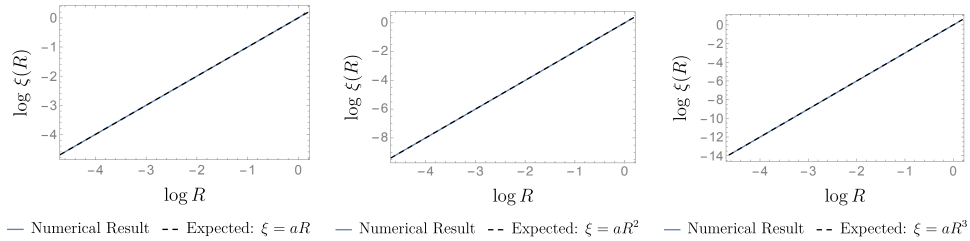


Figure 5.4: Comparison between the numerical result for the $k = \ell = m = 0$ wavefunctions with $j = 1, 2, 3$, corresponding to short states, and the expected analytical result: the modulus of (5.57) with (5.59).

where $n = 0$ corresponds to the massless graviton. Curiously, for the CPW geometry, the operators (5.56) are also short [93]. Their physical properties remain as in (5.51)–(5.55) with R_2 still given by the R-charge of $\mathcal{Z}^4$, which now takes on the value (5.7). The group theory result (5.56) is in agreement with our numerics, and in fact allows us to obtain the corresponding eigenfunctions analytically. The eigenfunction of (5.32) with (5.31), dual to the operator (5.56), is given by

$$\mathcal{Y}_j = (\xi_1)^j e^{ij\psi} \,, \tag{5.57}$$

in terms of $\xi_1(r)$, which is the r-dependent function $\xi_{\ell,m,j}(r)$ in (5.33) with $j = 1$, $\ell = m = 0$ and $k = 0$ so that $n = 1$ via (5.49). The subscript in $\xi_1(r)$ refers to the fact that this function corresponds to an SU(3) singlet: the SU(3) singlet at KK level $n = 1$ in table 5.3. Inserting the eigenfunction $\xi_1(r)$ and its analytic eigenvalue (5.51) into (5.36) with the above choice of quantum numbers, the ODE (5.36) reduces to

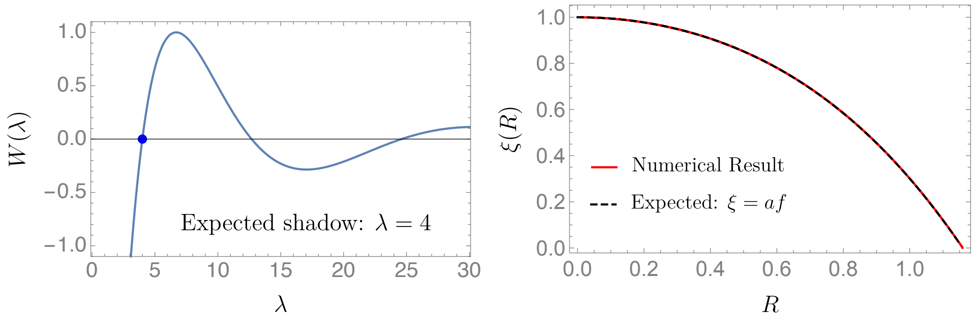
$$(\xi_1')^2 = \frac{1}{9r^2}\xi_1^2 \,. \tag{5.58}$$

This equation can be analytically solved as

$$\xi_1 = r^{1/3} \equiv R \,, \tag{5.59}$$

in exact agreement with our numerical integration, see figure 5.4. A similar analysis for CPW leads to $\xi_1 = r^{1/2}$.

Our numerics strongly suggest that all other gravitons belong to long multiplets, with masses M^2L^2 leading to conformal dimensions Δ through (5.53) that are above the bound (5.55), $\Delta > |R| + 3$. Group theory allows us to determine the structure of the dual operators as reported in table 5.3, but in general we can only access the mass eigenvalues numerically. There is an exception: for a certain series of long gravitons starting at SO(8) KK level $n = 2$, we can determine the masses analytically and relate the corresponding



(a) (b)

Figure 5.5: (a): Wronskian at $R = R_0/2$ of the functions $\xi^L_\lambda(R)$ and $\xi^R_\lambda(R)$ corresponding to shadow solutions with $\ell = 2$, $m = j = 0$. A blue dot signals the expected mass of a shadow octet state. (b): Wavefunction $\xi_8(R)$ for the lightest shadow mode with $\ell = 2$, $m = j = 0$. The agreement of the numerical result ξ_8 with the background function af is excellent, with the proportionality constant a fixed to $a = 2/9$.

eigenfunctions to precise metric functions. These modes have $\mathrm{SU}(3) \times \mathrm{U}(1)_3$ charges $[1, 1]_{\pm R_2(n-2)}$, with R_2 again given in (5.8), and are dual to operators of the schematic form

$$\mathcal{T}^{(0)}_{\alpha\beta}\Big(\mathcal{Z}^A \bar{\mathcal{Z}}_B - \tfrac{1}{3}\delta^A_B \mathcal{Z}^C \bar{\mathcal{Z}}_C\Big)(\mathcal{Z}^4)^{n-2}\,, \quad n = 2, 3, \dots \tag{5.60}$$

In [93] it was observed that the analogue tower of modes for CPW has dimensions

$$\Delta_n = (n-2)R_2 + 4 \tag{5.61}$$

(with R_2 accordingly given in (5.7) above). The authors of [93] suggested that this apparent protection of the conformal dimensions in terms of the R-charges for these modes occurs, despite being long, because they are *shadows* [155] of the massless vector at KK level $n = 0$, which lies in the $\mathbf{8}_0$ of $\mathrm{SU}(3) \times \mathrm{U}(1)_2$.

Our numerical routine finds a massive KK graviton over GMPS with quantum numbers $\ell = 2$, $k = j = m = 0$ and mass that can be very well approximated by the analytic value $L^2 M^2 = 4$. In terms of the quantum numbers (5.50) associated to the branching (5.28), this state is attained at KK level $n = 2$ with quantum numbers $\ell = 2$, $p = 1$, $t = 0$. From (5.53), the conformal dimension of this state is $\Delta = 4$, which agrees with (5.61) for $n = 2$. This suggests that this state lies at the bottom of a tower of shadow gravitons with dual operators (5.60) and conformal dimensions (5.61), exactly as for CPW but now with R_2 given by (5.8). The numerical integration confirms this expectation. We do find numerically a tower of masses that can be very well approximated by the analytic expression

$$L^2 M_n^2 = \big((n-2)R_2 + 4\big)\big((n-2)R_2 + 1\big)\,, \qquad n = 2, 3, \dots \tag{5.62}$$

with R_2 as in (5.8). These masses indeed correspond to the conformal dimension (5.61) through (5.53).

For these shadow gravitons we can also relate their eigenfunctions to a precise metric function. The eigenfunctions (5.33) corresponding to this tower of states can be written as

$$\mathcal{Y}_j = \xi_8 \, r^{j/3} \, Y_{2,0} \, e^{ij\psi} \,, \qquad j = 0, 1, \dots \,, \tag{5.63}$$

where $\xi_8(r)$ is the r-dependent part of the eigenfunction of the lightest state in the tower, with $\ell = 2$, $k = j = m = 0$. The subscript in $\xi_8(r)$ refers to the fact this function corresponds to an SU(3) octet: the SU(3) octet, $[1, 1]$, at KK level $n = 2$ in table 5.3. In (5.63) we have assumed that the $(\mathcal{Z}^4)^j$ contributions in (5.60) amount to factors of $(r^{1/3} e^{i\psi})^j$ in the eigenfunction by virtue of (5.57), (5.59). The function ξ_8 satisfies the ODE (5.36) for all j and with the other quantum numbers suitably fixed, with mass eigenvalue (5.62) with n there related to j and $\ell = 2$ through (5.49). This discrete, j-dependent set of ODEs can be shown to be equivalent to the following set of two ODEs:

$$\xi_8 + \frac{2}{r\alpha^2}\xi_8' = 0 \,, \qquad \xi_8 - \frac{3\sqrt{1+(1+r^2)\alpha^2}}{f \cdot \alpha}\xi_8 + \frac{1}{r\alpha^2 f^3}\left(r f^3 \xi_8'\right)' = 0 \,. \tag{5.64}$$

Now, the first ODE in (5.64) is the same as the first of the ODEs in (5.16) that characterise the background geometry. We thus conclude that ξ_8 is proportional to the metric function f. Having used this proportionality, it can then be shown that the second ODE in (5.64) can be deduced from (5.16). The complete set of eigenfunctions for the tower of long shadow multiplets is thus given by (5.63) with $\xi_8 \propto f$. See figure 5.5. We have verified that $\xi_8 \propto f$ also holds for the CPW case, with f now given analytically in (5.20).

5.3 Geometry and spectrum

For all other long gravitons on the GMPS background, we do not have an argument to fix analytically their mass eigenvalues from our numerical results. Still, for the triplet, $[1, 0]_{R_1}$, of long gravitons at KK level $n = 1$ we may ask whether the corresponding eigenfunction is $\xi_3 \propto \sqrt{f}$. This suspicion is based on the previous observation that $\xi_8 \propto f$, and that the radial part of the octet eigenfunction should be quadratically related to that of the triplet, in agreement with the group theory branching (5.28). Figure 5.6 shows that this is indeed the correct picture, as our numerically integrated ξ_3 perfectly matches $\sqrt{f}$ up to a numerical constant. Using the analytic expression (5.20), it is straightforward to check that $\xi_3 \propto \sqrt{f}$ also holds for the CPW solution.

It is also easy to verify for the CPW solution that the triplet, ξ_3, and singlet, ξ_1, radial eigenfunctions at KK level $n = 1$ are related through the

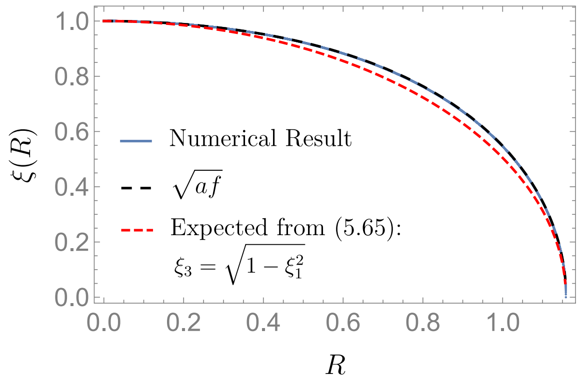


Figure 5.6: The radial wavefunction $\xi_3(R)$ of the triplet of long gravitons at KK level $n = 1$. The numerically integrated result is matched by $\sqrt{f}$ up to a proportionality constant $a = 2/9$, but not by the expression that would be expected if the S^7 constraint (5.65) held.

quadratic constraint that realises S^7 as a geometric locus in $\mathbb{R}^8$:

$$\bar{Z}_C Z^C + \bar{Z}_4 Z^4 = 1\,. \tag{5.65}$$

Somewhat surprisingly, this relation does not hold for GMPS, as we will now show building on our results from section 5.2. To see this, let us assume (5.65) and reach a contradiction. Equation (5.65) implies

$$\xi_3 \propto \sqrt{1-\xi_1^2} \tag{5.66}$$

by identifying the modulus of Z^4 with ξ_1 and that of Z^C with ξ_3. Using $\xi_1 = (r/r_0)^{1/3}$ as follows from a constant rescaling of (5.59), $\xi_3 \propto \sqrt{f}$ as verified in figure 5.6, and $\xi_8 \propto \xi_3^2 \propto f$ as shown in figure 5.5, we conclude from (5.66) that

$$\xi_8 \propto 1 - \left(\frac{r}{r_0}\right)^{2/3} \tag{5.67}$$

for the octet at level $n = 2$. Using (5.64), we finally manage to obtain the following explicit expression for the α metric function:

$$\alpha^2 = \frac{4}{3r^2\left[\left(\frac{r}{r_0}\right)^{-2/3} - 1\right]}\,. \tag{5.68}$$

Remarkably, this expression obeys the correct asymptotics (5.17). Unfortunately, the function α in (5.68) does not satisfy the second ODE in (5.16) for any value of r_0 and thus cannot be the correct GMPS metric function. In contrast, for CPW the same logic starting from (5.65) allows one to recover the correct α in (5.20). The failure of the argument for GMPS leads us to abandon the hypothesis that (5.65) should hold in the latter case. Equations (5.66)–(5.68) for GMPS are false, as must be the original assumption (5.65). Indeed, figure 5.6 manifestly shows that (5.66) as derived from the S^7 contraint (5.65) does not reproduce our numerical result for ξ_3.

From this discussion, we infer that the GMPS geometry is defined on a topological S^7 that, however, fails to satisfy the relation (5.65) and thus is not embedded isometrically in $\mathbb{R}^8$. Another example of an $\mathrm{AdS}_4 \times S^7$ solution for which (5.65) does not hold is provided by the squashed S^7 of [154].

Chapter 6

Spectra from maximal truncations

When the solutions of string theory that serve as background for the KK perturbations can be obtained from the uplift of a maximal gauged supergravity, new methods based on duality have been very recently put forward. In this chapter, we first discuss in the lines of [C] how the embedding tensor formalism and the truncation ansätze for maximal gauged supergravities in chapter 3 can be used to turn the differential equation (4.7) relevant for the spin-2 sector into an $\mathrm{SL}(8,\mathbb{R})$-covariant algebraic problem.

The full higher-dimensional supergravities can in fact be reformulated in a manifestly duality-covariant way at the cost of breaking Lorentz symmetry in higher dimensions. This reformulation is known as Exceptional Field Theory (ExFT). For lower spins, ExFT proves fundamental, and we review its basics and application to consistent truncations and spectroscopy in section 6.2, following [E].

6.1 Spin-2 spectrum from $\mathrm{SL}(8,\mathbb{R})$ matrices

We would like to determine the KK graviton mass matrix corresponding to string/M-theory AdS_4 solutions that uplift from the $\mathrm{SL}(8,\mathbb{R})$ gaugings in figure 3.1. In [143], an SO(7)-covariant mass matrix was derived for KK gravitons about solutions that uplift from the ISO(7) gauging. Here, we extend those results into a mass matrix that is formally $\mathrm{SL}(8,\mathbb{R})$ covariant, in agreement with the covariance that the gauged supergravitites take on using the embedding tensor formalism [69] particularised to gaugings contained in $\mathrm{SL}(8,\mathbb{R}) \subset \mathrm{E}_{7(7)}$.

As discussed before, the KK graviton spectrum can be computed by considering a perturbed metric of the form in (4.4), with the perturbation factorising as in (4.5) into a transverse traceless part which only depends on the external coordinates and satisfies the Fierz-Pauli equation (4.6), and a function $\mathcal{Y}(y)$ on the internal space. In all previously studied cases where the solution uplifts from a gauging within $\mathrm{SL}(8,\mathbb{R})$, these functions are symmetric polynomials

$$\mathcal{Y}^{A_1\dots A_m} = \mu^{(A_1}\dots\mu^{A_m)}\,, \qquad m = 0,1,2,\dots\,, \tag{6.1}$$

of the $\mathbb{R}^8$ coordinates μ^A, $A = 1,\dots,8$. The latter are formally in the fundamental of $\mathrm{SL}(8,\mathbb{R})$ and constrained as

$$\theta_{AB}\,\mu^A\mu^B = 1\,, \tag{6.2}$$

with $\theta_{AB} = \delta_{AB}$ for the SO(8) gauging, $\theta_{AB} = \mathrm{diag}(\mathbb{1}_7, 0)$ for ISO(7), and $\theta_{AB} = \mathrm{diag}(\mathbb{1}_6, 0, 0)$ for the $\big[\mathrm{SO}(6)\times\mathrm{SO}(1,1)\big]\ltimes\mathbb{R}^{12}$ gauging. Strictly speaking, the eigenfunctions (6.1) appearing in these gaugings are traceless with respect to these SO(n)-invariant metrics, but it is often convenient to retain the traces during the computation.

The KK graviton mass equation (4.7) can be written as a differential operator in the μ^A coordinates by means of the truncation ansätze given in [26, 30, 156]. In all these cases, the internal metric is related to gauged supergravity data via (*c.f.* (3.111))

$$4\,g^{-2}\,\bar{g}^{mn} = \mathcal{M}^{MN}\Theta_M{}^A{}_B\,\Theta_N{}^C{}_D\,K^m{}_A{}^B\,K^n{}_C{}^D\,, \tag{6.3}$$

with the vectors $K^m{}_A{}^B\partial_m = -\mu^B\partial_{\mu^A}$ related to the Killing vectors of the round spheres by

$$K^m{}_{AB} = 2K^m{}_{[A}{}^C\theta_{B]C}\,, \qquad K^{mAB} = 2K^m{}_C{}^{[A}\xi^{B]C}\,, \tag{6.4}$$

in terms of the components of the embedding tensor in (3.40)

$$\Theta_{[AB]}{}^C{}_D = 2\,\delta^C{}_{[A}\theta_{B]D}\,, \qquad \Theta^{[AB]C}{}_D = 2\,\delta_D{}^{[A}\xi^{B]C}\,, \tag{6.5}$$

for gaugings within $\mathrm{SL}(8,\mathbb{R})$.[1] For the specific cases considered here, the choices for θ and ξ are given in (3.44)-(3.46).

Then, the operator in (4.7) can be transformed as

$$\begin{aligned}\mathcal{L}\mathcal{Y} &= -\frac{g^2}{4}\mathcal{M}^{MN}\Theta_M{}^A{}_B\,\Theta_N{}^C{}_D\frac{1}{\sqrt{\mathring{g}}}\partial_m\big[\sqrt{\mathring{g}}K^m{}_A{}^B K^n{}_C{}^D\partial_n\mathcal{Y}\big]\\ &= -\frac{g^2}{4}\mathcal{M}^{MN}\Theta_M{}^A{}_B\,\Theta_N{}^C{}_D\,K^m{}_A{}^B\partial_m\big(K^n{}_C{}^D\partial_n\mathcal{Y}\big)\,,\end{aligned} \tag{6.6}$$

upon using Killing equation. When (6.6) acts on the harmonics (6.1), the PDE (4.7) simplifies into an algebraic eigenvalue problem by reading off an infinite-dimensional block-diagonal mass matrix,

$$M^2 = \mathrm{diag}\big(M^2_{(0)}, M^2_{(1)}, M^2_{(2)},\dots, M^2_{(m)},\dots\big)\,, \tag{6.7}$$

[1] In (6.4), the geometric Killing vectors are understood to have their indices appropriately restricted. For instance, in the S^6 case, $K_{AB} = \{K_{IJ}, 0\}$ with $I, J = 1,\dots,7$, and for the $S^5\times S^1$, $K_{AB} = \{K_{ij}, 0\}$ and $K^{AB} = \{0, K^{78}\}$ with $i, j = 1,\dots,6$.

labelled by an SL(8, ℝ) Kaluza-Klein level m. Each block in this decomposition is a square matrix of size

$$\dim M^2_{(m)} \equiv [m,0,0,0,0,0,0]_{\mathrm{SL}(8)} = \binom{m+7}{m} \,. \tag{6.8}$$

They are given explicitly by the following expressions. For $m=0$, expectedly,

$$M^2_{(0)} = 0 \,. \tag{6.9}$$

For $m=1$,

$$(M^2_{(1)})_A{}^B = -g^2 \mathcal{M}^{MN}\, \Theta_M{}^B{}_C\, \Theta_N{}^C{}_A \,, \tag{6.10}$$

for $m=2$,

$$(M^2_{(2)})_{A_1A_2}{}^{B_1B_2} = -2g^2\mathcal{M}^{MN}\Big[\Theta_M{}^{(B_1|}{}_C\,\Theta_N{}^C{}_{(A_1}\delta_{A_2)}{}^{|B_2)} + \Theta_M{}^{(B_1}{}_{(A_1}\,\Theta_N{}^{B_2)}{}_{A_2)}\Big] \,, \tag{6.11}$$

and for $m \geq 3$,

$$\begin{aligned}(M^2_{(m)})_{A_1\ldots A_m}{}^{B_1\ldots B_m} = -m\, g^2\mathcal{M}^{MN}\Big[&\Theta_M{}^{(B_1|}{}_C\,\Theta_N{}^C{}_{(A_1}\delta_{A_2}{}^{|B_2}\ldots\delta_{A_m)}{}^{|B_m)} \\ &+ (m-1)\Theta_M{}^{(B_1}{}_{(A_1}\,\Theta_N{}^{B_2}{}_{A_2}\delta_{A_3}{}^{B_3}\ldots\delta_{A_m)}{}^{B_m)}\Big] \,.\end{aligned} \tag{6.12}$$

Like the bosonic mass matrices of $D=4$ $\mathcal{N}=8$ gauged supergravity (*c.f.* section 4.4 of [64]), the KK graviton mass matrices (6.9)-(6.12) are quadratic in the $D=4$ embedding tensor and depend on the $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$ (inverse) scalar matrix $\mathcal{M}^{MN}$.

Since we have refrained ourselves from removing traces on same-level indices of the KK graviton mass matrices (6.9)–(6.12), the latter are manifestly SL(8, ℝ)-covariant. Proceeding like this, though, the price one pays is that every fixed SL(8, ℝ) KK level $m \geq 2$ contains repeated physical modes. For instance, for the SO(8) gauging the spectrum at SL(8, ℝ) level m includes modes of all SO(8) KK levels $n = m - 2s$ according to

$$[m,0,0,0,0,0,0] \overset{\mathrm{SO}(8)}{\longrightarrow} \sum_{s=0}^{[\frac{m}{2}]} [m-2s,0,0,0] \,. \tag{6.13}$$

For the other gaugings, there is an even larger overcounting. In the ISO(7) case, every SL(8, ℝ) level $m \geq 0$ formally contains the SO(8) levels n specified in (6.13), and each of these, in turn, includes all SO(7) levels $k = 0, 1, \ldots, n$ following

$$[n,0,0,0] \overset{\mathrm{SO}(7)_v}{\longrightarrow} \sum_{k=0}^{n} [k,0,0] \,, \tag{6.14}$$

and similarly for the $[\mathrm{SO}(6)\times\mathrm{SO}(1,1)] \ltimes \mathbb{R}^{12}$ gauging by branching SO(8) representations into SO(6) × SO(2), see appendix E for details. The repeated states can be projected out unambiguously leaving only physical modes by computing these levels subsequently, *e.g.* in (6.13)

$$[m, 0, 0, 0] = [m, 0, 0, 0, 0, 0, 0] \ominus [m-2, 0, 0, 0, 0, 0, 0] \,. \tag{6.15}$$

We also remark that it is the full embedding tensor for the dyonic gaugings, including the magnetic contributions from ξ^{AB}, that enters (6.9)-(6.12) for these gaugings.

The eigenvalues of the mass matrices (6.9)-(6.12) corresponding to the critical points in the SU(3) sector of the gaugings in figure 3.1 have been checked to reproduce the spectra obtained by directly solving the eigenvalue equation (4.7) in each case. This was only possible because of the availability and relative simplicity of the higher-dimensional solutions in, *e.g.* (3.147)-(3.154). There are other cases where solutions of the $D = 4$ gauged supergravity are known but their fully-fledged uplift has not been constructed. Despite the existence of the higher-dimensional solution by the consistency of the truncations, the lack of explicit expressions prevents the computation of the spin-2 spectra using (4.7) directly. Nonetheless, the mass formulae (6.9)-(6.12) can still be used in these cases, and the results can be checked to be compatible with the symmetry of the solutions.

6.2 ExFT spectroscopy

In general, the computation of the masses of KK modes with spin less than 2 is a much harder problem, as they do not decouple from the background fluxes and mix with one another. Lacking a relatively simple formula like (4.7), it would be desirable to have mass matrices similar to (6.9)-(6.12) that applied to other solutions apart from the ones of coset type. Again, such matrices exist when the higher-dimensional solutions are uplifted from lower-dimensional maximal supergravities, and their use goes beyond the cases where these solutions have been explicitly constructed.

The construction of these matrices relies on the language of Exceptional Field Theory to describe consistent truncations via generalised Scherk-Schwarz factorisations, which we now turn to succinctly discuss. Then, we will describe in detail how to address spectroscopy in this framework. The discussion in sections 6.2.1 and 6.2.2 mainly borrows from [125] and [28, 132] respectively. We have included them here to set the notation and introduce the concepts relevant for section 6.2.3.

6.2.1 Fundamentals of Exceptional Field Theory

We have seen in section 3.1 that the global symmetry group of the maximal ungauged supergravity organises the different gaugings that the theory allows by means of the embedding tensor. The full matter content and action of

$D = 4\, \mathcal{N} = 8$ supergravity can also be given in a manifestly duality-covariant manner by means of the tensor hierarchy. Subsequently, in section 3.2, we have shown that this tensor hierarchy proves instrumental to provide explicit oxidation formulae for the SO(8)-gauged theory into eleven-dimensional supergravity.

One does not need to stop there, but can reformulate the entire $D = 11$ and type II supergravities in a duality-covariant fashion, and not only the lower-dimensional truncations thereof. These reformulations, known as Generalised Geometry [45–54, 56] and Exceptional Field Theory [123–130] (see also [131] for a review), have provided much insight in the understanding of string unification.

We are interested here in the $\mathrm{E}_{7(7)}$-covariant formulation of M-theory, massive type IIA and type IIB supergravities based on a 4+56-dimensional generalised spacetime. The fields in this ExFT, are the following bosons

$$\{\boldsymbol{e}_\mu{}^a,\ \boldsymbol{\mathcal{M}}_{\hat{M}\hat{N}},\ \boldsymbol{\mathcal{A}}_\mu^{\hat{M}},\ \boldsymbol{\mathcal{B}}_{\mu\nu\,\hat{\alpha}},\ \boldsymbol{\mathcal{B}}_{\mu\nu\,\hat{M}}\}\,, \tag{6.16}$$

and fermions

$$\{\boldsymbol{\psi}_\mu{}^i,\ \boldsymbol{\chi}^{ijk}\}\,. \tag{6.17}$$

This set of fields is very similar to the tensor hierarchy found in (3.47) for maximal $D = 4$ supergravity, with all bosons (fermions) carrying $\mathrm{E}_{7(7)}$ (SU(8)) indices and $\boldsymbol{\mathcal{M}}_{\hat{M}\hat{N}} = (\boldsymbol{\mathcal{V}}\boldsymbol{\mathcal{V}}^T)_{\hat{M}\hat{N}}$ an $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$ coset representative. However, in ExFT all of them depend on both external x^μ, $\mu = 0, \ldots, 3$, and internal coordinates $Y^{\hat{M}}$, the latter also in the fundamental of $\mathrm{E}_{7(7)}$.

The theory is constructed to be invariant under both local internal $\mathrm{E}_{7(7)}$ and external $\mathrm{GL}(4, \mathbb{R})$ transformations which, similar to the toroidal reduction of the invitation, combine the group of diffeomorphisms and gauge transformations of the different differential forms in the $D = 10, 11$ supergravities. The internal transformations act through a generalised Lie derivative that, on a generalised vector in the $\mathbf{56}_\lambda$ of $\mathbb{R}^+ \times \mathrm{E}_{7(7)}$, is given by

$$L_\Lambda V^{\hat{M}} = \Lambda^{\hat{K}} \partial_{\hat{K}} V^{\hat{M}} - 12\, \mathbb{P}^{\hat{M}}{}_{\hat{N}}{}^{\hat{K}}{}_{\hat{L}}\, \partial_{\hat{K}}\, \Lambda^{\hat{L}}\, V^{\hat{N}} + \lambda\, \partial_{\hat{K}} \Lambda^{\hat{K}}\, V^{\hat{M}}\,, \tag{6.18}$$

with $\Lambda^{\hat{K}}(x, Y)$ an $\mathrm{E}_{7(7)}$ gauge parameter local in all 60 coordinates, and $\mathbb{P}^{\hat{M}}{}_{\hat{N}}{}^{\hat{K}}{}_{\hat{L}}$ the projector onto the adjoint representation of $\mathrm{E}_{7(7)}$

$$\begin{aligned} \mathbb{P}^{\hat{M}}{}_{\hat{N}}{}^{\hat{K}}{}_{\hat{L}} &= (t_{\hat{\alpha}})_{\hat{N}}{}^{\hat{M}}\, (t^{\hat{\alpha}})_{\hat{L}}{}^{\hat{K}} \\ &= \tfrac{1}{24} \delta^{\hat{M}}_{\hat{N}} \delta^{\hat{K}}_{\hat{L}} + \tfrac{1}{12} \delta^{\hat{M}}_{\hat{L}} \delta^{\hat{K}}_{\hat{N}} + (t_{\hat{\alpha}})_{\hat{N}\hat{L}}\, (t^{\hat{\alpha}})^{\hat{M}\hat{K}} - \tfrac{1}{24} \Omega_{\hat{N}\hat{L}} \Omega^{\hat{M}\hat{K}}\,, \end{aligned} \tag{6.19}$$

with the adjoint index $\hat{\alpha} = 1, \ldots, 133$ raised and lowered with the Cartan-Killing metric. Similarly, on a tensor $W_{\hat{\alpha}}$ in the $\mathbf{133}_\lambda$, the generalised Lie derivative is given by

$$L_\Lambda W_{\hat{\alpha}} = \Lambda^{\hat{K}} \partial_{\hat{K}} W_{\hat{\alpha}} - 12\, f_{\hat{\alpha}\hat{\beta}}{}^{\hat{\gamma}} (t^{\hat{\beta}})_{\hat{L}}{}^{\hat{K}}\, \partial_{\hat{K}}\, \Lambda^{\hat{L}}\, W_{\hat{\gamma}} + \lambda\, \partial_{\hat{K}} \Lambda^{\hat{K}}\, W_{\hat{\alpha}}\,, \tag{6.20}$$

with $f_{\hat{\alpha}\hat{\beta}}{}^{\hat{\gamma}}$ the $\mathrm{E}_{7(7)}$ structure constants. These definitions guarantee that $f_{\hat{\alpha}\hat{\beta}}{}^{\hat{\gamma}}$, $(t_{\hat{\alpha}})_{\hat{M}}{}^{\hat{N}}$ and $\Omega^{\hat{M}\hat{N}}$ are invariant tensors of weight zero under internal gauge transformations. The derivatives just introduced define a consistent gauge algebra only when the coordinate dependence of the fields and gauge parameters on the internal coordinates is restricted by the section conditions

$$(t_{\hat{\alpha}})^{\hat{M}\hat{N}}\partial_{\hat{M}} \otimes \partial_{\hat{N}} = 0\,, \qquad \Omega^{\hat{M}\hat{N}}\partial_{\hat{M}} \otimes \partial_{\hat{N}} = 0\,, \tag{6.21}$$

acting in any order on any fields or gauge parameters. The two-form $\boldsymbol{\mathcal{B}}_{\mu\nu\,\hat{M}}$ also needs to be covariantly constrained and compatible with the derivatives $\partial_{\hat{M}}$ through similar conditions. Finally, the external gauge transformations take the form of covariantised diffeomorphisms with additional compensating field-dependent transformations.

The fields in (6.16) and (6.17) carry the following weights under $\mathbb{R}^+$

$$\begin{array}{c|ccccccc} & \boldsymbol{e}_\mu{}^a & \boldsymbol{\mathcal{M}}_{\hat{M}\hat{N}} & \boldsymbol{\mathcal{A}}_\mu^{\hat{M}} & \boldsymbol{\mathcal{B}}_{\mu\nu\,\hat{\alpha}} & \boldsymbol{\mathcal{B}}_{\mu\nu\,\hat{M}} & \boldsymbol{\psi}_\mu{}^i & \boldsymbol{\chi}^{ijk} \\ \hline\hline \lambda & \frac{1}{2} & 0 & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{4} & -\frac{1}{4}\,. \end{array} \tag{6.22}$$

Requiring invariance under internal and external gauge transformations given by $\Lambda^{\hat{M}}(x,Y)$ and $\xi^\mu(x,Y)$ completely fixes the dynamics, which can be encoded into a twisted self-duality condition analogous to (3.9)

$$\boldsymbol{\mathcal{F}}_{\mu\nu}^{\hat{M}} = -\frac{1}{2}\boldsymbol{e}\,\epsilon_{\mu\nu\rho\sigma}\,\Omega^{\hat{M}\hat{N}}\,\boldsymbol{\mathcal{M}}_{\hat{N}\hat{K}}\,\boldsymbol{\mathcal{F}}^{\rho\sigma\,\hat{K}}\,, \tag{6.23}$$

and an action

$$S = \int d^4x\, d^{56}Y\, \boldsymbol{e}\Big(\hat{\boldsymbol{R}} + \tfrac{1}{48}\boldsymbol{g}^{\mu\nu}\mathcal{D}_\mu\boldsymbol{\mathcal{M}}^{\hat{M}\hat{N}}\mathcal{D}_\nu\boldsymbol{\mathcal{M}}_{\hat{M}\hat{N}} \\ - \tfrac{1}{8}\boldsymbol{\mathcal{M}}_{\hat{M}\hat{N}}\boldsymbol{\mathcal{F}}^{\mu\nu\hat{M}}\boldsymbol{\mathcal{F}}_{\mu\nu}{}^{\hat{N}} - V\big(\boldsymbol{\mathcal{M}}_{\hat{M}\hat{N}}, \boldsymbol{g}_{\mu\nu}\big)\Big) + \mathcal{L}_{\rm top}\,. \tag{6.24}$$

Here, the covariant derivatives are given by

$$\mathcal{D}_\mu = \partial_\mu - L_{\boldsymbol{\mathcal{A}}_\mu}\,, \tag{6.25}$$

and the vector field strengths involve Stückelberg couplings to the two-forms $\boldsymbol{\mathcal{B}}_{\mu\nu\,\hat{\alpha}}$, $\boldsymbol{\mathcal{B}}_{\mu\nu\,\hat{M}}$. These forms also appear in the topological term. Given that the vierbein is an $\mathrm{E}_{7(7)}$ scalar of weight $\frac{1}{2}$, the Einstein-Hilbert term is built out of a covariantised Riemann tensor which also includes a non-minimal coupling to the vectors that assures covariance under local Lorentz transformations. See section 3 of [125] for further details.

The section constraints (6.21) admit two inequivalent solutions involving a maximal subgroups of $\mathrm{E}_{7(7)}$. The first one, relevant for M-theory, decomposes the ExFT fields under $\mathrm{GL}(7,\mathbb{R})$ and only keeps dependence under the coordinates y^m in

$$
\begin{array}{ccc}
\mathrm{E}_{7(7)} & \supset & \mathrm{GL}(7,\mathbb{R}) \\
\hline\hline
\mathbf{56} & \to & \mathbf{7}_{+3} \oplus \mathbf{21'}_{+1} \oplus \mathbf{21}_{-1} \oplus \mathbf{7'}_{-3} \ , \\
Y^M & \to & \{y^m,\, y_{mn},\, y^{mn},\, y_m\} \ .
\end{array}
\tag{6.26}
$$

This decomposition is analogous to the one in section 3.2.1 and provides a bridge between the fields in ExFT and eleven-dimensional supergravity.

The other inequivalent solution of the section constraints relies on the decomposition of $\mathrm{E}_{7(7)}$ under $\mathrm{GL}(6,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. In this case,

$$
\begin{array}{ccc}
\mathrm{E}_{7(7)} & \supset & \mathrm{GL}(6,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \\
\hline\hline
\mathbf{56} & \to & (\mathbf{6},\mathbf{1})_{+2} \oplus (\mathbf{6'},\mathbf{2})_{+1} \oplus (\mathbf{20},\mathbf{1})_{0} \oplus (\mathbf{6},\mathbf{2})_{-1} \oplus (\mathbf{6'},\mathbf{1})_{-2} \ , \\
Y^M & \to & \{y^m,\, y_m{}^a,\, y_{mnp},\, y^{m\,a},\, y_m\} \ ,
\end{array}
\tag{6.27}
$$

and only the dependence on the six y^m is allowed for the fields. This decomposition describes the embedding of the full type IIB supergravity in ten dimensions, with $\mathrm{GL}(6,\mathbb{R})$ understood as a subgroup of the ten-dimensional diffeomorphisms, and $\mathrm{SL}(2,\mathbb{R})$ being the S-duality group.

Let us finally note that to account for type IIA supergravity with a non-vanishing Romans mass, the generalised Lie derivatives (6.18) and (6.20) must be deformed by an extra non-derivative term [129], which induces further constraints onto (6.21).

6.2.2 Generalised Scherk-Schwarz ansätze

The consistent truncations of the eleven-dimensional and type II theories down to $D = 4$ maximal gauged supergravities can be described in terms of a generalised Scherk-Schwarz ansatz generalising (6.3). The dependence on external and internal coordinates is taken to factorise, with the former carried by the $D = 4$ fields and the later by an $\mathrm{E}_{7(7)}$ twist matrix $U_{\hat{M}}{}^N(Y)$ and a scale factor $\rho(Y)$ [55, 132]. The indices $\hat{M}, \hat{N} = 1, \dots, 56$ are understood as local $\mathrm{E}_{7(7)}$ indices, with group transformations acting through the generalised Lie derivative (6.18). On the other hand, the indices $M, N = 1, \dots, 56$ are the global $\mathrm{E}_{7(7)}$ indices of the lower dimensional supergravity appearing in chapter 3.

The precise factorisation for most of the fields is given by their index structure and the $\mathbb{R}^+$ weights in (6.22)

$$
\begin{aligned}
\boldsymbol{g}_{\mu\nu}(x,Y) &= \rho(Y)^{-2} g_{\mu\nu}(x)\,,\\
\boldsymbol{\mathcal{M}}_{\hat{M}\hat{N}}(x,Y) &= U_{\hat{M}}{}^{M}(Y) U_{\hat{N}}{}^{N}(Y) M_{MN}(x)\,,\\
\boldsymbol{\mathcal{A}}_{\mu}^{\hat{M}}(x,Y) &= \rho(Y)^{-1} (U^{-1})_{N}{}^{\hat{M}}(Y) A_{\mu}^{N}(x)\,,\\
\boldsymbol{\mathcal{B}}_{\mu\nu\,\hat{\alpha}}(x,Y) &= \rho(Y)^{-2} U_{\hat{\alpha}}{}^{\beta}(Y) \mathcal{B}_{\mu\nu\,\beta}(x)\,,
\end{aligned}
\tag{6.28}
$$

and[2]

$$
\boldsymbol{\psi}_{\mu}{}^{i}(x,Y) = \rho^{-\frac{1}{2}}(Y)\, \psi_{\mu}{}^{i}(x)\,, \qquad \boldsymbol{\chi}^{ijk}(x,Y) = \rho^{\frac{1}{2}}(Y)\, \chi^{ijk}(x)\,. \tag{6.29}
$$

The only exception is the covariantly constrained two-form, whose reduction is given by

$$
\boldsymbol{\mathcal{B}}_{\mu\nu\,\hat{M}}(x,Y) = -2\rho(Y)^{-2} (U^{-1})_{S}{}^{\hat{P}}(Y) \partial_{\hat{M}} U_{\hat{P}}{}^{R}(Y) (t^{\alpha})_{R}{}^{S} \mathcal{B}_{\mu\nu\,\alpha}(x)\,. \tag{6.30}
$$

The factorisations in (6.28)-(6.30) define consistent truncations if, when inserted in (6.23) and (6.24), the dependence on Y completely factorises. This is the case provided that the scale factor and twist matrix satisfy the consistency conditions [132]

$$
\begin{aligned}
\left[(U^{-1})_{M}{}^{\hat{P}} (U^{-1})_{N}{}^{\hat{Q}} \partial_{\hat{P}} U_{\hat{Q}}{}^{K}\right]_{\mathbf{912}} &= \rho\, X_{MN}{}^{K}\,,\\
\partial_{\hat{N}} (U^{-1})_{M}{}^{\hat{N}} - 3\rho^{-1} (\partial_{\hat{N}}\rho) (U^{-1})_{M}{}^{\hat{N}} &= 0\,,
\end{aligned}
\tag{6.31}
$$

with $X_{MN}{}^{K}$ a constant tensor in the $\mathbf{912}$ of $\mathrm{E}_{7(7)}$ playing the rôle of the embedding tensor of the $D = 4$ gauged supergravity.[3] These conditions can be nicely expressed as a generalised parallelisability condition [55]

$$
L_{\mathcal{U}_M} \mathcal{U}_N = X_{MN}{}^{P} \mathcal{U}_P\,, \tag{6.32}
$$

in terms of the generalised Lie derivative acting on the generalised vectors $\mathcal{U}_M = \rho^{-1}(U^{-1})_M$ of weight $\frac{1}{2}$.

Different consistent truncations are associated to different expressions for the scale factor and twist matrix in the Scherk-Schwarz factorisation. See *e.g.* [27, 28, 132].

[2] Different to the case in section 3.2.1, we fix a gauge in which the SU(8) transformations of the (4+56)-dimensional and 4-dimensional spinors coincide

[3] This can be generalised to include trombone gaugings, but we will take these components of the embedding tensor to be absent in line with footnote 2 of chapter 3.

6.2.3 KK mass matrices

Given a background of eleven-dimensional or type II supergravity specified by a Scherk-Schwarz ansatz (6.28) and (6.29) and a set of $D=4$ fields

$$\{g_{\mu\nu},\ \mathcal{M}_{MN},\, \mathcal{A}_\mu^N,\, \mathcal{B}_{\mu\nu\,\alpha},\, ,\psi_\mu{}^i\,,\chi^{ijk}\} = \{\bar{g}_{\mu\nu},\ \Delta_{MN},\, 0,\, 0,\, 0,\, 0\}\,, \quad (6.33)$$

its associated Kaluza-Klein spectrum can be obtained by making the ExFT fields depend on the linearised perturbations as an extension to the Scherk-Schwarz ansatz [133, 134, 137]

$$\begin{aligned}
\boldsymbol{g}_{\mu\nu}(x,y) &= \rho(y)^{-2}\big(\bar{g}_{\mu\nu} + h_{\mu\nu}(x,y)\big)\,,\\
\boldsymbol{\mathcal{M}}_{\hat{M}\hat{N}}(x,y) &= U_{\hat{M}}{}^{M}(y)U_{\hat{N}}{}^{N}(y)\big(\Delta_{MN} + j_{MN}(x,y)\big)\,,\\
\boldsymbol{\mathcal{A}}_\mu^{\hat{M}}(x,y) &= \rho(y)^{-1}(U^{-1})_N{}^{\hat{M}}(y)\, a_\mu^N(x,y)\,,\\
\boldsymbol{\mathcal{B}}_{\mu\nu\,\hat{\alpha}}(x,y) &= \rho(y)^{-2}U_{\hat{\alpha}}{}^{\beta}(y)\, b_{\mu\nu\,\beta}(x,y)\,,\\
\boldsymbol{\mathcal{B}}_{\mu\nu\,\hat{M}}(x,y) &= -2\rho(y)^{-2}(U^{-1})_S{}^{\hat{P}}(y)\,\partial_{\hat{M}}U_{\hat{P}}{}^{R}(y)(t^\alpha)_R{}^S\, b_{\mu\nu\,\alpha}(x,y)\,,
\end{aligned} \quad (6.34)$$

and [136]

$$\boldsymbol{\psi}_\mu{}^i(x,y) = \rho^{-\frac{1}{2}}(y)\,\tilde{\psi}_\mu{}^i(x,y)\,, \qquad \boldsymbol{\chi}^{ijk}(x,y) = \rho^{\frac{1}{2}}(y)\,\tilde{\chi}^{ijk}(x,y)\,, \quad (6.35)$$

with the bosonic $h_{\mu\nu}(x,y)$, $j_{\bar{M}\bar{N}}(x,y)$, $a_\mu^N(x,y)$, $b_{\mu\nu\,\alpha}(x,y)$, $b_{\mu\nu\,\alpha}(x,y)$, and fermionic $\tilde{\psi}_\mu{}^i(x,y)$, $\tilde{\chi}^{ijk}(x,y)$ fields understood as small perturbations. These linear perturbations have a natural tower structure when expanded in terms of the harmonics of the background solution. In fact, the expansion only requires the harmonics corresponding to the maximally symmetric case, and the perturbations become simply

$$\begin{aligned}
h_{\mu\nu}(x,y) &= h_{\mu\nu}{}^{\Lambda}(x)\,\mathcal{Y}^\Lambda(y)\,, & j_{\bar{M}\bar{N}}(x,y) &= j_{\bar{M}\bar{N}}{}^{\Lambda}(x)\,\mathcal{Y}^\Lambda(y)\,,\\
a_\mu^N(x,y) &= a_\mu^{N\Lambda}(x)\,\mathcal{Y}^\Lambda(y)\,, & b_{\mu\nu\,\alpha}(x,y) &= b_{\mu\nu\,\alpha}{}^{\Lambda}(x)\,\mathcal{Y}^\Lambda(y)\,,\\
\tilde{\psi}_\mu{}^i(x,y) &= \tilde{\psi}_\mu{}^{i\Lambda}(x)\,\mathcal{Y}^\Lambda(y)\,, & \tilde{\chi}^{ijk}(x,y) &= \tilde{\chi}^{ijk\Lambda}(x)\,\mathcal{Y}^\Lambda(y)\,,
\end{aligned} \quad (6.36)$$

where Λ denotes Kaluza-Klein indices in the tower of symmetric traceless representations of the maximal isometry group the corresponding sphere. For the present cases, the index Λ runs over the infinite towers formed by the symmetric-traceless representations

$$\begin{aligned}
S^7 \text{ of M-theory} &\quad : \quad \textstyle\bigoplus_{n=0}^{\infty}\,[n,0,0,0] \text{ of } \mathrm{SO}(8)\,\,,\\
S^6 \text{ of massive IIA} &\quad : \quad \textstyle\bigoplus_{k=0}^{\infty}\,[k,0,0] \text{ of } \mathrm{SO}(7)\,\,,\\
S^5\times S^1 \text{ S-fold of IIB} &\quad : \quad \textstyle\bigoplus_{\ell=0}^{\infty}\bigoplus_{n=-\infty}^{\infty}\,[0,\ell,0]_n \text{ of } \mathrm{SU}(4)\times\mathrm{SO}(2)\,\,.
\end{aligned} \quad (6.37)$$

These indices are raised and lowered with $\delta_{\Lambda\Sigma}$ and their position thus has no meaning. See appendix E.1 for discussion of this group theory structure.

The choice of $\mathcal{Y}^\Lambda$ as the harmonics corresponding to the configuration with maximal symmetry translates into the fact that they close under the action of the relevant Killing vector fields. This leads to the definition of $\mathcal{T}_M{}^{\Lambda\Sigma}$ as the (6.37) representation matrices encoded in the twist matrix as

$$\rho^{-1} U_N{}^{\hat{M}}\, \partial_{\hat{M}} \mathcal{Y}^\Lambda = -\mathcal{T}_N{}^{\Lambda\Sigma} \mathcal{Y}^\Sigma \,. \tag{6.38}$$

The properties of the twist matrix (6.32) guarantee that the $\mathcal{T}_M{}^{\Lambda\Sigma}$ represent the gauge algebra, with the commutator normalised as

$$[\mathcal{T}_M, \mathcal{T}_N] = -X_{MN}{}^P\, \mathcal{T}_P \,. \tag{6.39}$$

For the backgrounds in figure 3.1, the matrices $\mathcal{T}_M$ can be decomposed into $\mathrm{SL}(8,\mathbb{R})$ blocks given by

$$\mathcal{T}_M = \big(\mathcal{T}_{AB},\, \mathcal{T}^{AB}\big) \,. \tag{6.40}$$

For the S^7 and S^6 cases, $\mathcal{T}^{AB} = 0$ and at lowest KK level in the towers (6.37), the non-vanishing components are

$$\begin{aligned} (\mathcal{T}_{AB})_C{}^D &= 2\delta^D{}_{[A}\delta_{B]C} \quad \text{ for } \ S^7 \,, \\ (\mathcal{T}_{AB})_K{}^L &= 2\delta^L{}_{[A}\delta_{B]K} \quad \text{ for } \ S^6 \,, \end{aligned} \tag{6.41}$$

with $A, B = 1, \ldots, 8$ and $I, J = 1, \ldots, 7$. At higher levels, these tensors can be constructed recursively from (6.41)

$$\begin{aligned} (\mathcal{T}_M)_{A_1 \ldots A_n}{}^{B_1 \ldots B_n} &= n\, (\mathcal{T}_M)_{\{A_1}{}^{\{B_1} \delta^{B_2}_{A_2} \ldots \delta^{B_n\}}_{A_n\}} \,, \\ (\mathcal{T}_M)_{I_1 \ldots I_k}{}^{J_1 \ldots J_k} &= k\, (\mathcal{T}_M)_{\{I_1}{}^{\{J_1} \delta^{J_2}_{I_2} \ldots \delta^{J_n\}}_{I_n\}} \,. \end{aligned} \tag{6.42}$$

where curly brackets denote traceless symmetrisation. Similarly, for the $S^5 \times S^1$ configuration they take the tensor product form

$$(\mathcal{T}_M)_{\Lambda c}{}^{\Sigma d} = (\mathcal{T}_M)_\Lambda{}^\Sigma\, \delta_c{}^d + \delta_\Lambda{}^\Sigma\, (\mathcal{T}_M)_c{}^d \,, \tag{6.43}$$

where the index $c = 1, 2$ is rotated by SO(2), while Λ ranges in $\oplus_{\ell=0}^{\infty} [0, \ell, 0]$ with fundamental index $i = 1, \ldots, 6$. The matrices on the r.h.s. of (6.43) can in turn be defined as

$$\begin{aligned} (\mathcal{T}_M)_\Lambda{}^\Sigma &= \big((\mathcal{T}_{AB})_\Lambda{}^\Sigma \,,\, (\mathcal{T}^{AB})_\Lambda{}^\Sigma \equiv 0\big) \,, \\ (\mathcal{T}_M)_c{}^d &= \Big((\mathcal{T}_{AB})_c{}^d \equiv 0 \,,\, (\mathcal{T}^{AB})_c{}^d\Big) \,, \end{aligned} \tag{6.44}$$

with $(\mathcal{T}_{AB})_\Lambda{}^\Sigma$ given by

$$(\mathcal{T}_{AB})_{i_1 \ldots i_\ell}{}^{j_1 \ldots j_\ell} = \ell\, (\mathcal{T}_{AB})_{\{i_1}{}^{\{j_1} \delta^{j_2}_{i_2} \ldots \delta^{j_\ell\}}_{i_\ell\}} \,, \tag{6.45}$$

in terms of the $\mathrm{SO}(6)_v$ and $\mathrm{SO}(2)$ generators

$$(\mathcal{T}_{ij})_k{}^l \equiv 2\,\delta^l_{[i}\delta_{j]k}\,, \qquad (\mathcal{T}^{ab})_c{}^d \equiv \tfrac{2\pi n}{T}\,\epsilon^{ab}\,\epsilon_c{}^d\,, \tag{6.46}$$

with all the rest, $(\mathcal{T}_{ib})_k{}^l$, $(\mathcal{T}_{ab})_k{}^l$, $(\mathcal{T}^{ij})_c{}^d$ and $(\mathcal{T}^{ib})_c{}^d$, identically zero.

The equations of motion that follow from (the supersymmetrised version of) (6.24) are linear and quadratic in derivatives for fermions and bosons respectively. Introducing the fluctuation ansätze (6.34) and (6.35) with (6.36) into these equations and linearising in the perturbations, we can easily read off the mass matrix for each field. Thanks to the choice of the internal harmonics as those satisfying (6.38) and the consistency of the truncation (6.32), these matrices are combinations of the embedding and background tensors suitably modulated by the scalar coset representatives. Schematically,

$$LM_{\text{ferm}} \sim X + \mathcal{T}\,, \qquad L^2 M^2_{\text{bos}} \sim X^2 + X\mathcal{T} + \mathcal{T}^2\,. \tag{6.47}$$

More precisely, the fermion mass matrices are

$$(LM_{\frac{3}{2}})_{i\Lambda,j\Sigma} = \tfrac{1}{2\sqrt{2}}\,A_{1\,i\Lambda,j\Sigma}\,, \qquad (LM_{\frac{1}{2}})_{ijk\Lambda,lmn\Sigma} = 2\sqrt{2}\,A_{3\,ijk\Lambda,lmn\Sigma}\,, \tag{6.48}$$

with the generalised shift tensors given by [136]

$$\begin{aligned} A_{1\,i\Lambda,j\Sigma} &= A_{1ij}\,\delta_{\Lambda\Sigma} - 8\,(\mathcal{V}^{-1})_{ij}{}^M(\mathcal{T}_M)_{\Lambda\Sigma}\,,\\ A_{3\,ijk\Lambda,lmn\Sigma} &= A_{3\,ijk,lmn}\,\delta_{\Lambda\Sigma} + \tfrac{\sqrt{2}}{18}\,\epsilon_{ijklmnpq}\,(\mathcal{V}^{-1})^{pqN}(\mathcal{T}_N)_{\Lambda\Sigma}\,. \end{aligned} \tag{6.49}$$

The coset representative $(\mathcal{V}^{-1})_{ij}{}^M$ and the quantities A_{1ij}, $A_{3\,ijk,lmn}$ were already encountered in section 3.1, as they pertain to $D=4$ $\mathcal{N}=8$ gauged supergravity [69]. As shown in (3.33), these shift tensors are determined by the $\mathcal{N}=8$ scalar fields and the embedding tensor.

Moving on into the bosonic sector, the KK graviton mass matrix coincides with the suitable $\mathrm{SO}(n)$ restrictions of (6.10)-(6.12), and can be given as [133]

$$(\mathcal{M}^2_{\text{grav}})_{\Lambda\Sigma} = M^{MN}\,(\mathcal{T}_M)_\Lambda{}^\Omega\,(\mathcal{T}_N)_\Sigma{}^\Omega\,. \tag{6.50}$$

Strictly speaking, this is not exactly the same as the mass matrix appearing in (6.10)-(6.12), as in the latter case there where magnetic contributions induced by ξ^{AB} for the ISO(7) gauging. However, these contributions drop out in the relevant contractions, and both results perfectly match.

The KK vector mass matrix was first presented in SU(8) covariant form in [137]. The $\mathrm{E}_{7(7)}$ covariant result obtained in [134] can be expressed as

$$\begin{aligned} (\mathcal{M}^2_{\text{vec}})_{M\Lambda}{}^{N\Sigma} = \tfrac{1}{12}\Big(&X_{M\Lambda\,(R|}{}^{T\Omega}X_{P\Sigma\,U}{}^{(R|\Omega} + X_{P\Sigma\,(R|}{}^{T\Omega}X_{M\Lambda\,U}{}^{(R|\Omega}\Big)\\ &\times M_{|S)T}\,M^{|S)U}\,M^{PN}\,, \end{aligned} \tag{6.51}$$

in terms of the quantity

$$X_{M\Lambda\,N}{}^{P\Sigma} \equiv \Big(X_{MN}{}^{P}\,\delta^{\Sigma}_{\Lambda} - 12\,\mathbb{P}^{P}{}_{N}{}^{Q}{}_{M}\,(\mathcal{T}_{Q})_{\Lambda}{}^{\Sigma} \Big)\,. \tag{6.52}$$

In (6.50) and (6.51), M_{MN} and its inverse M^{MN} are the $D=4$ supergravity scalar representatives and $X_{MN}{}^{P}$ the usual embedding tensor [69]. The tensor $\mathbb{P}^{P}{}_{N}{}^{Q}{}_{M}$ is the projector given in (6.19) but now for the global $\mathrm{E}_{7(7)}$ corresponding to the lower dimensional maximal supergravity. A mass matrix for the KK scalar perturbations has also been presented in [137]. Nonetheless, we will refrain from providing its concrete (fairly involved) expression, as it will not be used in the following. In fact, if we focus on supersymmetric solutions, this computation is not needed as the scalar masses can be deduced from supersymmetry.

The SU(8) form of the KK vector mass matrix (6.51) presented in [137] can be brought to the form

$$\left(M^2_{\mathrm{vec}}\right)_{\bar{A}\Lambda}{}^{\bar{B}\Sigma} = \tfrac{1}{12}\,T_{\bar{A}\Lambda\,\bar{C}}{}^{\bar{D}\Omega}\left(T_{\bar{G}\Sigma\,\bar{D}}{}^{\bar{C}\Omega} + T_{\bar{G}\Sigma\,\bar{E}}{}^{\bar{F}\Omega}\,\eta_{\bar{F}\bar{D}}\eta^{\bar{C}\bar{E}}\right)\eta^{\bar{G}\bar{B}}\,, \tag{6.53}$$

where $\eta^{\bar{G}\bar{B}}$ is the SO(28,28) invariant metric such that the $\mathrm{E}_{7(7)}$ coset representative can be written as $\mathcal{M}_{MN} = \mathcal{V}_{M}{}^{A}\mathcal{V}_{N}{}^{B}\eta_{AB}$. In (6.53) we have defined

$$T_{\bar{A}\Lambda\,\bar{B}}{}^{\bar{C}\Sigma} \equiv T_{\bar{A}\bar{B}}{}^{\bar{C}}\,\delta^{\Sigma}_{\Lambda} - 12\,\mathbb{P}^{\bar{C}}{}_{\bar{B}}{}^{D}{}_{\bar{A}}\,(\mathcal{T}_{D})_{\Lambda}{}^{\Sigma}\,, \tag{6.54}$$

featuring the T-tensor $T_{\bar{A}\bar{B}}{}^{\bar{C}}$ of $D=4$ $\mathcal{N}=8$ supergravity [69]. The quantity (6.54) is the dressed version of (6.52), obtained through contractions of the latter with the $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$ coset representative and its inverse,

$$T_{\bar{A}\Lambda\,\bar{B}}{}^{\bar{C}\Sigma} = (\mathcal{V}^{-1})_{\bar{A}}{}^{M}\,(\mathcal{V}^{-1})_{\bar{B}}{}^{N}\,\mathcal{V}_{P}{}^{\bar{C}}\,X_{M\Lambda\,\bar{N}}{}^{P\Sigma}\,, \tag{6.55}$$

exactly as for the relation between the X- and T-tensors of $D=4$ $\mathcal{N}=8$ supergravity reviewed in (3.30). This makes apparent the equivalence between the two vector mass matrices (6.51) and (6.53), and also to (4.31) of [137].

On a related note, the vector mass matrix of $D=4$ $\mathcal{N}=8$ gauged supergravity can be written in terms of the $\mathcal{N}=8$ fermion shifts A_{1ij} and $A_{2\,h}{}^{ijk}$ in (3.33), *c.f.* (4.83), (4.84) of [64]. Similarly, splitting the indices in the $\mathbf{28}+\overline{\mathbf{28}}$ of SU(8) in terms of fundamental indices as $\bar{A} = \left({}_{[ij]},\,{}^{[ij]}\right)$, the KK vector mass matrix (6.53) with (6.54) takes on the block structure

$$\left(M^2_{\mathrm{vec}}\right)_{\bar{A}\Lambda}{}^{\bar{B}\Sigma} = \begin{pmatrix} \left(M^2_{\mathrm{vec}}\right)_{ij\Lambda}{}^{lm\Sigma} & \left(M^2_{\mathrm{vec}}\right)^{ij\,\Omega\,lm\,\Sigma}\,\delta_{\Omega\Lambda} \\ \left(M^2_{\mathrm{vec}}\right)_{ij\Lambda\,lm\Omega}\,\delta^{\Omega\Sigma} & \left(M^2_{\mathrm{vec}}\right)^{ij\Omega}{}_{lm\Omega'}\,\delta^{\Omega'\Sigma}\,\delta_{\Lambda\Omega} \end{pmatrix} \tag{6.56}$$

with

$$\left(M^2_{\mathrm{vec}}\right)_{ij\Lambda}{}^{lm\Omega} = \left(\left(M^2_{\mathrm{vec}}\right)^{ij\Lambda}{}_{lm\Omega}\right)^{*}\,, \qquad \left(M^2_{\mathrm{vec}}\right)_{ij\Lambda\,lm\Omega} = \left(\left(M^2_{\mathrm{vec}}\right)^{ij\Lambda\,lm\Omega}\right)^{*}\,, \tag{6.57}$$

and

$$
\begin{aligned}
\left(M^2_{\text{vec}}\right)^{ij\Lambda}{}_{lm\Omega} = \frac{1}{12}\Big(&-A_{2\,pqr}^{[i}\delta_{[l}^{j]}A_{2\;m]}{}^{pqr} + 3A_{2\,pq[l}^{[i}A_{2\;m]}{}^{j]pq}\Big)\,\delta^{\Lambda}_{\Omega} \\
&+\Big(\delta^{[i}_{[l}A_{2\;m]}{}^{j]pq}\mathcal{T}_{pq}{}^{\Lambda}{}_{\Omega} + A_{2\;[m}{}^{ijq}\mathcal{T}_{l]q}{}^{\Lambda}{}_{\Omega} \\
&\quad -\delta^{[i}_{[l}A^{j]}_{2\;m]pq}\mathcal{T}^{pq\Lambda}{}_{\Omega} - A^{[j}_{2\;lmq}\mathcal{T}^{i]q\Lambda}{}_{\Omega}\Big) \\
&-12\Big[\tfrac{2}{3}\delta^{[i}_{[l}\mathcal{T}^{j]r\Lambda}{}_{\Sigma}\mathcal{T}_{m]r}{}^{\Sigma}{}_{\Omega} - \tfrac{1}{2}\delta^{[i}_{[l}\mathcal{T}_{m]r}{}^{\Lambda}{}_{\Sigma}\mathcal{T}^{j]r\Sigma}{}_{\Omega} + \tfrac{1}{4}\mathcal{T}_{lm}{}^{\Lambda}{}_{\Sigma}\mathcal{T}^{ij\Sigma}{}_{\Omega} \\
&\quad -\tfrac{1}{12}\mathcal{T}^{ij\Lambda}{}_{\Sigma}\mathcal{T}_{lm}{}^{\Sigma}{}_{\Omega} + \tfrac{1}{4}\delta^{ij}_{lm}\mathcal{T}_{rs}{}^{\Lambda}{}_{\Sigma}\mathcal{T}^{rs\Sigma}{}_{\Omega}\Big]\,,
\end{aligned}
\tag{6.58}
$$

$$
\begin{aligned}
\left(M^2_{\text{vec}}\right)^{ij\,\Lambda\,lm\Omega} &= \frac{1}{144}A^{[i}_{2\;qrs}\epsilon^{j]qrsuvw[l}A^{m]}_{2\;\;uvw}\delta^{\Lambda\Omega} \\
&+\frac{1}{12}\Big(A^{[i}_{2\;uvw}\epsilon^{j]uvwlmpq}\mathcal{T}_{pq}{}^{\Lambda\Omega} - \epsilon^{ijpquvw[l}A^{m]}_{2\;\;uvw}\mathcal{T}^{\Lambda\Omega}_{pq}\Big) \\
&+12\Big[\tfrac{1}{4}\Big(\mathcal{T}^{im\Lambda}{}_{\Sigma}\mathcal{T}^{jl\Sigma\Omega} - \mathcal{T}^{jm\Lambda}{}_{\Sigma}\mathcal{T}^{il\Sigma\Omega} \\
&\quad +\mathcal{T}^{il\Lambda}{}_{\Sigma}\mathcal{T}^{mj\Sigma\Omega} - \mathcal{T}^{jl\Lambda}{}_{\Sigma}\mathcal{T}^{mi\Sigma\Omega}\Big) \\
&\quad -\tfrac{1}{24}\mathcal{T}^{ij\Lambda}{}_{\Sigma}\mathcal{T}^{lm\Sigma\Omega} - \tfrac{1}{48}\epsilon^{ijpqlmrs}\mathcal{T}_{pq}{}^{\Lambda}{}_{\Sigma}\mathcal{T}_{rs}{}^{\Sigma\Omega}\Big]\,.
\end{aligned}
\tag{6.59}
$$

In order to arrive at these expressions, some calculation involving the identities given in appendix B of [127] is necessary. Against naive expectations, the blocks (6.58), (6.59) cannot be rewritten exclusively in terms of the combined KK fermion shifts $A_{1\,i\Lambda,j\Sigma}$ in (6.49) above and $A_{2\,i\Lambda}{}^{jkl\Sigma}$ in (2.26) of [136]. This is reminiscent of the situation for the vector mass matrix, (5.27), (5.28) of [157], of $D=4$ $\mathcal{N}=8$ supergravity with a trombone gauging, which cannot be written either in terms of the relevant fermion shifts solely. Indeed, the "KK embedding tensor" (6.52) does bear some resemblance with the trombone embedding tensor of [157], along with some crucial differences as the presence of extra KK indices, which allow for the existence of actions like (I.2), although defined in higher dimensions and not $D=4$.

The KK mass matrices (6.48)-(6.51) reduce to their counterparts within $D=4$ $\mathcal{N}=8$ gauged supergravity (see [64]), and extend those to higher KK levels. Diagonalisation of these four mass matrices is enough to determine the spectrum of any supersymmetric AdS_4 solutions to any desired KK level.

All the eigenvalues of the graviton and gravitino mass matrices, (6.50), (6.48), at a given AdS_4 vacuum correspond to physical spin-2 and spin-3/2 KK modes in the spectrum. In contrast, the vector and fermion mass matrices in (6.51) and (6.48), contain spurious eigenvalues at all KK levels corresponding to the magnetic vectors (in the former case), along with Goldstone and

Goldstino states eaten by the spin-2 and spin-3/2 states in the super-Higgs mechanism upon taking into account the off-diagonal couplings between modes of different spin.

For the explicit computations, it is useful to observe that the naive eigenvalues of the above mass matrices corresponding to Goldstone modes (ignoring their off-diagonal couplings) are related to the masses of the corresponding gravitons and gravitini at the same KK level through

$$L^2 M^2_{1\,\mathrm{Goldstone}} = 3L^2 M_2^2 + 6\,, \qquad L M_{\frac{1}{2}\,\mathrm{Goldstino}} = 2 L M_{\frac{3}{2}}\,. \tag{6.60}$$

Similar relations have also been observed to hold for the KK spectra in other dimensions (*c.f.* [G]) and are very helpful to identify the unphysical vector and fermion states to be removed from the spectra.

The KK spectral techniques discussed in this section, despite only being roughly two-year old, have already been applied in a variety of cases. See *e.g.* [C, E–G] and [134–141, 158]. In the next chapter, we will analyse the results in [C] and [F] to provide tight consistency checks of these methods based on compatibility with independent group theory results and the graviton spectrum based on (4.7) when possible. These analyses will yield interesting consequences for these examples.

Chapter 7

Applications

In this chapter, I will particularise the formalism discussed in the previous sections to specific solutions in ten- and eleven-dimensional supergravity. We will focus on three different classes. First, in section 7.1 we analyse the set of $\mathcal{N} = 1$ AdS$_4$ solutions of M-theory and massive type IIA that respectively uplift from the SO(8) and ISO(7) gaugings but are not in their SU(3) invariant sector. The symmetry groups of these solutions are very small or even empty, and this therefore constitutes a remarkable instance of the power of the methods explained in chapter 6.

Then, in section 7.2 a two-parameter family of AdS$_4$ $\mathcal{N} = 2$ that uplifts to an S-fold configuration of type IIB supergravity will be presented, and its spectrum analysed. This family is holographically dual to the conformal manifold of a strongly coupled CFT$_3$, and the global structure of this moduli space at large N can be inferred out of the KK spectrum. Finally, in section 7.3 we will scan over all known solutions in the SU(3) invariant sectors of the gauged supergravities in figure 3.1. The spectra thus obtained point at a curious universality property.

7.1 $\mathcal{N} = 1$ spectra in massive IIA and M-theory

In this section we compute the KK spectrum of the solutions of $D = 11$ supergravity that uplift from the vacua of $D = 4$ $\mathcal{N} = 8$ SO(8) supergravity preserving $\mathcal{N} = 1$ and SO(3) [159, 160] or U(1) $\times$ U(1) [161, 162] residual symmetry, and the solutions of massive IIA supergravity that uplift from the $\mathcal{N} = 1$ vacua of $D = 4$ $\mathcal{N} = 8$ ISO(7) supergravity with U(1) (two of these) [163] and no leftover continuous symmetry [164]. As mentioned in chapter 6, a powerful feature of the ExFT-based techniques of [133] is that the resulting KK mass matrices only depend on data of the relevant $D = 4$ $\mathcal{N} = 8$ gauged supergravity along with the generators of SO(8) or SO(7) in (6.41) for the uplifts on S^7 and S^6, respectively. None of the aforementioned AdS$_4$ solutions has actually been constructed in fully-fledged

susy	G	$g^{-2}V$	$D=4$ solution	$D=11$ uplift	Kaluza-Klein spectrum
$\mathcal{N}=8$	SO(8)	-24	[73]	[92]	[105, 110, 165]
$\mathcal{N}=2$	SU(3) × U(1)	$-18\sqrt{3}$	[77]	[31]	[82, 137]
$\mathcal{N}=1$	G_2	$-\frac{2^{11/2}\,3^{13/4}}{5^{5/2}}$	[77]	[94, 166]	[136]
$\mathcal{N}=1$	SO(3)	-55.363855	[159, 160]	N.A.	[E]
$\mathcal{N}=1$	U(1) × U(1)	-48	[161, 162]	N.A.	[E]

susy	G	$g^{-2}c^{\frac{1}{3}}V$	$D=4$ solution	IIA uplift	Kaluza-Klein spectrum
$\mathcal{N}=3$	SO(3) × SO(3)	$-\frac{2^{16/3}}{3^{1/2}}$	[167]	[168, 169]	[134]
$\mathcal{N}=2$	SU(3) × U(1)	$-2^2\,3^{3/2}$	[26]	[26]	[134]
$\mathcal{N}=1$	G_2	$-\frac{2^{28/3}\,3^{1/2}}{5^{5/2}}$	[170]	[171, 172]	[136]
$\mathcal{N}=1$	SU(3)	$-\frac{2^{8}\,3^{3/2}}{5^{5/2}}$	[74]	[172]	[136]
$\mathcal{N}=1$	U(1)	-25.697101	[163]	N.A.	[E]
$\mathcal{N}=1$	U(1)	-35.610235	[163]	N.A.	[E]
$\mathcal{N}=1$	$\emptyset$	-35.598340	[164]	N.A.	[E]

Table 7.1: All known supersymmetric AdS_4 solutions that consistently uplift to $D=11$ (left) and massive IIA (right) supergravities from $D=4$ $\mathcal{N}=8$ supergravity with SO(8) and dyonic ISO(7) gaugings, respectively. For every solution it is shown its residual supersymmetry $\mathcal{N}$, bosonic symmetry G, and $D=4$ cosmological constant V in units of the gauge coupling g and dyonic parameter c (if applicable). Our conventions for these differ by a factor of 4 with those of the SO(8) survey [159], but agree with the ISO(7) survey [164]. Pointers are given to the references where the solutions were found within $D=4$ and $D=11$ or IIA (if available), and to their KK spectra.

ten- or eleven-dimensional form.

Together with the complete KK spectra of the $D=11$ and type IIA supersymmetric AdS_4 solutions summarised in table 7.1 that have been previously computed in [82, 105, 110, 134, 136, 137, 165], our results exhaust the spectra for all known such supersymmetric AdS_4 solutions. We discuss at length these new KK spectra in the following. The first few KK levels thereof have been tabulated appendix A of [E] and will not be repeated here.

7.1.1 Salient features of the new spectra

Except for the U(1) × U(1)-invariant vacuum of SO(8)-gauged supergravity [161, 162] which was reported ten years or so ago, all the other solutions that we will cover here have been discovered fairly recently. These include another vacuum of SO(8)-gauged supergravity with residual SO(3) invariance [159, 160]; and, in the dyonic ISO(7) gauging, two vacua with U(1) symmetry [163] and one more vacuum with no continuous symmetry at all [164]. These solutions are only known as critical points of the corresponding $D=4$ $\mathcal{N}=8$ gauged supergravities, but associated higher-dimensional solutions are guaranteed to exist by the consistency of the truncations in figure 3.1. All these higher-dimensional solutions will be warped, supported by internal supergravity forms, and equipped with inhomogeneous metrics on the internal spheres with isometry groups containing the residual symmetry groups G of their associated $D=4$ critical points.

All these AdS_4 solutions are $\mathcal{N}=1$ and preserve a (possibly empty) subgroup G of SO(8) or SO(7) in the $D=11$ or IIA cases, respectively. Accordingly, their KK spectra must organise themselves in representations of OSp(1|4) $\times$ G. For all five cases, we have translated the individual KK masses into conformal dimensions via (4.2), and we have indeed been able to allocate these into OSp(1|4) supermultiplets [173] KK level by KK level. This is a successful crosscheck of our diagonalisations of the mass matrices (6.48)-(6.51) of the previous chapter. See table 1 of [136] for a summary of the state content of the OSp(1|4) supermultiplets, which we refer to here as (M)GRAV, GINO, (M)VEC and CHIRAL. The OSp(1|4) content of the spectra at lowest KK level, $n = 0$, is known for all five solutions.[1]

We recover these results and extend them to higher KK levels, $n \geq 1$. In all cases, we find one and only one massless graviton (MGRAV) multiplet, arising at KK level $n = 0$, as expected. Also at level $n = 0$, and only at this level, we find a number of massless vector (MVEC) multiplets compatible with the dimension of the residual symmetry group G of each solution: three, two, one or none for the solutions with SO(3), U(1) $\times$ U(1), U(1) or no continuous symmetry. For all the solutions, KK level $n = 0$ is completed with a number of massive gravitino (GINO), vector (VEC) and scalar (CHIRAL) multiplets. At every KK level $n \geq 1$, all four generic massive multiplets, GRAV, GINO, VEC and CHIRAL, of OSp(1|4) appear with suitable dimensions E_0 for all solutions. As usual, singleton multiplets are absent in all spectra.

The $D = 4$ scalar vevs for all solutions under consideration are only known numerically, except for the U(1) $\times$ U(1) solution, where they are known analytically [162]. Thus, our results for the spectra of all four solutions different than this one are necessarily numerical. For U(1) $\times$ U(1), most of our results are numerical as well, although we have determined analytically some masses and dimensions. Some conformal dimensions stand out as rational or integer within numerical precision. For example, there is a GRAV with $E_0 = \frac{9}{2}$ in the SO(3) spectrum at level $n = 2$. The U(1) $\times$ U(1) spectrum also shows a GINO and a CHIRAL, both at $n = 2$, with $E_0 = 3$ and $E_0 = 2$ respectively. Perhaps more curiously, the U(1) solution with $g^{-2}c^{1/3}V = -35.610235$ contains a doubly-degenerate GINO and CHIRAL, both of them with $E_0 = 3$, and a single GINO with $E_0 = 4$, all of them at KK level $n = 1$; in contrast, the U(1) solution with $g^{-2}c^{1/3}V = -25.697101$ does not seem to contain multiplets with rational or integer dimensions.

Contrary to all previously known cases in table 7.1, it has not been possible to recover the values of these masses at each solution from a single formula, at least, of the general type considered in [136]. For gravitons, the formulae in [136] are naturally associated to geometrical data via (4.7) provided this equation can be turned into an ODE by expanding the $\mathcal{Y}$

[1] Strictly speaking, for the $D = 11$ U(1) $\times$ U(1) solution, the $n = 0$ OSp(1|4) spectrum does not seem to have been given in the literature, but it follows from the individual mass states given in [161, 162]. The $n = 0$ bosonic spectrum for the IIA U(1)-invariant solutions was given in [174] and allocated into OSp(1|4) supermultiplets in [164]. The OSp(1|4) spectrum for the solutions with SO(3) symmetry in $D = 11$ and no continuous symmetry in IIA can be respectively found at KK level $n = 0$ in [160, 164].

eigenfunctions in spherical harmonics exploiting the isometry of the internal manifold. Given the little amount of preserved symmetry for the present solutions, it is not surprising that this cannot be accomplished here.

Let us now be more specific about each one of these spectra, particularly regarding degeneracies in them. Degeneracy, or lack thereof, in the conformal dimensions E_0 of the generic OSp(1|4) supermultiplets present in the spectra arises in a way compatible with the additional bosonic symmetry G preserved by each solution. Accidental degeneracies also occur for the $G = \mathrm{U}(1)$ and $G = \mathrm{U}(1) \times \mathrm{U}(1)$ invariant solutions, as do for the $\mathcal{N} = 2$ [134, 137] and $\mathcal{N} = 3$ [134] solutions of table 7.1, and for the $\mathcal{N} = 1$ cases covered in [136].

$D = 11$ solution with SO(3) symmetry

The only degeneracies that appear in the $\mathcal{N} = 1$ spectrum of the $D = 11$ SO(3)-invariant solution are those demanded by its SO(3) representation content. In other words, the spectrum arranges itself in OSp(1|4) × SO(3) representations, with no accidental degeneracies between different representations, either at the same or across different KK levels. This feature singles out this solution together with the $\mathcal{N} = 8$ SO(8) solution, as the only ones in table 7.1 with a continuous residual symmetry and completely non-degenerate supersymmetric spectrum. Except for the type IIA solution with no residual continuous symmetry (which exhibits complete non-degeneracy), the OSp($\mathcal{N}$|4) spectrum of any other solution with residual continuous symmetry in table 7.1 contains accidental degeneracies.

Another peculiar feature of the KK spectrum of the $\mathcal{N} = 1$ SO(3)-invariant solution is that all the individual states within every OSp(1|4) multiplet have the same charges, not only under SO(3) (as of course they must) but also, somewhat unexpectedly, under a larger $\mathrm{SU}(3) \times \mathrm{U}(1)_s$. The actual symmetry group SO(3) is embedded into this SU(3) as the real subgroup of the latter (so that the fundamental representation is irreducible), while $\mathrm{SU}(3) \times \mathrm{U}(1)_s$ is embedded into SO(8) through $\mathrm{SO}(7)_s$, with $\mathbf{8}_s \to \mathbf{1} + \mathbf{7}$ and $\mathbf{8}_v$, $\mathbf{8}_c \to \mathbf{8}$ under $\mathrm{SO}(7)_s$ so that[2]

$$\mathbf{8}_s \longrightarrow \mathbf{3}_{-\frac{2}{3}} + \bar{\mathbf{3}}_{\frac{2}{3}} + \mathbf{1}_0 + \mathbf{1}_0 \,, \qquad \mathbf{8}_v,\, \mathbf{8}_c \longrightarrow \mathbf{3}_{\frac{1}{3}} + \bar{\mathbf{3}}_{-\frac{1}{3}} + \mathbf{1}_1 + \mathbf{1}_{-1} \tag{7.1}$$

under $\mathrm{SU}(3) \times \mathrm{U}(1)_s$. This is notable for a couple of reasons. Firstly, the symmetry group SO(3) is SO(8)–triality invariant as noted in [160]; yet, the KK spectrum shows some preference for the $\mathbf{8}_s$. Secondly, the spectrum exhibits a qualitative $\mathrm{OSp}(1|4) \times \mathrm{SU}(3) \times \mathrm{U}(1)_s$ structure, even if this group is certainly not a symmetry of the solution and the spectrum

[2] Our conventions are such that, at the $\mathcal{N} = 8$ SO(8) point, the (graviton, gravitini, vectors, spinors, scalars, pseudoscalars) of $\mathcal{N} = 8$ supergravity lie in the $(\mathbf{1}, \mathbf{8}_s, \mathbf{28}, \mathbf{56}_s, \mathbf{35}_v, \mathbf{35}_c)$ of SO(8), as reviewed in appendix E.1. In these conventions, the bosonic symmetries of the $\mathcal{N} = 2$ solutions in table 7.1 are, more precisely, $\mathrm{SU}(3) \times \mathrm{U}(1)_c$ and $\mathrm{SU}(3) \times \mathrm{U}(1)_v$ in the SO(8) and ISO(7) gaugings, respectively.

does not organise itself in representations of this larger group (because of the SO(3) non-degeneracy just noted). More concretely, the OSp(1|4) $\times$ SO(3) representations in the spectrum branch down from OSp(4|8) via

$$\mathrm{OSp}(4|8) \supset \mathrm{OSp}(1|4) \times \mathrm{SU}(3) \times \mathrm{U}(1)_s \supset \mathrm{OSp}(1|4) \times \mathrm{SO}(3) \,, \qquad (7.2)$$

so that, in order to form OSp(1|4) multiplets KK level by KK level, it is enough to split the SO(8) content at each level only under SO(8) $\supset$ SU(3) $\times$ U$(1)_s$.

We are unaware of anything similar happening in the KK spectrum of any other AdS_4 solution in table 7.1. For example, in the KK spectrum of the $\mathcal{N} = 2$ or $\mathcal{N} = 1$ SU(3)–invariant solutions [82, 134, 136, 137], all the individual states within a given OSp($\mathcal{N}$|4) supermultiplet have the same charges under SU(3) (as of course they must). However, different states within the same OSp($\mathcal{N}$|4) multiplet will typically lie in different representations of any larger group containing SU(3), say SU(4) or G_2. Of course, this is not surprising, because these larger groups are not symmetries of these solutions. In these cases, one can only form OSp($\mathcal{N}$|4) multiplets KK level by KK level when the SO(8) in $D = 11$ [105] or SO(7) in type IIA [134] state content at each level has already been broken down to the actual residual symmetry group SU(3). In section 7.2 we will see this Zeeman-like effect happening for other solutions in type IIB.

$D = 11$ solution with U(1) $\times$ U(1) symmetry

The spectrum for the $D = 11$ U(1) $\times$ U(1)-invariant solution displays frequent degeneracies 1, 2 and 4 for the OSp(1|4) dimensions, but also 8 and even 3. The former set of degeneracies, 1, 2 and 4, seems natural for multiplets charged under none, one or both U(1)'s. Any other degeneracy can only be accidental. For example, the spectrum must contain seven U(1) $\times$ U(1)-neutral GRAVs at level $n = 2$, as these descend from the $\mathbf{35}_v$ of SO(8) [105] and, under SO(8) $\supset$ U(1) $\times$ U(1) [162],

$$\mathbf{35}_v \longrightarrow 7(0,0) + 4(\pm1,0) + 4(0,\pm1) + 3(\pm1,\pm1) + 3(\pm1,\mp1) \,. \qquad (7.3)$$

However, there are only two non-degenerate $n = 2$ GRAVs, with dimensions $E_0 = \frac{9}{2} = 4.5$ and $E_0 = 1 + \frac{\sqrt{21}}{2} \approx 3.2912878$. The five remaining singlet GRAV multiplets have dimensions $E_0 = 1 + \frac{\sqrt{37}}{2} \approx 4.0413813$ and $E_0 = 1 + \frac{\sqrt{29}}{2} \approx 3.6925824$ with accidental degeneracies 3 and 2, respectively.

Type IIA solutions with U(1) symmetry

Inspection of the spectra around these solutions shows that, for both type IIA solutions with U(1) symmetry, the dimensions E_0 are either non-degenerate or doubly-degenerate. An analysis of the U(1) charges present in both spectra suggests that OSp(1|4) multiplets with non-vanishing, opposite U(1) charges are always degenerate. These appear as doubly-degenerate multiplets in the tables. All non-degenerate multiplets are in turn U(1)–neutral. The converse is not true, however: some U(1)–neutral multiplets are accidentally doubly-degenerate.

In order to see this, let us look for example at the spectrum of GRAV and GINO multiplets at KK level $n = 1$. The individual spin–2 and spin–3/2 states contained therein have U(1) charges that respectively descend from the representations **7** and **8** + **48** of SO(7) [134]. Under the embedding SO(7) ⊃ U(1) described in [163],

$$\mathbf{7} \to 3(0)+2\left(\pm\tfrac{1}{2}\right)\ , \quad \mathbf{8} \to 4(0)+2\left(\pm\tfrac{1}{2}\right)\ , \quad \mathbf{48} \to 16(0)+12\left(\pm\tfrac{1}{2}\right)+4(\pm 1)\ , \tag{7.4}$$

in line with the 3 non-degenerate and 2 doubly degenerate $n = 1$ GRAVs present in the spectrum of either solution. Each of these spectra also shows 12 non-degenerate and 15 doubly-degenerate GINOs at level $n = 1$. The branchings (7.4) are compatible with all 12 non-degenerate GINOs being U(1)-neutral, 14 doubly-degenerate GINOs being charged, and two further U(1)-neutral GINO multiplets being accidentally degenerate.

Type IIA solution with no continuous symmetry

The KK spectrum of the type IIA solution with no continuous symmetry is completely non-degenerate. Indeed the conformal dimension E_0 of every single OSp(1|4) multiplet present in the spectrum is different. This spectrum thus plays by the book, making no concessions whatsoever to accidental degeneracies.

We conclude by emphasising that, by $\mathcal{N} = 1$ supersymmetry, our results also contain the KK scalar spectrum above all these solutions, even if we did not explicitly diagonalise the KK scalar mass matrix.

7.2 S-fold conformal manifolds

The specific AdS/CFT instances with an associated maximal gauged supergravity interpretation are few and far between and, for that reason, must be treasured. All the string theory configurations in figure 3.1 enjoy such an understanding, with the well-known S^7 case dual to the ABJM theory reviewed in section 5.1 [39, 175], and the S^6 configuration dual to the Guarino-Jafferis-Varela (GJV) super Chern-Simons theory [26].

The proposed holographic dual the $\mathcal{N} = 4$ AdS$_4$ solution of type IIB supergravity constructed in [28] is the three-dimensional CFT described in [176]. This CFT arises as an $\mathcal{N} = 4$ infrared fixed point of the T[U(N)] field theory of [177], enhanced with an adjoint Chern-Simons term at level

k, and with its $\mathrm{U}(N) \times \mathrm{U}(N)$ global symmetry gauged with an $\mathcal{N} = 4$ vector multiplet. This field theory can be also thought to arise as a limit of four-dimensional $\mathcal{N} = 4$ super-Yang-Mills at a co-dimension one interface [178, 179]. Contrary to the other supergravity solutions in figure 3.1, the type IIB configuration of [28] is non-geometric, of the form $\mathrm{AdS}_4 \times S^5 \times S^1$, with non-trivial $\mathrm{SL}(2, \mathbb{Z})$ S-duality monodromy on S^1, and with S^5 and S^1 radii related to the gauge group rank and CS level, N and k, in the CFT. This AdS_4 solution can be thought as a limit of a Janus solution of type IIB [180, 181], compatible with the interface interpretation of the CFT.

As mentioned in section 3.1, this type IIB solution enjoys a maximal consistent truncation with dyonic $[\mathrm{SO}(6) \times \mathrm{SO}(1,1)] \ltimes \mathbb{R}^{12}$ gauge group [75, 76]. The $D = 4$ gauge couplings g and $m \equiv gc$ are related to N and k. Due to the consistency of the truncation, the vacua of this gauged supergravity (all of which are AdS, see [75, 156, 167, 182–184] for examples) give rise to (non-geometric) $\mathrm{AdS}_4 \times S^5 \times S^1$ solutions of type IIB, with the S^5 possibly fibred trivially over the S^1. The above $D = 4$ $\mathcal{N} = 8$ supergravity has an $\mathcal{N} = 4$, $\mathrm{SO}(4)$-invariant critical point [167] that uplifts to the $\mathcal{N} = 4$ type IIB S-fold solution of [28]. The $\mathcal{N} = 8$ gauged supergravity also has a two-parameter family of $\mathcal{N} = 2$ AdS vacua [183] continuously connected to the $\mathcal{N} = 4$ point, with the same cosmological constant as the latter. These features led the authors of [183] to put forward the interpretation of this family of AdS_4 solutions as the holographic realisation of the (necessarily $\mathcal{N} = 2$ [185]) conformal manifold (CM) of the $\mathcal{N} = 4$ CFT of [176].

A convenient subsector of the $D = 4$ $\mathcal{N} = 8$ gauged supergravity to address this family was constructed in [182]. It contains seven scalars, φ_i, and seven pseudoscalars, χ_i, $i = 1, \ldots, 7$, that parameterise an $(\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2))^7$ submanifold of $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$. A one-parameter family of $\mathcal{N} = 2$ vacua was identified in [182] (and referred to as Family I in [183]) located, in our conventions, at

$$
\begin{aligned}
c^{-1}e^{-\varphi_1} = c^{-1}e^{-\varphi_2} = e^{-\varphi_6} = e^{-\varphi_7} = \tfrac{1}{\sqrt{2}}\,, \quad c^{-1}e^{-\varphi_3} = e^{-\varphi_4} = e^{-\varphi_5} = 1\,, \\
\chi_1 = \chi_2 = c\,\chi\,, \quad \chi_3 = \chi_4 = \chi_5 = 0\,, \quad \chi_6 = -\chi_7 = \tfrac{1}{\sqrt{2}}\,. \qquad (7.5)
\end{aligned}
$$

The free parameter here is the pseudoscalar χ. A second one-parameter family of $\mathcal{N} = 2$ vacua was found in [183], where it was named Family II. This occurs at the locus

$$
\begin{aligned}
\varphi_1 = \varphi_2 = \varphi\,, \quad e^{-\varphi_3} = c\,, \quad e^{-\varphi_6} = e^{-\varphi_7} = \tfrac{1}{\sqrt{2}}\,, \quad e^{-\varphi_4} = e^{-\varphi_5} = \tfrac{c}{\sqrt{2}}\,e^{\varphi}\,, \\
\chi_1 = \chi_2 = \chi_3 = 0\,, \quad \chi_6 = -\chi_7 = \tfrac{1}{\sqrt{2}}\,, \quad \chi_4^2 = \chi_5^2 = 1 - \tfrac{1}{2}\,c^2\,e^{2\varphi}\,, \qquad (7.6)
\end{aligned}
$$

parameterised by the scalar φ. These families contain the $\mathcal{N} = 4$ point at $\varphi = \chi = 0$. In (7.5), (7.6), $c = m/g \neq 0$, with g and m the electric and magnetic gauge couplings of the parent $D = 4$ $\mathcal{N} = 8$ supergravity. We henceforth set $c = 1$ without loss of generality. A series of dualities can be performed on the $E_{7(7)}/SU(8)$ coset representative corresponding to the vacua (7.5), (7.6), in order to generate a larger set of vacua with both parameters (φ, χ) turned on [183]. This local family of AdS vacua parameterised by (φ, χ) was proposed in [183] as the holographic CM of the $\mathcal{N} = 4$ CFT of [176] at large N. Generically, it is still $\mathcal{N} = 2$ (with supersymmetry enhancement at the $\mathcal{N} = 4$ point) and lies outside the $(SL(2,\mathbb{R})/SO(2))^7$ submanifold of [182]. When restricted to this two-dimensional surface, the $\mathcal{N} = 8$ non-linear sigma model on $E_{7(7)}/SU(8)$ gives rise to the leading contribution to the Zamolodchikov metric on the CM [183]. This metric is Kähler and reads, with our parameterisation,[3]

$$ds^2 = (4 - e^{2\varphi})\Big[(2 - e^{2\varphi})^{-1}\, d\varphi^2 + d\chi^2\Big] \,. \tag{7.7}$$

The corresponding Riemann tensor and Ricci scalar are

$$R_{mnpq} = -R\, g_{m[p}g_{q]n} \,, \qquad R = \frac{2\, e^{2\varphi}\left(e^{4\varphi} - 12\, e^{2\varphi} + 16\right)}{(4 - e^{2\varphi})^3} \,. \tag{7.8}$$

The local $D = 4$ $\mathcal{N} = 8$ supergravity scalars originally range on the entire real line, but we find the CM construction to be only well defined if the parameters are restricted as:

$$0 < e^{2\varphi} \leq 2 \quad , \qquad 0 \leq \chi < \tfrac{2\pi}{T}, \text{ and periodic: } \chi \sim \chi + \tfrac{2\pi}{T} \ . \tag{7.9}$$

with T the inverse radius of the S^1 factor of the associated type IIB S-fold solutions. Within the intervals (7.9), both the metric (7.7) and the curvature (7.8) are smooth and finite. The Ricci scalar is in fact bounded, $-2 \leq R \leq \frac{10}{27}$, and the Riemann tensor vanishes at $e^{2\varphi} = 2(3 - \sqrt{5})$ and in the limit $e^{2\varphi} \to 0$, with χ arbitrary within its allowed interval.

The range of φ specified in (7.9) must be enforced already at the gauged supergravity level, so that the solution (7.6) (with $c = 1$) is well defined and singularity-free. This is further confirmed by the KK analysis of section 7.2.1, as only within the range (7.9) for φ are the KK spectra on the CM free from tachyonic modes, as required by supersymmetry. The periodicity in χ cannot be seen at the $D = 4$ gauged supergravity level, but is an intrinsically higher-dimensional feature of the corresponding type IIB S-folds, as will be argued in section 7.2.2.

The CM is generically $\mathcal{N} = 2$ and $U(1)_F \times U(1)_R$–invariant, except at the locations specified below. Here, $U(1)_F \times U(1)_R$ is the subgroup of $SO(6) \sim SU(4)$ (the isometry of the round S^5 in type IIB, or the R-symmetry of the parent dual $\mathcal{N} = 4$ super-Yang-Mills) defined by

$$SU(4) \supset SO(4) \sim SU(2)_1 \times SU(2)_2 \supset U(1)_1 \times U(1)_2 \,, \tag{7.10}$$

[3] Our moduli and those in [183] are related as $\chi_{\text{here}} = \chi_{\text{there}}$ and $e^{-2\varphi_{\text{here}}} = \frac{1}{2}(1+\varphi^2_{\text{there}})$.

with SO(4) the real subgroup of SU(4), $\mathrm{SU}(2)_i \supset \mathrm{U}(1)_i$, $i = 1, 2$, and $\mathrm{U}(1)_R$ and $\mathrm{U}(1)_F$ respectively corresponding to the diagonal and antidiagonal combinations of $\mathrm{U}(1)_1$ and $\mathrm{U}(1)_2$. Alternatively, $\mathrm{U}(1)_F \times \mathrm{U}(1)_R$ is equivalently defined through

$$\mathrm{SU}(4) \supset \mathrm{SU}(3) \times \mathrm{U}(1)_b \supset \mathrm{SU}(2)_F \times \mathrm{U}(1)_a \times \mathrm{U}(1)_b \supset \mathrm{U}(1)_F \times \mathrm{U}(1)_R\,, \tag{7.11}$$

with $\mathbf{3} \to \mathbf{2} \oplus \mathbf{1}$ under $\mathrm{SU}(3) \supset \mathrm{SU}(2)_F$; then, $\mathrm{SU}(2)_F \supset \mathrm{U}(1)_F$ and $\mathrm{U}(1)_a \times \mathrm{U}(1)_b \supset \mathrm{U}(1)_R$, so that if p, q, y_0 are $\mathrm{U}(1)_a$, $\mathrm{U}(1)_b$, $\mathrm{U}(1)_R$ charges, then $y_0 = \frac{1}{3}(p - q)$. The SO(6) in (7.10) and (7.11) is in turn embedded inside the SU(8) compact subgroup of the $\mathcal{N} = 8$ supergravity scalar manifold as the $\mathrm{SO}(6)_v$ subgroup of the real subgroup SO(8) of SU(8) [140, 186]. The commutant of $\mathrm{SO}(6)_v$ inside SO(8) will be denoted as SO(2) and is broken by the type IIB configuration. The labels F and R in the U(1) and SU(2) groups above refer to the flavour and R-symmetry of the dual CFTs. An R label could also be added to the SO(4) in (7.10), but is omitted for notational simplicity. Note, for later reference, that the embeddings (7.10), (7.11) are globally defined and independent of the $D = 4$ supergravity scalars.

The CM exhibits symmetry or supersymmetry enhancements at specific points. The $\mathcal{N} = 4$ SO(4)-invariant vacuum [167] of the $\mathcal{N} = 8$ supergravity, which uplifts to the $\mathrm{AdS}_4 \times S^5 \times S^1$ type IIB S-fold solution of [28] with the CFT dual of [176] is attained in our parameterisation at

$$\mathcal{N} = 4 \quad \mathrm{SO}(4) \text{ point}\ : \quad \varphi = 0\,, \qquad \chi = \tfrac{2\pi}{T} n'\,,\ n' = 0, \pm 1, \pm 2, \ldots \tag{7.12}$$

with the SO(4) symmetry group being the one that appears in the branching (7.10). Strictly speaking, only the $n' = 0$ (super)symmetry enhancement to $\mathcal{N} = 4$ SO(4) can be seen at the gauged supergravity level: the periodicity for $|n'| \geq 1$ will be shown in section 7.2.2 out of the spectrum in section 7.2.1.

All other points in the CM are $\mathcal{N} = 2$, with generic $\mathrm{U}(1)_F \times \mathrm{U}(1)_R$ symmetry. The latter is enhanced to the $\mathrm{SU}(2)_F \times \mathrm{U}(1)_R$ defined in (7.11) at two specific locations within Family I, (7.5), of [182, 183], which corresponds to the upper boundary $e^{2\varphi} = 2$ in (7.9). The first such symmetry enhancement occurs, in our parameterisation, for

$$\mathcal{N} = 2 \quad \mathrm{SU}(2)_F \times \mathrm{U}(1)_R \text{ point 1}\ : \quad e^{2\varphi} = 2\,, \quad \chi = \tfrac{\pi}{T} n'\,,\ \ n' \text{ even}\,. \tag{7.13}$$

Again, only the $n' = 0$ realisation is visible in gauged supergravity, and corresponds to the $\mathcal{N} = 2$ $\mathrm{SU}(2) \times \mathrm{U}(1)$ critical point found in [182]. The second such enhancement occurs at

$$\mathcal{N} = 2 \quad \mathrm{SU}(2)_F \times \mathrm{U}(1)_R \text{ point 2}\ : \quad e^{2\varphi} = 2\,, \quad \chi = \tfrac{\pi}{T} n'\,,\ \ n' \text{ odd}\,. \tag{7.14}$$

and has no counterpart in gauged supergravity [140] (note the different ranges of n' in (7.14) and (7.13)). Of course, the generic $\mathrm{U}(1)_F \times \mathrm{U}(1)_R$

symmetry group of the CM is a subgroup of both enhanced symmetry groups SO(4) and $\mathrm{SU}(2)_F \times \mathrm{U}(1)_R$ as indicated in (7.10) and (7.11), but the latter $\mathrm{SU}(2)_F \times \mathrm{U}(1)_R$ is not a subgroup of the former SO(4).

There are no (super)symmetry enhancements across the CM other than (7.12), (7.13) and (7.14). A couple of other notable loci within the CM are the following one-parameter families. The following locus parameterised by φ,

$$\text{Family II}\ :\ \left(0 < e^{2\varphi} \leq 2\,,\ \chi = \tfrac{2\pi}{T} n'\right)\,,\ n' = 0, \pm 1, \pm 2, \dots \tag{7.15}$$

was discussed for $n' = 0$ in [183] (see also [187]) and, for this value of n', corresponds to the gauged supergravity solution (7.6). Family II is the geodesic of the metric (7.7) that passes through the $\mathcal{N} = 4$ SO(4) point (7.12) and ends at the $\mathcal{N} = 2$ $\mathrm{SU}(2)_F \times \mathrm{U}(1)_R$ point 1, (7.13), with zero winding number on the cylindrical CM. On the other hand, the lower endpoint of the φ range in (7.9) in this curve is at infinite distance with respect to the Zamolodchikov metric (7.7). In fact the lower inequality in (7.9) is strict and the singular locus $e^{2\varphi} = 0$ does not belong to the CM, as will be argued in section 7.2.2. Family II provides a useful way to visualise the global aspects of the CM. If the latter is first represented as a rectangle in $\mathbb{R}^2$ with sides defined by (7.9), the cylinder is constructed by identifying the geodesics corresponding to Family II at $\chi = 0$ and $\chi = \frac{2\pi}{T}$. In spite of this periodicity, the large-N CM in (7.9) non-compact, with infinite volume w.r.t. the leading contribution (7.7) to the Zamolodchikov metric.

Finally, the following circumference, parameterised by χ, in the interior of the CM is also interesting

$$\text{Family III}:\quad e^{2\varphi} = 1\,,\quad 0 \leq \chi < \tfrac{2\pi}{T},\ \text{and periodic: } \chi \sim \chi + \tfrac{2\pi}{T}, \tag{7.16}$$

as the complete KK spectrum on this locus can be given in closed form and its type IIB uplift provided. See figure 7.1 for a visual summary of the CM.

7.2.1 KK towers on the two-parameter $\mathcal{N} = 2$ family

The existence of a maximal gauged supergravity description of the $\mathrm{AdS}_4/\mathrm{CFT}_3$ dualities at hand allows one to apply to the present two-dimensional holographic CM the ExFT-based KK spectral methods of [133, 136, 137] discussed in section 6.2. It is noteworthy that the ten-dimensional picture of this conformal manifold is unavailable for most of it. However, as mentioned at the end of section 6.1, this is not required to obtain the spectrum using the new duality-based techniques.

The spectrum can be labelled by the two independent KK levels, ℓ and n, appearing in (6.37) respectively associated with the internal S^5 and S^1 of the IIB S-folds. They range as

$$\ell = 0, 1, 2, \dots \qquad n = 0,\ \pm 1,\ \pm 2,\ \dots \tag{7.17}$$

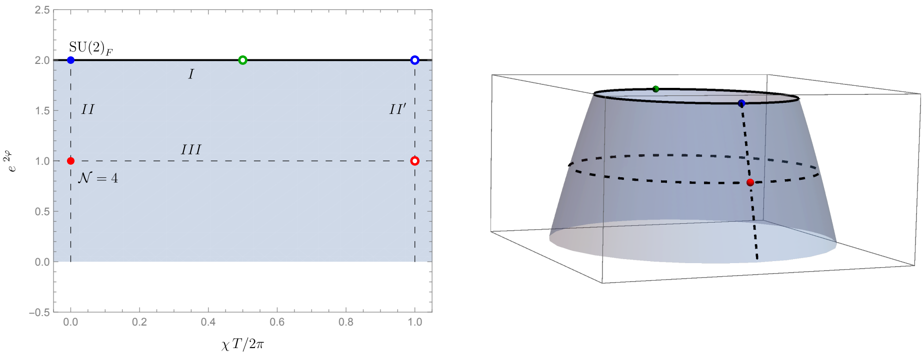


Figure 7.1: The large-N holographic CM. The left plot indicates the location of the upper boundary (solid black line), Family I (7.5), along with the (super)symmetry enhanced points: (7.12) in solid ($n'=0$) and hollow ($n'=1$) red, (7.13) in solid ($n'=0$) and hollow ($n'=1$) blue, and (7.14) in hollow green ($n'=1$). The dashed lines marked as II and II$'$ correspond to Family II, (7.15), at $n'=0$ and $n'=1$, respectively. These two lines are identified per the periodicity (7.9) of χ, rendering the topological cylinder on the right plot. Family III, (7.16), in the interior is also indicated. The locus $e^{2\varphi}=0$ lies outside the CM and is at infinite distance of the upper boundary w.r.t. the metric (7.7).

At generic $\mathcal{N}=2$ points in this two-dimensional holographic CM, the KK spectrum organises itself in representations of $\mathrm{OSp}(2|4)\times \mathrm{U}(1)_F$, with $\mathrm{U}(1)_R \subset \mathrm{OSp}(2|4)$ and $\mathrm{U}(1)_F$ defined by either branching rule (7.10) or (7.11). At the $\mathcal{N}=2$ points (7.13) and (7.14) with enhanced flavour symmetry, the KK spectrum lies in representations of $\mathrm{OSp}(2|4)\times \mathrm{SU}(2)_F$, with the latter factor defined in (7.11). Finally, at the $\mathcal{N}=4$ point (7.12) the KK spectrum is organised in $\mathrm{OSp}(4|4)$ multiplets, with R-symmetry given by the SO(4) group defined in (7.10). Multiplets of these supergroups whose superconformal primary has dimension E_0 and $\mathrm{U}(1)_R$ or SO(4) R-charges y_0 or (ℓ_1,ℓ_2) will be labelled as

$$\begin{aligned} \mathrm{OSp}(2|4)\times \mathrm{U}(1)_F \quad &: \quad \mathrm{MULT}_2\,[E_0,\, y_0;\; f]\ , \\ \mathrm{OSp}(2|4)\times \mathrm{SU}(2)_F \quad &: \quad \mathrm{MULT}_2\,[E_0,\, y_0]\otimes[k]\ , \\ \mathrm{OSp}(4|4) \quad &: \quad \mathrm{MULT}_4\,[E_0,\, \ell_1,\ell_2]\ , \end{aligned} \tag{7.18}$$

with f and k the additional $\mathrm{U}(1)_F$ charge and $\mathrm{SU}(2)_F$ (half-integer) spin, common to all states in a given $\mathrm{OSp}(2|4)$ multiplet MULT_2. The subindices in MULT_2 and MULT_4 are used to distinguish $\mathcal{N}=2$ and $\mathcal{N}=4$ multiplets. For the former, we follow the notation and conventions of appendix A of [82]. See also that reference for their state contents. For the $\mathcal{N}=4$ multiplets, we record some relevant aspects in appendix E.3. See also appendix E.1 for more details on the setup and calculations behind the results reported in this section. Previous results on the spectra of these solutions may be found in [140, 167, 182, 183] as well as [C].

Before diagonalising the relevant mass matrices, we will use this group theory information to obtain the algebraic structure of the complete KK spectrum across the entire CM, and then use the latter to obtain explicit information about the dimensions of the $\mathcal{N}=2$ supermultiplets. We provide closed-form, analytic expressions for the multiplet dimensions of the complete spectrum at specific loci, and for specific multiplets at all points in the CM. However, for most of the CM the computation needs to be performed numerically and we only highlight some aspects here. The complete results on a lattice of points across the entire moduli space can be found as an ancillary file of [F].

Algebraic structure of the complete spectrum

The algebraic structure of the complete spectrum at all points in the CM, including the protected spectrum, is inherited from that at the $\mathcal{N}=4$ point. The KK spectrum at this point was given for lowest KK levels $\ell=n=0$ in [167] and was extended to all higher levels in [140]. (see also [C] for previous partial results).

At fixed $\mathrm{SO}(6)_v \times \mathrm{SO}(2)$ KK levels (ℓ, n) ranging as in (7.17), the KK spectrum at the $\mathcal{N}=4$ point is composed of OSp(4|4) long graviton multiplets

$$\mathrm{LGRAV}_4\big[E_0, \ell_1, \ell_2\big]\,, \tag{7.19}$$

whose scalar superconformal primaries have SO(4) Dynkin labels and dimensions specified as follows. The Dynkin labels correspond to all possible pairs (ℓ_1, ℓ_2) that appear on the r.h.s. of the following branching under the first inclusion in the chain (7.10), namely,

$$[0,\ell,0] \;\to\; \bigoplus_{a=0}^{[\ell/2]} \bigoplus_{k=0}^{\ell-2a} (\ell-2a-k, k)\,. \tag{7.20}$$

At fixed ℓ, each of these $\big(\ell+1-\big[\frac{\ell}{2}\big]\big)\big(1+\big[\frac{\ell}{2}\big]\big)$ pairs of integers (ℓ_1,ℓ_2) defines a multiplet (7.19) present in the spectrum if $n=0$, or two if $n\neq 0$, corresponding to the two signs of n.[4] The conformal dimension for each of these depends on the KK levels ℓ, n and on the SO(4) Dynkin labels ℓ_1, ℓ_2, restricted as in (7.20), through the formula

$$E_0 = -\tfrac{1}{2} + \sqrt{\tfrac{9}{4} + \tfrac{1}{2}\ell(\ell+4) + \ell_1(\ell_1+1) + \ell_2(\ell_2+1) + \tfrac{1}{2}\big(\tfrac{2\pi n}{T}\big)^2}\,. \tag{7.21}$$

[4] This is the only effect of the S^1 KK level n in the algebraic structure of the $\mathcal{N}=4$ spectrum. The spectrum, though, does not come in $\mathrm{OSp}(4|4)\times\mathrm{SO}(2)$ representations (7.19) with definite SO(2) charge $2n$, because different states in a given OSp(4|4) multiplet carry different charges under the (broken) SO(2), see *e.g.* table E.3 in appendix E.1. The S^1 level n also affects the spectrum through the dimensions E_0, see (7.21), with degeneracy for both signs of n at all other quantum numbers held equal. On the rest of the CM, similar remarks apply about the dependence of the algebraic structure of the multiplet spectrum with n. The dimensions also acquire an n dependence, and the sign degeneracy is lifted for flavoured multiplets.

The l.h.s. in (7.20) corresponds to the $\mathrm{SO}(6)_v$ representations of the putative graviton states discussed in appendix E.1. The dimensions (7.21), computed in [140] using ExFT methods, agree with those that follow from the individual KK graviton masses found in (3.9) of [C] (with $n_{\text{here}} = j_{\text{there}}$). See appendix E.3 for the state content of the $\mathcal{N} = 4$ multiplets (7.19).

For specific values of the quantum numbers some of the multiplets (7.19) in the spectrum become short, and split into a SGRAV_4 (or MGRAV_4 for $\ell = 0$) and a SGINO_4 via (E.17). Specifically, this happens for [C]

$$n = 0\,, \qquad \ell_1 = \ell_2 = \tfrac{1}{2}\ell\,, \text{ with } \ell \text{ even}\,. \tag{7.22}$$

Indeed, when (7.22) holds, the dimension (7.21) saturates the $\mathcal{N} = 4$ unitarity bound, (E.16) with $s_0 = 0$.

The algebraic structure of the complete KK spectrum across the entire CM turns out to be determined by the spectrum at the $\mathcal{N} = 4$ point, through the branching (E.18) of the multiplets (7.19) under

$$\mathrm{OSp}(4|4) \supset \mathrm{OSp}(2|4) \times \mathrm{U}(1)_F\,. \tag{7.23}$$

More concretely, at fixed ℓ and n, the spectrum at an arbitrary point (φ, χ) in the CM contains $\big(1 + H(|n|)\big)$ contributions of the form

$$\begin{aligned}
&\bigoplus_{m_1=-\ell_1}^{\ell_1} \bigoplus_{m_2=-\ell_2}^{\ell_2} \Big\{ \mathrm{LGRAV}_2\big[E^{(1)}_{m_1 m_2}, y_{m_1 m_2}; f_{m_1 m_2}\big] \\
&\oplus \mathrm{LGINO}_2\big[E^{(2)}_{m_1 m_2}, y_{m_1 m_2}; f_{m_1 m_2} + 1\big] \oplus \mathrm{LGINO}_2\big[E^{(3)}_{m_1 m_2}, y_{m_1 m_2}; f_{m_1 m_2} - 1\big] \\
&\oplus \mathrm{LGINO}_2\big[E^{(4)}_{m_1 m_2}, y_{m_1 m_2}; f_{m_1 m_2} + 1\big] \oplus \mathrm{LGINO}_2\big[E^{(5)}_{m_1 m_2}, y_{m_1 m_2}; f_{m_1 m_2} - 1\big] \\
&\oplus \mathrm{LVEC}_2\big[E^{(6)}_{m_1 m_2}, y_{m_1 m_2}; f_{m_1 m_2}\big] \\
&\oplus \mathrm{LVEC}_2\big[E^{(7)}_{m_1 m_2}, y_{m_1 m_2}; f_{m_1 m_2} + 2\big] \oplus \mathrm{LVEC}_2\big[E^{(8)}_{m_1 m_2}, y_{m_1 m_2}; f_{m_1 m_2}\big] \\
&\oplus \mathrm{LVEC}_2\big[E^{(9)}_{m_1 m_2}, y_{m_1 m_2}; f_{m_1 m_2} - 2\big] \oplus \mathrm{LVEC}_2\big[E^{(10)}_{m_1 m_2}, y_{m_1 m_2}; f_{m_1 m_2}\big] \Big\}\,,
\end{aligned} \tag{7.24}$$

for each of the $\big(\ell + 1 - \big[\frac{\ell}{2}\big]\big)\big(1 + \big[\frac{\ell}{2}\big]\big)$ pairs of integers (ℓ_1, ℓ_2) defined by the r.h.s. of (7.20). All of the multiplets in (7.24) are typically long. In (7.23), the $\mathcal{N} = 2$ $\mathrm{U}(1)_R \subset \mathrm{OSp}(2|4)$ R-symmetry and the $\mathrm{U}(1)_F$ flavour symmetry are embedded into the $\mathcal{N} = 4$ $\mathrm{SO}(4) \subset \mathrm{OSp}(4|4)$ R-symmetry as indicated in (7.10) and below that equation. As remarked below (7.11), these embeddings are independent of the $D = 4$ supergravity scalars. For this reason, the R- and flavour charges of the $\mathcal{N} = 2$ multiplets in the spectrum do not depend on the position on the CM. Indeed, the quantities $y_{m_1 m_2}$ and $f_{m_1 m_2}$ in (7.24) that govern these charges are simply given, in our conventions, by the integers

$$y_{m_1 m_2} = m_1 + m_2\,, \qquad f_{m_1 m_2} = m_1 - m_2\,. \tag{7.25}$$

$\mathrm{SGRAV}_2\big[\ell+2,\,\pm\ell;\,0\big]$	$\mathrm{SVEC}_2\big[\ell+1,\,\pm\ell;\,0\big]$
$\mathrm{SGINO}_2\big[\ell+\frac{5}{2},\,\pm(\ell+1);\,0\big]$	$\mathrm{HYP}_2\big[\ell+2,\,\pm(\ell+2);\,0\big]$

Table 7.2: The protected (short, moduli independent) OSp(2|4) spectrum on the CM at KK levels $n=0$ and $\ell \geq 0$ even. At $\ell=0$, there is only one graviton and one vector multiplets, both of them massless.

The dimensions $E^{(1)}_{m_1 m_2}$, etc., in (7.24) do depend on the moduli (φ, χ) and, except for Family III, do not follow in any obvious way from the $\mathcal{N}=4$ dimensions (7.21).

At particular points in the CM and for specific choices of quantum numbers, the dimension of some of the multiplets in (7.24) might saturate the corresponding $\mathcal{N}=2$ unitarity bounds. In those cases, these long multiplets may be formally written in terms of short $\mathcal{N}=2$ multiplets. In general, though, these accidental saturations will not lead to multiplet protection: the dimensions will typically remain moduli dependent and the short multiplets will tend to recombine into long ones. For the concrete choice of quantum numbers

$$n=0\,, \qquad |m_1+m_2| = 2\ell_1 = 2\ell_2 = \ell\,, \text{ with } \ell \text{ even}\,, \tag{7.26}$$

which encompasses the $\mathcal{N}=4$ shortening condition (7.22), some of the multiplet dimensions in (7.24) both saturate the $\mathcal{N}=2$ unitarity bound and become moduli independent. This series, labelled by even ℓ, is protected in the sense that the multiplet dimensions are independent of the moduli. The series includes, at $\ell=0$, a MGRAV_2 and a MVEC_2, respectively dual to the energy-momentum tensor and the $\mathrm{U}(1)_F$ flavour current of the CFT, as well as two SGINO_2's and two HYP_2's. The latter contain the two real moduli on the CM, dual to a superpotential deformation [183]. For each $\ell = 2, 4, \ldots$, the protected series includes two of each of the possible short multiplets of OSp(2|4), with ℓ-dependent opposite R-charges. All of these protected multiplets are $\mathrm{U}(1)_F$ flavour neutral (the converse is not true, though). See table 7.2 for a summary.

After these generalities, let us highlight the main features of the spectra at specific loci within the two-parameter family.

Spectrum on the upper boundary

The complete KK spectrum on the upper boundary, Family I (7.5), of the CM has already been determined in [140] for all KK levels ℓ and n (see also [182] for the $\ell=n=0$ spectrum). Our presentation will therefore be brief.

The main new observation is that the KK spectrum on this locus follows the algebraic pattern just presented, which is valid across the CM on general grounds. At fixed ℓ and for all n, the spectrum of $\mathrm{OSp}(2|4)\times\mathrm{U}(1)_F$ multiplets on the upper boundary of the CM contains contributions of the form (7.24), with R- and flavour charges controlled by (7.25). Expressions may be found for the multiplet dimensions in terms of the quantum numbers, adapted to the branching (7.10), that appear in those expressions. For example, the dimension on the upper boundary of the $\mathrm{LGRAV}_2\big[E^{(1)}_{m_1m_2}, y_{m_1m_2}; f_{m_1m_2}\big]$ in (7.24) can be written, suppressing the subindices on the l.h.s. for simplicity, as

$$\begin{aligned} E^{(1)} = \tfrac{1}{2} + \Big[& \tfrac{9}{4} + \ell(\ell+4) + \tfrac{1}{2}(m_1+m_2)^2 + \big(\tfrac{2\pi n}{T} + (m_1-m_2)\chi\big)^2 \\ & - \tfrac{1}{2}\big(|m_1|+|m_2|\big)\big(|m_1|+|m_2|+2\ell - 2\ell_1 - 2\ell_2 + 2\big) \\ & - \tfrac{1}{2}\big(\ell-\ell_1-\ell_2\big)\big(\ell-\ell_1-\ell_2+2\big)\Big]^{\frac{1}{2}} . \end{aligned} \tag{7.27}$$

As usual, the χ dependence is introduced by a non-zero flavour, *e.g.* $f_{m_1m_2}$ in (7.25) for the LGRAV_2 dimension in (7.27). This dimension saturates the relevant $\mathcal{N}=2$ unitarity bound for the choice of quantum numbers (7.26), and the multiplet becomes short as indicated in table 7.2. For all other multiplets in (7.24), we also find the protected shortening patterns of that table and, for generic points in this family with only $\mathrm{U}(1)_F\times\mathrm{U}(1)_R$ symmetry, we find no further shortenings beyond the protected ones in table 7.2.

At the points (7.13) and (7.14) on this boundary, the flavour symmetry is enhanced to $\mathrm{SU}(2)_F$, and the spectrum accordingly recombines into representations of $\mathrm{OSp}(2|4)\times\mathrm{SU}(2)_F$ [140]. The algebraic structure and the dimensions (in particular (7.27)) at these symmetry-enhanced points are the same as in the rest of the upper boundary, only with $\mathrm{U}(1)_F$ charges now labelling $\mathrm{SU}(2)_F$ representations. This reassembling into $\mathrm{SU}(2)_F$ multiplets occurs at every fixed S^5 KK number ℓ, with the same (at $\chi=0$) or possibly different (at $\chi=\pi/T$ and $\chi=2\pi/T$) S^1 KK levels n [140]. Though all these three (up to periodicity) locations exhibit $\mathrm{SU}(2)_F$ symmetry enhancement, only $\chi=0$ and $\chi=2\pi/T$ have the same KK spectrum, and this differs from that at $\chi=\pi/T$ [140]. These $\mathrm{SU}(2)_F$ symmetry enhancements are somewhat peculiar from the point of view of the parent $\mathcal{N}=4$ point of the CM in the sense that $\mathrm{SU}(2)_F$ is not a subgroup of its SO(4) R-symmetry group. The symmetry breaking from SO(4) to $\mathrm{SU}(2)_F$ proceeds by first breaking the former into $\mathrm{U}(1)_F\times\mathrm{U}(1)_R$ via (7.10) and then recombining back up through (7.11). In fact, an alternate dimension formula adapted to the quantum numbers of the latter branching also exists [140]. Similarly to the $\mathcal{N}=4$ point, the spectrum on the $\mathrm{SU}(2)_F$-enhanced points has ($\mathrm{U}(1)_F$-charged) short multiplets [140] besides the ones in table 7.2: see the discussion around equation (E.19) in appendix E.3.

Spectrum on Family III

At this one-parameter locus defined in (7.16), which contains the $\mathcal{N}=4$ SO(4) point at (7.12), the spectrum can be given in closed form at all KK levels. The contributions (7.24) to the spectrum at KK levels ℓ and n take on the specific form:

$$
\begin{aligned}
&\bigoplus_{m_1=-\ell_1}^{\ell_1}\bigoplus_{m_2=-\ell_2}^{\ell_2}\Big\{\mathrm{LGRAV}_2\big[1+E_0^f,y;f\big]\\
&\oplus \mathrm{LGINO}_2\big[\tfrac{1}{2}+E_0^{f+1},y;f+1\big]\oplus \mathrm{LGINO}_2\big[\tfrac{1}{2}+E_0^{f-1},y;f-1\big]\\
&\oplus \mathrm{LGINO}_2\big[\tfrac{3}{2}+E_0^{f+1},y;f+1\big]\oplus \mathrm{LGINO}_2\big[\tfrac{3}{2}+E_0^{f-1},y;f-1\big]\\
&\oplus \mathrm{LVEC}_2\big[E_0^f,y;f\big]\\
&\oplus \mathrm{LVEC}_2\big[1+E_0^{f+2},y;f+2\big]\oplus \mathrm{LVEC}_2\big[1+E_0^{f},y;f\big]\\
&\oplus \mathrm{LVEC}_2\big[1+E_0^{f-2},y;f-2\big]\oplus \mathrm{LVEC}_2\big[2+E_0^{f},y;f\big]\Big\}\,,
\end{aligned}
\tag{7.28}
$$

with $y=y_{m_1m_2}$ and $f=f_{m_1m_2}$ given in (7.25), and dimensions $E^{(1)}_{m_1m_2}\equiv 1+E_0^f$, etc., specified as follows. The quantity E_0^f that determines the dimension of a multiplet in (7.28) with $\mathrm{U}(1)_F$ flavour f is simply obtained from the $\mathcal{N}=4$ expression (7.21) with the same ℓ, n, ℓ_1, ℓ_2 quantum numbers by replacing the contribution $\big(\frac{2\pi n}{T}\big)^2$ there as

$$
E_0^f=-\tfrac{1}{2}+\sqrt{\tfrac{9}{4}+\tfrac{1}{2}\ell(\ell+4)+\ell_1(\ell_1+1)+\ell_2(\ell_2+1)+\tfrac{1}{2}\big(\tfrac{2\pi n}{T}+f\chi\big)^2}\,.
\tag{7.29}
$$

At $\chi=0$, the contributions to the spectrum (7.28) with (7.29) straightforwardly recombine KK level by KK level into the contributions at the $\mathcal{N}=4$ point, (7.19) with (7.21), via the branching (**??**) under the supergroup embedding (7.23).

It is instructive to write the above expressions for a few particular cases. The lowest lying, $\ell=n=0$, case contains simply $(\ell_1,\ell_2)=(0,0)$, and becomes

$$
\begin{aligned}
&\mathrm{MGRAV}_2\big[2,\,0;\,0\big]\,\oplus\,\mathrm{SGINO}_2\big[\tfrac{5}{2},\,\pm1;\,0\big]\\
&\quad\oplus\,\mathrm{LGINO}_2\big[\tfrac{1}{2}\sqrt{9+2\chi^2},\,0;\,\pm1\big]\,\oplus\,\mathrm{LGINO}_2\big[1+\tfrac{1}{2}\sqrt{9+2\chi^2},\,0;\,\pm1\big]\\
&\quad\quad\oplus\,\mathrm{LVEC}_2\big[\tfrac{1}{2}+\tfrac{1}{2}\sqrt{9+8\chi^2},\,0;\,\pm2\big]\,\oplus\,\mathrm{LVEC}_2\big[2,\,0;\,0\big]\\
&\quad\quad\quad\oplus\,\mathrm{LVEC}_2\big[3,\,0;\,0\big]\,\oplus\,\mathrm{MVEC}_2\big[1,\,0;\,0\big]\,\oplus\,\mathrm{HYP}_2\big[2,\,\pm2;\,0\big]\,,
\end{aligned}
\tag{7.30}
$$

after writing all possible long multiplets at the $\mathcal{N}=2$ unitarity bounds in terms of short ones. This agrees with the gauged supergravity result, (4.3), (4.4) of [183] with $\varphi_{\text{there}}=1$, after some dimensions there are square-completed. In (7.30) and elsewhere, a flavour or R-symmetry charge with $\pm$ sign indicates the existence of multiplets with both charges. The short multiplets in (7.30) are those appearing in table 7.2 for $\ell = 0$. In agreement with the general discussion, the dimensions of all flavoured multiplets develop a χ dependence and thus the multiplets remain necessarily long. Not all long multiplets are flavoured, though, and those that are not have χ-independent dimensions.

Spectrum at generic points on the interior

Away from the origin and the upper boundary, the spectrum on the CM remains organised for all ℓ and n strictly in the collections (7.24) of $\mathrm{OSp}(2|4)\times \mathrm{U}(1)_F$ multiplets with moduli-independent charges controlled by (7.25). The multiplet dimensions typically depend on both moduli.

Diagonalising analytically the KK mass matrices at generic locations of φ and χ requires formidable computer power even at first S^5 KK level $\ell = 1$. The tower with $\ell = 0$ and n arbitrary is still tractable analytically, and so are the first few KK levels of the graviton mass matrix. We report on these results here. More generally, we have resorted to numerics to obtain the multiplet spectrum on a (Euclidean) lattice on the CM, with the database provided as an attachment to [F].

The spectrum at lowest, $\ell = n = 0$, levels has already been computed from gauged supergravity at generic points in the CM [183]:

$$
\begin{aligned}
&\mathrm{MGRAV}_2\big[2,\, 0;\, 0\big] \,\oplus\, \mathrm{SGINO}_2\big[\tfrac{5}{2},\, \pm 1;\, 0\big] \\
&\oplus\, \mathrm{LGINO}_2\left[\tfrac{1}{2} - s + \tfrac{1}{2}\sqrt{e^{-2\varphi}\left(2+e^{2\varphi}\right)^2 + 2e^{2\varphi}\chi^2},\, 0;\, \pm 1\right] \\
&\oplus\, \mathrm{LGINO}_2\left[\tfrac{1}{2} + s + \tfrac{1}{2}\sqrt{e^{-2\varphi}\left(2+e^{2\varphi}\right)^2 + 2e^{2\varphi}\chi^2},\, 0;\, \pm 1\right] \\
&\oplus \mathrm{LVEC}_2\left[\tfrac{1}{2} + \sqrt{-\tfrac{7}{4} + 4e^{-2\varphi} + 2e^{2\varphi}\chi^2}, 0; \pm 2\right] \oplus \mathrm{LVEC}_2\left[\tfrac{1}{2} + \sqrt{\tfrac{1}{4} + 2e^{2\varphi}}, 0; 0\right] \\
&\oplus \mathrm{LVEC}_2\left[\tfrac{1}{2} + \sqrt{\tfrac{33}{4} - 2e^{2\varphi}},\, 0;\, 0\right] \oplus\, \mathrm{MVEC}_2\big[1,\, 0;\, 0\big] \,\oplus\, \mathrm{HYP}_2\big[2,\, \pm 2;\, 0\big]\,.
\end{aligned}
\tag{7.31}
$$

with the shorthand $s = \frac{1}{2}\sqrt{2 - e^{2\varphi}}$. This reduces to the $\ell = n = 0$ spectra on the upper boundary, [140, 182], and on Family III, (7.30). It also contains the protected multiplets of table 7.2 at $\ell = 0$ and no other short multiplet. The dimensions of all the long multiplets depend on φ, and also on χ for flavour-charged multiplets.

Still at $\ell = 0$ but now at all n, (7.31) extends into the following tower of generically long multiplets:

$$\mathrm{LGRAV}_2[\tfrac{1}{2} + \beta_1,\, 0;\, 0]$$

$$
\begin{aligned}
&\oplus \mathrm{LGINO}_2\left[\tfrac{1}{2}-s+\beta_2^+,\,0;+1\right] \oplus \mathrm{LGINO}_2\left[\tfrac{1}{2}-s+\beta_2^-,0;-1\right]\\
&\oplus \mathrm{LGINO}_2\left[\tfrac{1}{2}+s+\beta_2^+,0;+1\right] \oplus \mathrm{LGINO}_2\left[\tfrac{1}{2}+s+\beta_2^-,0;-1\right]\\
&\oplus\ \mathrm{LVEC}_2\left[\tfrac{1}{2}+\beta_3^+,\,0;\ 0\right] \oplus \mathrm{LVEC}_2\left[\tfrac{1}{2}+\beta_3^-,\,0;\ 0\right] \oplus\ \mathrm{LVEC}_2\left[\tfrac{1}{2}+\beta_4,\,0;\ 0\right]\\
&\oplus\ \mathrm{LVEC}_2\left[\tfrac{1}{2}+\beta_5^+,\,0;\ +2\right] \oplus\ \mathrm{LVEC}_2\left[\tfrac{1}{2}+\beta_5^-,\,0;\ -2\right]\,, \qquad (7.32)
\end{aligned}
$$

where we have introduced the shorthands

$$
\begin{aligned}
\beta_1^2 &= \tfrac{9}{4}+\tfrac{1}{2}e^{2\varphi}\big(\tfrac{2\pi n}{T}\big)^2\,,\\
(\beta_2^\pm)^2 &= \tfrac{1}{4}e^{-2\varphi}\left(2+e^{2\varphi}\right)^2+\tfrac{1}{2}e^{2\varphi}\left(\tfrac{2\pi n}{T}\pm\chi\right)^2\,,\\
(\beta_3^\pm)^2 &= \tfrac{17}{4}+\tfrac{1}{2}e^{2\varphi}\Big[\big(\tfrac{2\pi n}{T}\big)^2-2\Big]\pm\sqrt{(4-e^{2\varphi})^2+2\,e^{2\varphi}(2-e^{2\varphi})\big(\tfrac{2\pi n}{T}\big)^2}\,,\\
\beta_4^2 &= \tfrac{1}{4}+2e^{2\varphi}+\tfrac{1}{2}e^{2\varphi}\left(\tfrac{2\pi n}{T}\right)^2\,,\\
(\beta_5^\pm)^2 &= -\tfrac{7}{4}+4\,e^{-2\varphi}+\tfrac{1}{2}e^{2\varphi}\left(\tfrac{2\pi n}{T}\pm 2\chi\right)^2\,. \qquad (7.33)
\end{aligned}
$$

These dimensions allow for no point in (7.9) such that all values are rational for any n, and this therefore excludes the presence of a free point.

At $n=0$, the multiplet content (7.32) with (7.33) reduces to (7.31). It also reproduces the $\ell=0$, $n=0,\pm1,\pm2,\dots$ towers at the upper boundary, (4.25) of [140], and on Family III when $e^{2\varphi}=2$ and $e^{2\varphi}=1$, respectively. In particular, as $\varphi\to0$, $\chi\to0$, the multiplets in (7.32), (7.33) yield

$$
\begin{aligned}
\mathrm{LGRAV}_2\left[\tfrac{1}{2}+\beta_1,0;0\right] &\to \mathrm{LGRAV}_2\left[\tfrac{1}{2}+\tfrac{1}{2}\sqrt{9+2\big(\tfrac{2\pi n}{T}\big)^2},0;0\right]\,,\\
\mathrm{LGINO}_2\left[\tfrac{1}{2}-s+\beta_2^\pm,0;\pm1\right] &\to \mathrm{LGINO}_2\left[\tfrac{1}{2}\sqrt{9+2\big(\tfrac{2\pi n}{T}\big)^2},0;\pm1\right],\\
\mathrm{LGINO}_2\left[\tfrac{1}{2}+s+\beta_2^\pm,0;\pm1\right] &\to \mathrm{LGINO}_2\left[1+\tfrac{1}{2}\sqrt{9+2\big(\tfrac{2\pi n}{T}\big)^2},0;\pm1\right],\\
\mathrm{LVEC}_2\left[\tfrac{1}{2}+\beta_3^\pm,0;0\right] &\to \mathrm{LVEC}_2\left[\tfrac{1}{2}\pm1+\tfrac{1}{2}\sqrt{9+2\big(\tfrac{2\pi n}{T}\big)^2},0;0\right],\\
\mathrm{LVEC}_2\left[\tfrac{1}{2}+\beta_4,0;0\right] &\to \mathrm{LVEC}_2\left[\tfrac{1}{2}+\tfrac{1}{2}\sqrt{9+2\big(\tfrac{2\pi n}{T}\big)^2},0;0\right],\\
\mathrm{LVEC}_2\left[\tfrac{1}{2}+\beta_5^\pm,0;\pm2\right] &\to \mathrm{LVEC}_2\left[\tfrac{1}{2}+\tfrac{1}{2}\sqrt{9+2\big(\tfrac{2\pi n}{T}\big)^2},0;\pm2\right], \qquad (7.34)
\end{aligned}
$$

and thus reproduce via (E.18) the $\ell=0$, $n=0,\pm1,\dots$ tower at the $\mathcal{N}=4$ point, (7.20) with $\ell=\ell_1=\ell_2=0$. When $e^{2\varphi}\to2$, $\chi\to0$, the multiplet content instead reproduces the $\ell=0$, $n=0,\pm1,\dots$ tower at $\mathrm{SU}(2)_F$ point 1, (7.13), in agreement with [140]. This occurs through the recombinations

$$
\begin{aligned}
\text{LGRAV}_2\left[\tfrac{1}{2}+\beta_1,0;0\right] \quad &\rightarrow \quad \text{LGRAV}_2\left[\tfrac{1}{2}+\tfrac{1}{2}\sqrt{9+2\big(\tfrac{2\pi n}{T}\big)^2},0\right]\otimes[0],\\
\left.\begin{array}{r}\text{LGINO}_2\left[\tfrac{1}{2}\pm s+\beta_2^+,0;+1\right]\\ \text{LGINO}_2\left[\tfrac{1}{2}\pm s+\beta_2^-,0;-1\right]\end{array}\right\} \quad &\rightarrow \quad 2\,\text{LGINO}_2\left[\tfrac{1}{2}+\sqrt{2+\big(\tfrac{2\pi n}{T}\big)^2},0\right]\otimes[\tfrac{1}{2}],\\
\left.\begin{array}{r}\text{LVEC}_2\left[\tfrac{1}{2}+\beta_3^+,0;0\right]\\ \text{LVEC}_2\left[\tfrac{1}{2}+\beta_4,0;0\right]\end{array}\right\} \quad &\rightarrow \quad 2\,\text{LVEC}_2\left[\tfrac{1}{2}+\sqrt{\tfrac{17}{4}+\big(\tfrac{2\pi n}{T}\big)^2},0\right]\otimes[0]\,,\\
\left.\begin{array}{r}\text{LVEC}_2\left[\tfrac{1}{2}+\beta_5^+,0;+2\right]\\ \text{LVEC}_2\left[\tfrac{1}{2}+\beta_3^-,0;0\right]\\ \text{LVEC}_2\left[\tfrac{1}{2}+\beta_5^-,0;-2\right]\end{array}\right\} \quad &\rightarrow \quad \text{LVEC}_2\left[\tfrac{1}{2}+\sqrt{\tfrac{1}{4}+\big(\tfrac{2\pi n}{T}\big)^2},0\right]\otimes[1]\,,
\end{aligned}
\tag{7.35}
$$

in the notation of (7.18), as usual. In our conventions, the $\text{U}(1)_F \subset \text{SU}(2)_F$ charges are normalised to be integers so that, for example, the $[\frac{1}{2}]$ of $\text{SU}(2)_F$ breaks into ± 1 $\text{U}(1)_F$ charges.

For the tower $\ell = 1$ with any integer n, we can provide analytic expressions for the dimensions of the LGRAV_2 multiplets contained therein. From (7.20) and (7.24), these gravitons are controlled by the SO(4) labels (ℓ_1, ℓ_2) given by $(1,0)$ or $(0,1)$. Altogether, the multiplets are

$$
\begin{aligned}
&2\times\text{LGRAV}_2\big[\tfrac{1}{2}+\gamma_1,\,0;0\big]\\
&\oplus\ \text{LGRAV}_2\big[\tfrac{1}{2}+\gamma_2^+,\,+1;+1\big]\oplus\text{LGRAV}_2\big[\tfrac{1}{2}+\gamma_2^-,\,+1;-1\big]\\
&\oplus\ \text{LGRAV}_2\big[\tfrac{1}{2}+\gamma_2^+,\,-1;+1\big]\oplus\text{LGRAV}_2\big[\tfrac{1}{2}+\gamma_2^-,\,-1;-1\big]\,,
\end{aligned}
\tag{7.36}
$$

where we have defined

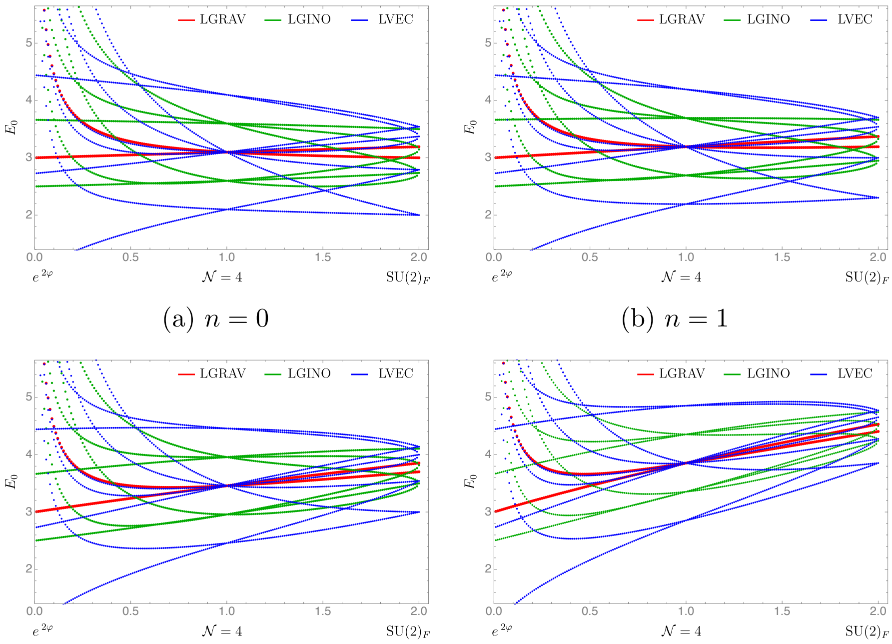
$$
(\gamma_1)^2=\tfrac{25}{4}+\tfrac{e^{2\varphi}}{2}\big[\big(\tfrac{2\pi n}{T}\big)^2+1\big]\,,\qquad (\gamma_2^{\pm})^2=\tfrac{23}{4}+e^{-2\varphi}+\tfrac{e^{2\varphi}}{2}\big(\tfrac{2\pi n}{T}\pm\chi\big)^2\,. \tag{7.37}
$$

It is again instructive to see how these expressions reduce to the known towers on the points with enhanced (super)symmetry. At the SO(4) point, all of the multiplets in (7.36) degenerate with dimension

$$
E_0=\tfrac{1}{2}+\sqrt{\tfrac{27}{4}+\big(\tfrac{2\pi n}{T}\big)^2}\,. \tag{7.38}
$$

This agrees with (7.21), with $\ell = 1$ and $(\ell_1, \ell_2) = (1,0)$ or $(0,1)$ there. At the SU(2) point $e^{2\varphi} = 2$, $\chi = 0$, the graviton multiplets (7.36), (7.37) recombine as

$$
\begin{aligned}
\text{LGRAV}_2\big[\tfrac{1}{2}+\gamma_1,0;0\big] \quad &\rightarrow \quad \text{LGRAV}_2\big[\tfrac{1}{2}+\sqrt{\tfrac{29}{4}+\big(\tfrac{2\pi n}{T}\big)^2},0\big]\otimes[0]\,,\\
\left.\begin{array}{r}\text{LGRAV}_2\big[\tfrac{1}{2}+\gamma_2^+,\pm1;+1\big]\\ \text{LGRAV}_2\big[\tfrac{1}{2}+\gamma_2^-,\pm1;-1\big]\end{array}\right\} \quad &\rightarrow \quad \text{LGRAV}_2\big[\tfrac{1}{2}+\sqrt{\tfrac{25}{4}+\big(\tfrac{2\pi n}{T}\big)^2},\pm1\big]\otimes[\tfrac{1}{2}],
\end{aligned}
\tag{7.39}
$$



(a) $n = 0$ (b) $n = 1$

(c) $n = 2$ (d) $n = 3$

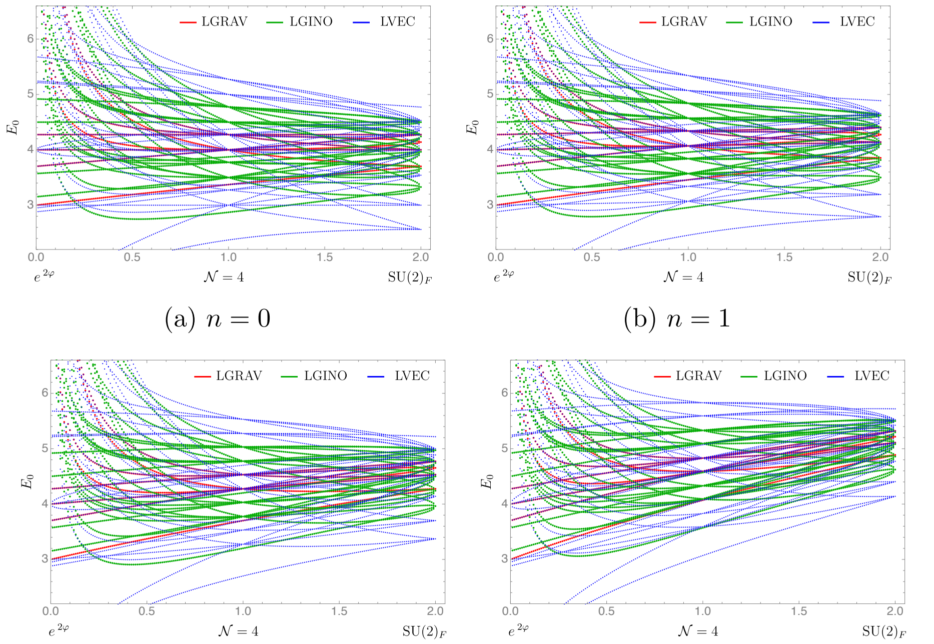
Figure 7.2: $\mathcal{N} = 2$ multiplets at KK levels $\ell = 1$ and $n = 0, \ldots, 3$ on Family II at $\chi = 0$, with $T = 2\pi$.

again matching the result in [140].

Moving up in S^5 KK level, the multiplet content at the tower $\ell = 2$ and any n follows from the (ℓ_1, ℓ_2) pairs $(0, 0)$, $(1, 1)$, $(2, 0)$ and $(0, 2)$ in (7.24). Analytic expressions for these graviton multiplets where given in [F], reducing appropriately again to the previously obtained formulae at the points with enhanced (super)symmetry.

For low values of the KK levels up to $\ell = |n| = 3$, we have recomputed numerically the spectrum of graviton multiplets at a grid of locations in the CM, and our results agree with the analytic expressions above. We have also determined numerically on this grid the remaining contributions, from LGINO_2's and LVEC_2's, to the KK spectrum at those levels. The complete results were presented in separate files in [F] (see appendix A therein). Here, we only provide figures 7.2 and 7.3 as graphical summaries of those calculations, on the representative one-parameter locus on the CM corresponding to Family II at $\chi = 0$. These plots show the dependence on the modulus φ of the dimensions of all long graviton, gravitino and vector OSp(2|4) multiplets present in the spectrum at S^5 levels $\ell = 1$ (in figure 7.2) and $\ell = 2$ (in figure 7.3), for various choices of the S^1 KK level n.

Our numerical results across the interior of the holographic CM are compatible with the shortening patterns of table 7.2. Reciprocally, we do not see any other accidental shortenings taking place, at least on our grid.



(a) $n=0$ (b) $n=1$

(c) $n=2$ (d) $n=3$

Figure 7.3: $\mathcal{N}=2$ multiplets at KK levels $\ell=2$ and $n=0,\ldots,3$ on Family II at $\chi=0$, with $T=2\pi$.

At level $\ell=2$ we indeed see moduli-independent multiplet dimensions given, within numerical precision, by the integers specified in the table. Curiously, we also see other integer multiplet dimensions arising on certain loci of the CM, although these should not be regarded as particularly significant, as they were obtained for fixed $T=2\pi$. For example, at fixed $e^{2\varphi}=\frac{3}{2}$ and all χ, there is a (flavour neutral) LVEC_2 with dimension $E_0=3$ that arises at KK levels $\ell=0$ and $n=2$. This multiplet contains classically marginal, $\Delta=3$, scalars which, however, cannot become exactly marginal because the multiplet lies above the unitarity bound and thus must be long. Also on this locus, and on the $e^{2\varphi}=\frac{6}{5}$, χ free locus, there are LGRAV_2's arising at $(\ell,n)=(2,2)$ and $(\ell,n)=(1,3)$, respectively, with $E_0=4$. The latter locus has an LVEC_2 with $E_0=6$ at $(\ell,n)=(3,3)$, and there is also a LVEC_2 with $E_0=4$ at $(\ell,n)=(2,2)$ on the family $e^{2\varphi}=\frac{8}{5}$ with χ free. The points $(e^{2\varphi},\chi)=(\frac{6}{5},0)$ and $(e^{2\varphi},\chi)=(\frac{5}{4},1)$ have LVEC_2's with $E_0=5$ and $E_0=7$, arising in both cases at $(\ell,n)=(3,1)$. This list is presumably not exhaustive. Finally, in the region $1\leq e^{2\varphi}\leq 2$ for all χ, all relevant or marginal, $\Delta\leq 3$, scalars arise at KK levels up to $\ell=2$: at KK levels $\ell=3$, all scalars have dimensions $\Delta>3$ for all n. For $0<e^{2\varphi}<1$, there are $\Delta\leq 3$ scalars even at $\ell=3$. Our numerical calculations fix $T=2\pi$ for simplicity, but the results do not differ qualitatively from those with the more realistic k-dependent choices for T in [28, 176].

Our numerics show that for all values of the parameters within the ranges specified in (7.9), the KK spectra are well behaved. As the singular limit $e^{2\varphi} = 0$ with χ arbitrary is approached, the multiplet dimensions become independent of χ even for flavoured multiplets. The dimensions also become independent of the S^1 KK level n. For example, from (7.32), the $\ell = 0$ spectrum on this asymptotic locus becomes, for all $n = 0, \pm 1, \pm 2, \ldots$,

$$\begin{aligned}\text{MGRAV}_2\big[2,\, 0;\, 0\big] \,\oplus\, \text{SGINO}_2\big[\tfrac{5}{2},\, \pm 1;\, 0\big] \,\oplus\, 2\times \text{LGINO}_2\big[e^{-\varphi} + \ldots,\, 0;\, \pm 1\big] \\ \oplus\, \text{LVEC}_2\big[2e^{-\varphi} + \ldots,\, 0;\, \pm 2\big] \,\oplus\, \text{LVEC}_2\big[\tfrac{1}{2}\big(1+\sqrt{33}\big),\, 0;\, 0\big] \\ \oplus\, 2\times\text{MVEC}_2\big[1,\, 0;\, 0\big] \,\oplus\, 2\times\text{HYP}_2\big[2,\, \pm 2;\, 0\big]\,, \qquad (7.40)\end{aligned}$$

after again writing all the multiplets at the $\mathcal{N} = 2$ unitarity bound as short. The dimensions of the flavour-neutral multiplets reduce to finite constants, but those with flavour f appear to grow without bound as $|f|\, e^{-\varphi}$. This is apparent from (7.40) for the $\ell = 0$, $n = 0, \pm 1, \pm 2, \ldots$ tower (the ellipses in (7.40) denote subleading terms with respect to that behaviour), and is further confirmed at higher ℓ, at least for the graviton multiplets, by the $e^{2\varphi} \to 0$ limit of (7.36). Note also that (7.40) contains an infinite tower, $n = 0, \pm 1, \pm 2, \ldots$, of massless scalars, vectors and gravitons.

7.2.2 The holographic conformal manifold

On the interior of the conformal manifold, it has not been possible to establish in general the analytical functional dependence of the KK dimensions either on the modulus φ or on the quantum numbers ℓ, ℓ_1, ℓ_2 (or possibly others). However, the dependence on χ of the dimension of a multiplet with flavour charge f arising at S^1 KK level n is always locked into the combination (*c.f.* (7.33) and (7.37))

$$\big(\tfrac{2\pi n}{T} + f\chi\big)^2\,, \qquad (7.41)$$

across the entire CM. This combination was noted in [140] to hold for the KK spectrum on Family I, but it does extend at all other points in the CM.

There are two immediate consequences of the χ-dependence (7.41) of the multiplet dimensions. Firstly, a multiplet in the spectrum is flavour neutral if and only if its dimension is independent of the modulus χ. Secondly, the dependence (7.41) establishes the periodic behaviour of the multiplet dimensions in χ advertised in (7.9) and depicted in figure 7.1. Indeed, for all fixed S^5 KK level ℓ, the dimension of any given multiplet with flavour f, evaluated at $\chi = \chi_0$ and S^1 KK level n, coincides with the dimension of the same multiplet evaluated at $\chi = \chi_0 + \frac{2\pi}{T}$ and S^1 level n', with

$$n' = n - f\,. \qquad (7.42)$$

Such integer n' always exists given n and f because, as (7.24), (7.25) show, the flavour charges are also integer (in our conventions). As noted in footnote

4 of this chapter, only the dimensions, but not the multiplet content (7.24) itself, depend on n. For this reason, the entire contribution (7.24) to the spectrum at KK level ℓ goes back to itself as χ ranges from 0 to $2\pi/T$. Only the S^1 KK level needs to be readjusted as χ reaches each endpoint of its cycle. As remarked in [140], this mixture of KK levels is reminiscent of the 'space invaders scenario' described in [24] and also encountered in chapter 5.

To see how this mechanism works, let us analyse the recovery of the $\mathcal{N}=4$ point [167] in detail. At $\chi=0$, (7.30) reduces to the $\ell=n=0$ spectrum at the $\mathcal{N}=4$ point [167], branched out under (7.23) into $\mathcal{N}=2$ representations through (E.24). At $\chi=2\pi/T$ the $\mathcal{N}=4$ spectrum at lowest KK levels is also reproduced, but with reshuffled S^1 levels. In order to make this more apparent, it is convenient to extract the KK tower with $\ell=0$ and $n=0,\pm1,\pm2,\ldots$, for Family III (7.28), (7.29). The result,

$$\begin{aligned}
&\mathrm{LGRAV}_2\big[\tfrac{1}{2}+\tfrac{1}{2}\sqrt{9+2\big(\tfrac{2\pi n}{T}\big)^2},0;0\big]\\
&\oplus\mathrm{LGINO}_2\big[\tfrac{1}{2}\sqrt{9+2\big(\tfrac{2\pi n}{T}\pm\chi\big)^2},0;\pm1\big]\oplus\mathrm{LGINO}_2\big[1+\tfrac{1}{2}\sqrt{9+2\big(\tfrac{2\pi n}{T}\pm\chi\big)^2},0;\pm1\big]\\
&\oplus\mathrm{LVEC}_2\big[\tfrac{1}{2}+\tfrac{1}{2}\sqrt{9+2\big(\tfrac{2\pi n}{T}\pm2\chi\big)^2},0;\pm2\big]\oplus\mathrm{LVEC}_2\big[\tfrac{1}{2}+\tfrac{1}{2}\sqrt{9+2\big(\tfrac{2\pi n}{T}\big)^2},0;0\big]\\
&\oplus\mathrm{LVEC}_2\big[\tfrac{3}{2}+\tfrac{1}{2}\sqrt{9+2\big(\tfrac{2\pi n}{T}\big)^2},0;0\big]\oplus\mathrm{LVEC}_2\big[-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{9+2\big(\tfrac{2\pi n}{T}\big)^2},0;0\big]\,,
\end{aligned}\tag{7.43}$$

reduces to (7.30) at $n=0$ and extends that equation to all other n. Here and elsewhere, the presence in a multiplet of two labels with $\pm$ signs indicates the existence of two (not four) multiplets with correlated upper and lower signs (note incidentally that, at $|n|\neq0$ fixed, each of these appears twice like any other multiplet, once for each sign of n). All the multiplets present in (7.43) are generically long, and the dimension of those with non-zero $\mathrm{U}(1)_F$ charge develops a χ dependence, as usual. At $\chi=0$, (7.43) reproduces the $\ell=0$, $n=0,\pm1,\pm2,\ldots$ tower at the $\mathcal{N}=4$ point, (7.19)-(7.21) with $\ell=\ell_1=\ell_2=0$ therein, through the branching (E.18). At $\chi=2\pi/T$, (7.43) also recombine into $\mathcal{N}=4$ multiplets through (E.18), possibly retrieved from different KK levels n. For example, the $\mathrm{LGINO}_2\big[\tfrac{1}{2}\sqrt{9+2\big(\tfrac{2\pi n}{T}\pm\chi\big)^2},\,0;\,\pm1\big]$ multiplets in (7.43) are indeed long at $\chi=0$, but at $\chi=2\pi/T$ become massless for KK levels $n=\mp1$. For that value of χ, these join the flavour-neutral (and thus χ-independent) $\mathrm{MGRAV}_2[2,0;0]$ and $\mathrm{MVEC}_2[1,0;0]$ that arise at level $n=0$ in (7.30) into an $\mathrm{MGRAV}_4[1,0,0]$ through (E.24). See figure 7.4 for a graphical account of these 'space invasion' patterns.

By the above analysis, the supermultiplets on Family III recombine into $\mathcal{N}=4$ supermultiplets at both endpoints of the χ range (7.9). It is also informative to look at the individual states contained in those multiplets, and see how two gravitino states become 'massless' (or rather, acquire AdS_4 mass $mL=1$ so that their dimension becomes $\Delta=\frac{5}{2}$) at $\chi=2\pi/T$, thus enhancing the generic $\mathcal{N}=2$ supersymmetry on Family III to $\mathcal{N}=4$. Four

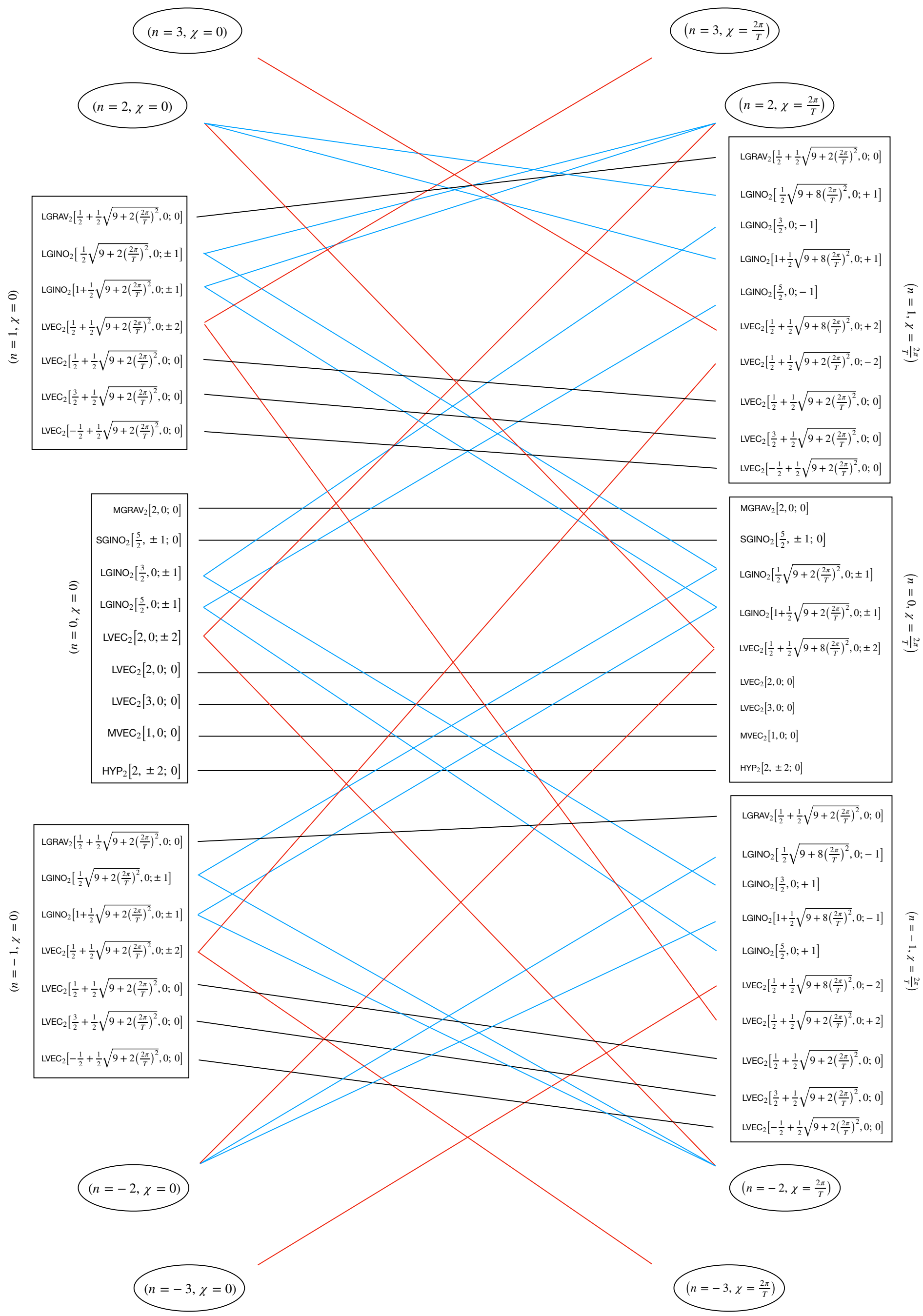
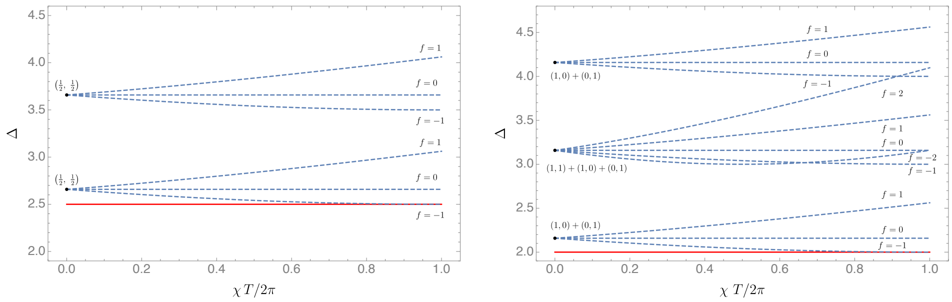


Figure 7.4: 'Space invasion' patterns for the reassembling of the $\mathrm{OSp}(2|4) \times \mathrm{U}(1)_F$ multiplets present in the KK spectrum on Family III, (7.16), within the CM at KK levels $\ell = 0$, $n = 0, \pm1, \pm2, \ldots$, into $\mathrm{OSp}(4|4)$ multiplets at the same S^5 level $\ell = 0$ but possibly different S^1 level n, at $\chi = 0$ (left) and $\chi = 2\pi/T$ (right). The boxes correspond to the multiplet content in (7.28) with $\ell_1 = \ell_2 = 0$ and n fixed as indicated. Black lines connect χ-independent, flavour-neutral $\mathcal{N} = 2$ multiplets. Blue and red lines respectively connect $\mathcal{N} = 2$ multiplets that need to be retrieved from one or two higher (or lower) S^1 KK levels.



(a) Gravitini (b) Vectors

Figure 7.5: Dimensions Δ (dashed blue lines) of individual gravitino (left) and vector (right) states with flavour f in the spectrum on Family III, at KK levels $\ell = 0$, $n = 1$, as functions of χ. The solid red lines stand at the massless threshold. At $\chi = 0$, the SO(4) representations from which the flavoured states branch down are shown.

KK vector states must also become massless, $\Delta = 2$, in order for the bosonic symmetry to get enhanced from $\mathrm{U}(1)_F \times \mathrm{U}(1)_R$ to SO(4). The evolution with χ of these gravitino and vector mass eigenstates on Family III, as they arise from the diagonalisation of the $\ell = 0$, $n = 1$ mass matrices of [134, 136], is depicted in figure 7.5. The left plot indeed identifies one gravitino with flavour $f = -1$ that branches out from the $(\frac{1}{2}, \frac{1}{2})$ SO(4) mode with $\Delta = 1 + \sqrt{\frac{9}{4} + \frac{2\pi^2}{T^2}}$ at $\chi = 0$ (and $T = 2\pi$ in the plot), and reaches $\chi = 2\pi/T$ with $\Delta = \frac{5}{2}$. The other relevant gravitino, not depicted, has $f = 1$ and becomes massless at $n = -1$. A similar story unfolds for the vectors on the right plot. Two vector states (superimposed in the plot) with flavour $f = -1$ branch out from the $(1, 0) + (0, 1)$ of SO(4) at $\chi = 0$ and become massless at $\chi = 2\pi/T$. Two more vectors, not depicted, with flavour $f = 1$ become massless for $n = -1$, while the two vectors that gauge $\mathrm{U}(1)_F \times \mathrm{U}(1)_R$ stay massless all along.

Type IIB uplift of Family III

Some aspects of the holographic CM, like the symmetry enhancement (7.14) and the periodicity in χ determined through the KK spectra cannot be seen in $D = 4$ gauged supergravity, which has fixed $n = 0$, but are an intrinsic feature of the fully-fledged type IIB ten-dimensional solution. Determining the type IIB uplift of the entire two-parameter family of $D = 4$ gauged supergravity vacua of [183] is beyond the scope of this work. Previously known uplifts into type IIB S-folds of points or loci in the holographic CM include that of the $\mathcal{N} = 4$ SO(4) point [28], the $\mathcal{N} = 2$ $\mathrm{SU}(2)_F \times \mathrm{U}(1)_R$ point [182], and Family I [140]. Here we will give the uplift of Family III to explain the $\mathcal{N} = 4$ SO(4) (super)symmetry enhancements (7.12) under complete cycles of χ.

The type IIB uplift of any solution of $D = 4$ $\mathcal{N} = 8$ $[\mathrm{SO}(6) \times \mathrm{SO}(1,1)] \ltimes \mathbb{R}^{12}$-gauged supergravity may be obtained using the ExFT approach reviewed in section 6.2.2 with the explicit formulae of [28]. However, we have not followed this route to obtain the type IIB solutions corresponding to Family III. Instead, we have used the following reverse engineering approach. Firstly, we wrote an educated guess for the ten-dimensional metric, and confirmed it by reproducing the graviton sector of the KK spectrum of section **??** using the formalism of [114]. Secondly, we wrote ansätze for the remaining supergravity fields, and enforced the type IIB field equations on the full configuration. In retrospect, the successful reproduction of the graviton spectrum for Family III using [114], together with the fact that all KK modes close into $\mathrm{OSp}(2|4) \times \mathrm{U}(1)_F$ representations, provides a solid crosscheck on our implementation of the ExFT spectral techniques [133, 136, 137] presented in section 6.2.3.

In order to write the type IIB solutions, it is convenient to employ the same coordinates, η on S^1 and (r, θ_i, ϕ_i), $i = 1, 2$, on S^5, used in [28] to express the $\mathcal{N} = 4$ SO(4)-invariant solution. These coordinates range as

$$0 \le \eta < T\,, \qquad 0 \le r \le 1\,, \qquad 0 \le \theta_i \le \tfrac{\pi}{2}, \qquad 0 \le \phi_i < 2\pi\,, \quad i = 1, 2\,. \tag{7.44}$$

In particular, η and ϕ_i are periodic with periods T and 2π,

$$\eta \sim \eta + T\,, \qquad \phi_i \sim \phi_i + 2\pi\,, \quad i = 1, 2\,. \tag{7.45}$$

It is also helpful to introduce the following χ-dependent one-

$$\boldsymbol{e}_1 = d\phi_1 - \chi\, d\eta\,, \qquad \boldsymbol{e}_2 = d\phi_2 + \chi\, d\eta\,, \tag{7.46}$$

and two-forms

$$\boldsymbol{v}_1 = \tfrac{r^2}{1+2r^2} \sin\theta_1\, d\theta_1 \wedge \boldsymbol{e}_1\,, \qquad \boldsymbol{v}_2 = \tfrac{1-r^2}{3-2r^2} \sin\theta_2\, d\theta_2 \wedge \boldsymbol{e}_2\,. \tag{7.47}$$

The relative signs in (7.46) have been chosen to match the flavour group in (7.10), but any other choice of signs will also yield a solution of the equations of motion by reparameterisation invariance.

With these definitions, the type IIB uplift of $D = 4$ Family III of vacua can be written as follows. The metric reads

$$\begin{aligned} ds^2_{10} = L^2 \Delta^{-1} \Big[& ds^2(\mathrm{AdS}_4) + 2d\eta^2 + \frac{2dr^2}{1-r^2} \\ & + \frac{2r^2}{1+2r^2} \left[d\theta_1^2 + \sin^2\!\theta_1\, \boldsymbol{e}_1^2 \right] + \frac{2(1-r^2)}{3-2r^2} \left[d\theta_2^2 + \sin^2\!\theta_2\, \boldsymbol{e}_2^2 \right] \Big]\,, \end{aligned} \tag{7.48}$$

and the self-dual five-form is

$$F_{(5)} = L^4 \Big[6\, \mathrm{vol}_4 \wedge \Big(\frac{4}{3} r dr - d\eta \Big) + \frac{6}{\sqrt{1-r^2}}\, dr \wedge \boldsymbol{v}_1 \wedge \boldsymbol{v}_2 + 8r\sqrt{1-r^2}\, \boldsymbol{v}_1 \wedge \boldsymbol{v}_2 \wedge d\eta \Big]\,. \tag{7.49}$$

Here, $ds^2(\text{AdS}_4)$ and vol_4 are the metric and volume form on unit radius AdS_4 space and L is related both to the electric gauge coupling g of the $D = 4$ $\mathcal{N} = 8$ supergravity as $L^2 \equiv \frac{1}{2}g^{-2}$, and to the dual gauge group rank N as $L^4 \sim N$ upon flux quantisation. The warp factor depends only on the coordinate r,

$$\Delta = \Big((1+2r^2)(3-2r^2)\Big)^{-\frac{1}{4}}\,, \tag{7.50}$$

while the dilaton and axion depend also on η:

$$\begin{aligned} e^{-\Phi} &= \frac{\sqrt{2}\sqrt{(1+2r^2)(3-2r^2)}}{(3+2r^3)\cosh 2\eta + 4r^2\sinh 2\eta}\,, \\ C_{(0)} &= \frac{4r^2\cosh 2\eta + (3+2r^3)\sinh 2\eta}{(3+2r^3)\cosh 2\eta + 4r^2\sinh 2\eta}\,. \end{aligned} \tag{7.51}$$

Finally, the Neveu-Schwarz and Ramond-Ramond three-form field strengths are

$$\begin{aligned} H_{(3)} &= 4L^2\Big[-3^{-\frac{1}{4}}\,e^{-\eta}\left(\frac{(3+2r^2)}{(1+2r^2)}dr - rd\eta\right)\wedge \boldsymbol{v}_1 \\ &\qquad + 3^{\frac{1}{4}}\,e^{\eta}\left(\frac{(5-2r^2)}{(3-2r^2)}\frac{r}{\sqrt{1-r^2}}dr\wedge\boldsymbol{v}_2 - \sqrt{1-r^2}\,\boldsymbol{v}_2\wedge d\eta\right)\Big]\,, \\ F_{(3)} &= \tilde{F}_{(3)} - C_{(0)}H_{(3)}\,, \end{aligned} \tag{7.52}$$

where

$$\begin{aligned} \tilde{F}_{(3)} &= 4L^2\Big[3^{-\frac{1}{4}}\,e^{-\eta}\left(\frac{(3+2r^2)}{(1+2r^2)}dr - r\,d\eta\right)\wedge \boldsymbol{v}_1 \\ &\qquad + 3^{\frac{1}{4}}\,e^{\eta}\left(\frac{(5-2r^2)}{(3-2r^2)}\frac{r}{\sqrt{1-r^2}}dr\wedge\boldsymbol{v}_2 - \sqrt{1-r^2}\,\boldsymbol{v}_2\wedge d\eta\right)\Big]\,. \end{aligned} \tag{7.53}$$

We also note the following expressions for the two-form potentials,

$$\begin{aligned} B_{(2)} &= 4L^2\Big(-3^{-\frac{1}{4}}\,e^{-\eta}r\,\boldsymbol{v}_1\; - 3^{\frac{1}{4}}\,e^{\eta}\sqrt{1-r^2}\,\boldsymbol{v}_2\Big)\,, \\ C_{(2)} &= 4L^2\Big(3^{-\frac{1}{4}}\,e^{-\eta}r\,\boldsymbol{v}_1\; - 3^{\frac{1}{4}}\,e^{\eta}\,\sqrt{1-r^2}\,\boldsymbol{v}_2\Big)\,, \end{aligned} \tag{7.54}$$

such that $H_{(3)} = dB_{(2)}$ and $\tilde{F}_{(3)} = dC_{(2)}$.

We have verified that (7.48)-(7.53) solve the equations of motion and Bianchi identities of type IIB supergravity, as given in *e.g.* appendix A of [188]. This configuration thus defines a one-parameter family, labelled by the constant χ, of non-geometric S-fold solutions of type IIB supergravity. At both endpoints of the interval in (7.44) for the S^1 coordinate η, the fields (7.51), (7.52) charged under $\text{SL}(2,\mathbb{R})$ (or $\text{SL}(2,\mathbb{Z})$ in the full string theory), are related by an $\mathrm{SL}(2,\mathbb{R})$ (or $\mathrm{SL}(2,\mathbb{Z})$) S-duality transformation, exactly as in [28, 176]. Further, as argued in the first of these references, supersymmetry is not upset by the uplifting process as long as such S-duality transformation lies in the hyperbolic $\mathrm{SL}(2,\mathbb{Z})$ conjugacy class. Thus, the type IIB solution (7.48)-(7.53) inherits the generic $\mathcal{N}=2$ supersymmetry of the $D=4$ Family III solution it uplifts from. It also contains the $\mathcal{N}=4$ SO(4) point at $\chi=0$ and at the other locations specified below.

The type IIB solution (7.48)-(7.53) depends on the parameter χ only through the one-forms (7.46) (and the two-forms (7.47) via their dependence on the former). For all values of χ and the specified coordinate ranges (7.44), the solution extends globally over $S^5 \times S^1$, with the S^5 trivially fibred over S^1. At $\chi=0$, our solution reduces to the $\mathcal{N}=4$ SO(4)-invariant solution on $S^5 \times S^1$, (3.35)-(3.41) of [28], upon identifying $\mathcal{Y}^p$, $\mathcal{Z}^p$, $p=1,2,3$ there as

$$\begin{aligned} \{\mathcal{Y}^1\,,\ \mathcal{Y}^2\,,\ \mathcal{Y}^3\} &= r\,\{\cos\theta_1\,,\ \sin\theta_1\cos\phi_1\,,\ \sin\theta_1\sin\phi_1\}\,, \\ \{\mathcal{Z}^1\,,\ \mathcal{Z}^2\,,\ \mathcal{Z}^3\} &= \sqrt{1-r^2}\,\{\cos\theta_2\,,\ \sin\theta_2\cos\phi_2\,,\ \sin\theta_2\sin\phi_2\}\,. \end{aligned} \tag{7.55}$$

This is the type IIB counterpart of the fact that the $D=4$ Family III reduces to the four-dimensional $\mathcal{N}=4$ SO(4) vacuum of [167] when $\chi=0$. For this value of χ, the brackets in the internal portion of the ten-dimensional metric (7.48) become the round metrics on two two-spheres, S_i^2, $i=1,2$. In turn, the two-forms (7.47) become the volume forms $\mathrm{vol}(S_i^2)$, up to overall functions of r. The $\chi=0$ metric on S^5 is thus a deformation of the round, Einstein metric on the join of the two S_i^2, $i=1,2$, such that only the $\mathrm{SO}(4)\sim\mathrm{SU}(2)_1\times\mathrm{SU}(2)_2$ subgroup of SO(6) in (7.10) is preserved. Each $\mathrm{SU}(2)_i$ rotates each S_i^2, for $i=1,2$. The SO(2) isometry of S^1 is broken by the supergravity fields.

When $\chi\neq 0$, the symmetry of the solution (7.48)–(7.53) generically reduces to the $\mathrm{U}(1)_1\times\mathrm{U}(1)_2$ defined in (7.10), with $\mathrm{U}(1)_i$ generated by ∂_{ϕ_i} for $i=1,2$. Equivalently, the generic symmetry when $\chi\neq 0$ is the $\mathrm{U}(1)_R\times\mathrm{U}(1)_F$ group generated by the diagonal and anti-diagonal combinations

$$\partial_R = \partial_{\phi_1}+\partial_{\phi_2}\,, \qquad \partial_F = \partial_{\phi_1}-\partial_{\phi_2}\,, \tag{7.56}$$

as specified below (7.10). Interestingly, the change of coordinates

$$\phi_1 \longrightarrow \phi_1' = \phi_1 - \chi\eta\,, \qquad \phi_2 \longrightarrow \phi_2' = \phi_2 + \chi\eta\,, \tag{7.57}$$

with η, r, θ_i, $i=1,2$, untouched, can be used to eliminate χ locally from the solution. Generically, though, the change (7.57) is not globally well defined, *i.e.* is not a diffeomorphism, and does not generically allow one to eliminate χ globally. For specific values of χ, the change (7.57) *is* globally well defined: these are the values that render ϕ_i' periodic, $\phi_i' \sim \phi_i' + 2\pi$. Given the periods (7.45) of the original coordinates, this induces a periodic identification $\chi \sim \chi + 2\pi/T$ such that, for $\chi = 2\pi n'/T$, with n' integer, the solution (7.48)-(7.53) becomes diffeomorphic to the $\chi = 0$, $\mathcal{N} = 4$ SO(4) solution. This explains the compactness of the χ direction for Family III observed below (7.43).

7.3 Universality of traces

The mass formulae discussed in section 6.2.3 lead to spectra which exhibit a curious phenomenon of universality. In this section we will first use the explicit higher-dimensional solutions available at the SU(3)-invariant sector of the $D = 11$ and type II uplifts of the gaugings in figure 3.1 to crosscheck again the mass matrices in chapter 6 using the alternate approach based on equation (4.7). Precise combinations of these masses, which we name traces following the intuitions of chapter 6, are shown to be common to different solutions of different theories when they preserve the same (super)symmetry at the gauged supergravity level. The discussion is extended for lower-spin fields for the first time, where this form of universality is still present.

7.3.1 Graviton spectra in string theory

The KK graviton spectrum about the AdS_4 solutions of $D = 11$ supergravity and type IIB supergravity that uplift from critical points of SO(8) supergravity and dyonic $\big(\mathrm{SO}(6) \times \mathrm{SO}(1,1)\big) \ltimes \mathbb{R}^{12}$ supergravity with at least SU(3) symmetry can be computed using (4.7). Around the solutions of massive type IIA that uplift from the dyonic ISO(7) gauging this computation was performed in [143], and we bring here their results for convenience.

We find it useful to collect here some facts about the SU(3)-invariant sector of the three different gaugings of $D = 4$ $\mathcal{N} = 8$ supergravity considered in the following. The field content is of course the same for all the gaugings considered but the interactions differ. The SU(3)-invariant sector contains three scalars, φ, ϕ, a, and three pseudoscalars χ, ζ, $\tilde{\zeta}$. All these are coordinates on the submanifold (3.53),

$$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SU}(2,1)}{\mathrm{SU}(2) \times \mathrm{U}(1)} \tag{7.58}$$

of $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$, with the first factor parametrised by (φ, χ), and the second by $(\phi, a, \zeta, \tilde{\zeta})$. The Lagrangian in this sector is (3.54), with the precise form of the minimal ($D\phi$, etc.) and non-minimal ($\mathcal{R}_{\Lambda\Sigma}$, $\mathcal{I}_{\Lambda\Sigma}$) couplings of the scalars to the vectors not needed in the following. The most relevant object for us is the scalar potential V, which fixes the radius L of its AdS_4 vacua (for which $V_0 < 0$ at a critical point) as

$$L^2 = -\frac{6}{V_0}\,. \tag{7.59}$$

$\mathcal{N}$	G_0	χ	$e^{-\varphi}$	$e^{-\phi}$	a	ζ	$\tilde{\zeta}$	V_0
1	SU(3)	0	$\frac{\sqrt{5}m}{3g}$	$\sqrt{\frac{5}{6}}$	0	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$-\frac{648g^3}{25\sqrt{5}m}$
0	$\mathrm{SO}(6)_v$	0	$\frac{m}{\sqrt{2}g}$	1	0	0	0	$-\frac{8\sqrt{2}g^3}{m}$
0	SU(3)	χ	$\frac{m}{\sqrt{2}g}$	$(1-a^2)^{1/4}$	a	0	0	$-\frac{8\sqrt{2}g^3}{m}$

Table 7.3: All critical loci of $D = 4$ $\mathcal{N} = 8$ $[\mathrm{SO}(6) \times \mathrm{SO}(1,1)] \ltimes \mathbb{R}^{12}$-gauged supergravity with at least SU(3) invariance. All of these are AdS. For each point we give the residual supersymmetry $\mathcal{N}$ and bosonic symmetry G_0 within the full $\mathcal{N} = 8$ theory, their location in the parametrisation that we are using and the cosmological constant V_0. The $\mathcal{N} = 0$ $\mathrm{SO}(6)_v$ vacuum is the $\chi = a = 0$ point of the $\mathcal{N} = 0$ SU(3) critical locus.

The potential is different for each gauging. For the SU(3)-invariant sector of the purely electric SO(8) gauging [73], the potential in our conventions was already given in (3.58). For the dyonic ISO(7) gauging the, SU(3)-invariant potential reads [74]

$$V = 6g^2\Big[\frac{1}{12}e^{4\phi-3\varphi}X^3 + e^{2\phi-\varphi}\Big(X^2(Y-1) - XY\Big) + e^{\varphi}\Big(3XY(Y-1) - 2Y^2\Big)\Big]$$
$$- g\,m\,\chi\,e^{3\varphi+2\phi}\Big[6(Y-1) + e^{2\phi-2\varphi}(X-1)\Big] + \frac{1}{2}m^2 e^{3\varphi+4\phi}\,. \tag{7.60}$$

Finally, the SU(3)-invariant potential [156] for the $[\mathrm{SO}(6) \times \mathrm{SO}(1,1)] \ltimes \mathbb{R}^{12}$ gauging is

$$V = 6\,g^2 e^{\varphi}\Big[3XY(Y-1) - 2Y^2\Big] + 6\,g\,m\,\chi\,e^{3\varphi+2\phi}(Y-1)\Big[1 - e^{-4\phi}\left(Y^2+Z^2\right)\Big]$$
$$+\frac{1}{2}m^2 e^{3\varphi}\Big[e^{4\phi} + e^{-4\phi}\left(Y^2+Z^2\right)^2 - 2\left(Y^2 - 2Y + Z^2\right)\Big]\,, \tag{7.61}$$

In these potential, as in (3.44)-(3.46), g and m are the electric and magnetic gauge couplings of the parent $\mathcal{N} = 8$ supergravities. For the latter two gaugings at hand, these can be set equal, $m = g$, without loss of generality [76], which we have done. We have also employed the shorthand notations in (3.56) for frequent combinations of the scalars.

The AdS vacua of the SO(8), ISO(7) and $[\mathrm{SO}(6) \times \mathrm{SO}(1,1)] \ltimes \mathbb{R}^{12}$ $\mathcal{N} = 8$ gaugings that preserve at least the SU(3) subgroup of those gauge groups were respectively investigated in [74, 77, 156]. In our conventions, these correspond to extrema of the scalar potentials (3.58), 7.60 and (7.61). The location of these vacua in scalar space, in the notation that we are using, can be respectively found in table 3.2, table 3 of [74] (with labels + there replaced with labels v here), and table 7.3.

M-theory configurations

The class of AdS_4 solutions of $D = 11$ supergravity [16] that we are interested in was the subject of section 3.2.2. These solutions are invariant, both in $D = 4$ and in $D = 11$, under a number of subgroups of SO(8) larger than SU(3), and display supersymmetries $\mathcal{N} = 0, 1, 2, 8$. See table 3.2 for a summary. The entire spectrum about the Freund-Rubin $\mathcal{N} = 8$ SO(8)-invariant AdS_4 solution [92] has long been known [105, 110, 165] (see also [24] for a review) thanks to its supersymmetry and homogeneity. The spectrum of gravitons about the $\mathcal{N} = 2$ $\mathrm{SU}(3) \times \mathrm{U}(1)_c$-invariant solution [31] is also known [93]. Here we compute the graviton spectra for the four other AdS_4 solutions in this sector.

The starting point for our analysis is the local geometries presented in section 3.2.2. In order to simplify the calculations, we will focus on two disjoint further subsectors with symmetries G_2 and $\mathrm{SU}(4)_c$ larger than SU(3). We will obtain the graviton spectra for arbitrary constant values of the $D = 4$ scalars in those sectors. Finding the actual spectra about each individual solution will simply entail an evaluation of those formulae at the corresponding scalar vevs.

G_2-sector The G_2-invariant sector of the $D = 4$ SO(8) supergravity is attained from the SU(3) sector through the identifications (3.91). It contains an $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ dilaton-axion pair (φ, χ), and its uplift was given in (3.137). The warp factor and internal $d = 7$ geometry that feature in (4.7) are given by

$$
\begin{aligned}
e^{2A} &= e^{-\varphi} X^{1/3} \Delta_1^{2/3} L^2 \,, \\
d\bar{s}_7^2 &= g^{-2} L^{-2} \Big(e^{3\varphi} X^{-3} d\beta^2 + e^{\varphi} \Delta_1^{-1} \sin^2\beta \, ds^2(S^6) \Big) \,.
\end{aligned}
\tag{7.62}
$$

Here β is an angle on S^7, with $0 \leq \beta \leq \pi$, and $ds^2(S^6)$ is the round Einstein metric on the unit radius S^6. The dilaton φ appears explicitly in (7.62) and the axion χ appear both explicitly and through the combinations X defined in equation (3.56) and

$$
\Delta_1 = X \big(e^{2\varphi} \sin^2\beta + e^{-2\varphi} X^2 \cos^2\beta \big) \,. \tag{7.63}
$$

The geometry (7.62) is in fact invariant under the $\mathrm{SO}(7)_v$ that rotates the round S^6. When $\chi \neq 0$, the symmetry of the full $D = 11$ configuration is broken to G_2 by the supergravity four-form field strength (3.139).

For the class of geometries (7.62), the differential equation (4.7) becomes

$$
\Big[e^{-3\varphi} X^3 (\partial_\beta^2 + 6 \cot\beta \, \partial_\beta) + e^{-\varphi} \Delta_1 \sin^{-2}\beta \, \Box_{S^6} \Big] \mathcal{Y} = -g^{-2} M^2 \, \mathcal{Y} \,, \tag{7.64}
$$

where $\Box_{S^6}$ is the S^6 Laplacian. Using separation of variables,

$$
\mathcal{Y} = f \, \mathcal{Y}_k \,, \tag{7.65}
$$

where $f = f(\beta)$ depends only on β and $\mathcal{Y}_k$ are the S^6 spherical harmonics,

$$\Box_{S^6}\mathcal{Y}_k = -k(k+5)\mathcal{Y}_k\,, \tag{7.66}$$

the PDE (7.64) reduces to an ODE for $f(\beta)$,

$$e^{-3\varphi}X^3(f''(\beta)+6\cot\beta f'(\beta))-e^{-\varphi}\Delta_1\sin^{-2}\beta\, k(k+5)f(\beta) = -g^{-2}M^2 f(\beta)\,, \tag{7.67}$$

where a prime denotes derivative with respect to β. Finally, it is convenient to introduce a further change of variables,

$$u = \cos^2\beta\,, \qquad f(u) = (1-u)^{\frac{k}{2}}H(u)\,. \tag{7.68}$$

The independent variable u now ranges in $0 \leq u \leq 1$, covering this interval twice given the range of β below (7.62). In the variables (7.68), the differential equation (7.67) takes on the standard hypergeometric form

$$u(1-u)H'' + (c-(1+a_+ + a_-)u)H' - a_+a_-H = 0\,, \tag{7.69}$$

with

$$\begin{aligned} a_\pm &= \tfrac{1}{2}(k+3) \mp \tfrac{1}{2}\sqrt{9+e^{3\varphi}X^{-3}\,g^{-2}M^2+(1-e^{4\varphi}X^{-2})k(k+5)}\,, \\ c &= \tfrac{1}{2}\,. \end{aligned} \tag{7.70}$$

The two linearly independent solutions to (7.69) are given by the hypergeometric functions

$${}_2F_1(a_+,a_-,c;u) \quad \text{and} \quad u^{1-c}\,{}_2F_1(1+a_+-c,1+a_--c,2-c;u)\,. \tag{7.71}$$

Both solutions are regular at $u=0$ for all values of the parameters (7.70). At $u=1$, however, regularity imposes restrictions on the parameters. Regularity of the first solution in (7.71) demands $a_+ = -j$ with j a non-negative integer. Bringing this condition to (7.70), we find a first tower of KK graviton squared masses:

$$g^{-2}M^2_{(1)\,j,k} = e^{-3\varphi}X^3(2j+k)(2j+k+6)+e^{-\varphi}X(e^{2\varphi}-e^{-2\varphi}X^2)k(k+5)\,. \tag{7.72}$$

The corresponding eigenfunctions are given by (7.65), (7.68), with $H(u)$ given by the first choice in (7.71), namely

$$\mathcal{Y}_{(1)\,j,k} = \mathcal{Y}_k\,\sin^k\beta\sum_{s=0}^{j}(-1)^s\binom{j}{s}\frac{(j+k+3)_s}{(\frac{1}{2})_s}\cos^{2s}\beta \tag{7.73}$$

(no sum in k), where

$$(x)_s = \begin{cases} 1 & ,\ \text{if } s=0 \\ x(x+1)\cdots(x+s-1) & ,\ \text{if } s>0 \end{cases} \tag{7.74}$$

is the Pochhammer symbol. Regularity of the second solution in (7.71) at $u = 1$ in turn requires $1 + a_+ - c = -j$, with j again a non-negative integer. Bringing this condition to (7.70), we find a second tower of KK graviton squared masses:

$$g^{-2}M^2_{(2)\,j,k} = e^{-3\varphi}X^3(2j+1+k)(2j+1+k+6)+e^{-\varphi}X(e^{2\varphi}-e^{-2\varphi}X^2)k(k+5)\,. \tag{7.75}$$

The associated eigenfunctions are now given by (7.65), (7.68), with $H(u)$ given by the second choice in (7.71):

$$\mathcal{Y}_{(2)\,j,k} = \mathcal{Y}_k \,\sin^k\beta \sum_{s=0}^{j}(-1)^s\binom{j}{s}\frac{(j+k+4)_s}{(\frac{3}{2})_s}\cos^{2s+1}\beta\,. \tag{7.76}$$

The eigenvalues (7.72) and (7.75) actually correspond to a unique tower of KK graviton masses. This is made apparent by introducing a new quantum number n defined as

$$n = \begin{cases} 2j+k & , \text{ for the first branch} \\ 2j+1+k & , \text{ for the second branch}\,. \end{cases} \tag{7.77}$$

In terms of (n, k), (7.72) and (7.75) can be combined into the single KK tower:

$$g^{-2}M^2_{n,k} = e^{-3\varphi}X^3 n(n+6) + e^{-\varphi}X(e^{2\varphi} - e^{-2\varphi}X^2)k(k+5)\,, \tag{7.78}$$

which is our final result. The quantum numbers range here as

$$n = 0, 1, 2, \ldots\,, \qquad k = 0, 1, \ldots, n\,. \tag{7.79}$$

Only n ranges freely over the non-negative integers, due to its definition (7.77)) in terms of the non-negative but otherwise unconstrained integer j. The range of k is limited to $k \leq n$ by (7.77). At fixed n, the eigenvalue (7.78) occurs with degeneracy

$$D_{k,7} \equiv \dim\,[k, 0, 0]_{\mathrm{SO}(7)}\,, \tag{7.80}$$

where, more generally, $D_{k,N}$ is the dimension of the symmetric traceless representation $[k, 0, \ldots, 0]$ of SO(N),

$$\begin{aligned} D_{k,N} &= \binom{k+N-1}{k} - \binom{k+N-3}{k-2} \\ &= \tfrac{1}{(N-2)!}\,(2k+N-2)(k+N-3)(k+N-4)\cdots(k+2)(k+1)\,, \end{aligned} \tag{7.81}$$

for $k \geq 2$ and

$$D_{0,N} = 1\,, \qquad D_{1,N} = N\,, \qquad \text{for all } N = 2, 3\ldots \tag{7.82}$$

It is also useful to note that

$$D_{n,N-1} = D_{n,N} - D_{n-1,N}\,, \qquad \text{for all } n = 1,2,\dots \text{ and all } N = 2,3\dots \tag{7.83}$$

The eigenfunctions (7.73), (7.76) can be similarly combined into

$$\mathcal{Y}_{n,k} = \mathcal{Y}_k \sin^k\beta \sum_{s=0}^{\left[\frac{n-k}{2}\right]} (-1)^s \binom{\left[\frac{n-k}{2}\right]}{s} \frac{\left(\left[\frac{n-k}{2}\right] + k + 3 + h_{n,k}\right)_s}{\left(\frac{1}{2} + h_{n,k}\right)_s} \cos^{2s+h_{n,k}}\beta\,, \tag{7.84}$$

where $[\cdot]$ means integer part and we define the symbol $h_{n,k}$ as

$$h_{n,k} = n - k - 2\left[\frac{n-k}{2}\right] = \begin{cases} 0 & , \quad n-k \text{ even (for the first branch)} \\ 1 & , \quad n-k \text{ odd (for the second branch)}\,. \end{cases} \tag{7.85}$$

At fixed n and k, the eigenfunctions (7.84) span the $[k,0,0]$ representation of $\mathrm{SO}(7)_v$. Moreover, it can be checked that these eigenfunctions at fixed n actually span the full symmetric traceless representation $[n,0,0,0]$ of SO(8). In other words, the eigenfunctions (7.84) turn out to be simply the SO(8) spherical harmonics of S^7, branched out into $\mathrm{SO}(7)_v$ representations through

$$[n,0,0,0] \overset{\mathrm{SO}(7)_v}{\longrightarrow} \sum_{k=0}^{n} [k,0,0]\,. \tag{7.86}$$

This is consistent with the quantum number ranges (7.79). This is also compatible with the internal geometry (7.62) being topologically S^7: it can be continuously deformed into the round SO(8)-invariant geometry by setting $\varphi = \chi = 0$. These arguments suggest that the spectrum (7.78), (7.84) is in fact complete. Thus, the quantum number n can be regarded as the Kaluza-Klein level in (6.37), as it coincides with the unique integer that characterises the KK spectrum of the $\mathcal{N} = 8$ SO(8)-invariant Freund-Rubin solution.

$\mathrm{SU}(4)_c$-sector The $\mathrm{SU}(4)_c$-invariant sector of SO(8)-gauged supergravity contains three pseudoscalars: χ, ζ, $\tilde{\zeta}$. In the Iwasawa parametrisation of the appendix, the SU(3)-invariant dilatons φ, ϕ become identified in terms of the pseudoscalars via equation (3.83). With the understanding that φ, ϕ depend on the independent fields χ, ζ, $\tilde{\zeta}$, the former can be conveniently used to parametrise the $\mathrm{SU}(4)_c$-invariant sector, as the resulting expressions are more compact. The embedding of this sector into the $D = 11$ warp factor and internal metric [78] given in (3.133) read

$$e^{2A} = e^{\frac{4}{3}\phi+\varphi}\,L^2\,, \qquad d\bar{s}_7^2 = g^{-2}L^{-2}\left[e^{-2\phi-\varphi}ds^2(\mathbb{CP}^3) + e^{-3\varphi}(d\psi+\sigma)^2\right]. \tag{7.87}$$

Here, $ds^2(\mathbb{CP}^3)$ is the Fubini-Study metric on the complex projective space, σ a one-form potential for the Kähler form on the latter, and $0 \leq \psi \leq 2\pi$ a coordinate on the Hopf fibre of S^7. Away from the $\mathrm{SO}(7)_c$-invariant locus, (3.89), where the symmetry is enhanced accordingly, the geometry (7.87) is invariant under $\mathrm{SU}(4)_c \times \mathrm{U}(1)$, with U(1) generated by ∂_ψ. This U(1) is broken by the $D = 11$ supergravity four-form (3.135).

The $D = 11$ configuration (7.87) is homogeneous: the warp factor depends only on the $D = 4$ scalars and not on the S^7 coordinates, and the metric $d\bar{s}_7^2$ corresponds to a homogeneous stretching of the S^7 geometry along its Hopf fibre. Therefore, the differential equation (4.7) simplifies for this geometry as

$$\Big[\big(e^{3\varphi} - e^{2\phi+\varphi}\big)\partial_\psi^2 + e^{2\phi+\varphi}\Box_{S^7}\Big]\mathcal{Y} = -g^{-2}M^2\,\mathcal{Y}\,, \tag{7.88}$$

with $\Box_{S^7}$ the Laplacian on the round, Einstein metric on S^7. The solutions of (7.88) are accordingly given by the SO(8) spherical harmonics on S^7, branched out into representations of the $\mathrm{SU}(4)_c \times \mathrm{U}(1)$ symmetry group of (7.87) and (7.88) via

$$[n,0,0,0] \xrightarrow{\mathrm{SU}(4)_c\times\mathrm{U}(1)} \sum_{r=0}^{n} [r,0,n-r]_{2r-n}\,, \tag{7.89}$$

with the subindex indicating the U(1) charge. More concretely, the S^7 spherical harmonics, in the $[n,0,0,0]$ of SO(8), split according to (7.89) as

$$\mathcal{Y}_{n,r}(z,\bar{z}) = c_{a_1\ldots a_r}{}^{b_1\ldots b_{n-r}}\, z^{a_1}\ldots z^{a_r}\,\bar{z}_{b_1}\ldots\bar{z}_{b_{n-r}}\,, \tag{7.90}$$

for

$$n = 0,1,2,\ldots\,, \qquad r = 0,1,\ldots,n\,. \tag{7.91}$$

In (7.90), $z^1 = \mu^1 + i\mu^2$, etc, are complexified embedding coordinates of $\mathbb{R}^8$ constrained as $\delta_{AB}\,\mu^A\mu^B = 1$, with $A, B = 1,\ldots,8$, and $c_{a_1\ldots a_r}{}^{b_1\ldots b_{n-r}}$ is a constant tensor in the $[n-r,0,r]$ of SU(4). The functions (7.90) obey

$$\Box_{S^7}\mathcal{Y}_{n,r} = -n(n+6)\mathcal{Y}_{n,r}\,, \qquad \partial_\psi^2\mathcal{Y}_{n,r} = -(n-2r)^2\mathcal{Y}_{n,r}\,, \tag{7.92}$$

and thus satisfy the differential equation (7.88) with eigenvalue

$$g^{-2}M_{n,r}^2 = e^{2\phi+\varphi}n(n+6) + \big(e^{3\varphi} - e^{2\phi+\varphi}\big)(n-2r)^2\,. \tag{7.93}$$

This occurs with multiplicity

$$d_{n,r} = \dim[r,0,n-r]_{\mathrm{SU}(4)} = \tfrac{1}{12}(n+3)(r+1)(r+2)(n-r+1)(n-r+2)\,. \tag{7.94}$$

To summarise, the complete spectrum of the eigenvalue equation (7.88) is (7.93), (7.90), with the quantum numbers ranging as in (7.91). The eigenvalues (7.93) have multiplicity (7.94) and the eigenfunctions (7.90) are simply the S^7 spherical harmonics split into $\mathrm{SU}(4)_c \times \mathrm{U}(1)$ representations through (7.89). The eigenvalues have been given in terms of $D = 4$ scalars. The massive KK graviton spectra about $D = 11$ AdS_4 solutions in this sector are obtained by fixing the $D = 4$ scalars to the corresponding vevs. Like in the case of the G_2 sector, the integer n is identified with the KK level by an argument similar to that put forward below (7.86).

Type IIB

We now move on to compute the graviton spectrum about the AdS_4 solutions of type IIB supergravity recently obtained in [156]. These geometries, in the same class as addressed in section 7.2 arise upon consistent uplift [28] on an S-fold geometry of AdS_4 vacua of $D = 4$ $\mathcal{N} = 8$ gauged supergravity with dyonic $[\mathrm{SO}(6) \times \mathrm{SO}(1,1)] \ltimes \mathbb{R}^{12}$ gauging [75, 76]. Again, we will focus in this section on solutions that preserve at least SU(3) symmetry classified in [156]. We will compute the generic graviton spectra for arbitrary constant values of the SU(3)-invariant scalars of the $D = 4$ supergravity.

The type IIB geometries under consideration are given by [156]

$$\begin{aligned} e^{2A} &= \sqrt{Y} e^{\varphi} L^2 \,, \\ d\bar{s}_6^2 &= \frac{e^{-\varphi}}{\sqrt{Y}} g^{-2} L^{-2} \Big[\sqrt{Y} e^{-2\varphi} d\eta^2 + \frac{1}{\sqrt{Y}} \big(ds^2(\mathbb{CP}^2) + Y (d\tau + \sigma')^2 \big) \Big] \,. \end{aligned} \tag{7.95}$$

The geometry inside the last parenthesis extends globally over a topological S^5, with $ds^2(\mathbb{CP}^2)$ the Fubini-Study metric on the complex projective plane within S^5 and $0 \le \tau < 2\pi$ the Hopf fibre angle. The local one-form σ' is a potential for the Kähler form on $\mathbb{CP}^2$. The sixth internal coordinate η will be taken to be periodic, $\eta \sim \eta + T$, with T a positive number. The ten-dimensional geometry (7.95) also depends on the SU(3)-invariant scalars both explicitly and through the combination Y defined in (3.56). For general values of the scalars, the geometry (7.95) displays an isometry group $\mathrm{SU}(3) \times \mathrm{U}(1)_\tau \times \mathrm{U}(1)_\eta$, with the $\mathrm{U}(1)_\eta$ factor broken by the type IIB fluxes. In particular, the type IIB fields charged under the S-duality group $\mathrm{SL}(2,\mathbb{R})$ undergo a monodromy transformation as η crosses through different periods [28]. The type IIB metric is neutral under S-duality and thus insensitive to this transformation.

Like (7.87), the type IIB embedding (7.95) is homogeneous. Accordingly, the differential equation (4.7) reduces for this geometry to

$$\Big[e^{3\varphi} \partial_\eta^2 + e^{\varphi} (1 - Y) \partial_\tau^2 + e^{\varphi} Y \, \Box_{S^5} \Big] \mathcal{Y} = -g^{-2} M^2 \, \mathcal{Y} \,, \tag{7.96}$$

where $\Box_{S^5}$ is the Laplacian on the round, Einstein metric on S^5. The complete set of eigenfunctions $\mathcal{Y} \equiv \mathcal{Y}_{\ell,p,n}$ that solve (7.96) can be taken to satisfy

$$\Box_{S^5}\mathcal{Y}_{\ell,p,n} = -\ell(\ell+4)\mathcal{Y}_{\ell,p,n}\,, \qquad \partial_\tau^2\mathcal{Y}_{\ell,p,n} = -(\ell-2p)^2\mathcal{Y}_{\ell,p,n}\,,$$
$$\partial_\eta^2\mathcal{Y}_{\ell,p,n} = -\left(\tfrac{2\pi}{T}n\right)^2\mathcal{Y}_{\ell,p,n}\,, \qquad (7.97)$$

for

$$\ell = 0,1,2,\dots\,, \qquad p = 0,1,\dots,\ell\,, \qquad n = 0,\pm1,\pm2,\dots\,, \qquad (7.98)$$

with ℓ and n unconstrained and p constrained by ℓ through $p \leq \ell$. In other words, the eigenfunctions $\mathcal{Y}_{\ell,p,n}$ come in representations of $\mathrm{SU}(3) \times \mathrm{U}(1)_\tau \times \mathrm{U}(1)_\eta$, and are explicitly given by products of harmonics on the S^1 generated by ∂_η and spherical harmonics $[0,\ell,0]_{\mathrm{SU}(4)}$ on S^5 branched out into representations of $\mathrm{SU}(3)\times\mathrm{U}(1)_\tau$ via

$$[0,\ell,0] \xrightarrow{\mathrm{SU}(3)\times\mathrm{U}(1)_\tau} \sum_{p=0}^{\ell}\,[p,\ell-p]_{\ell-2p}\,. \qquad (7.99)$$

Bringing (7.97) to (7.96), we find the eigenvalues

$$g^{-2}M^2_{\ell,p,n} = e^{\varphi}Y\ell(\ell+4) + e^{\varphi}(1-Y)(\ell-2p)^2 + e^{3\varphi}\left(\tfrac{2\pi}{T}j\right)^2\,, \qquad (7.100)$$

occurring with degeneracy

$$d_{\ell,p,j} = \begin{cases} \dim[p,\ell-p]_{\mathrm{SU}(3)} & = \tfrac{1}{2}\,(p+1)(\ell-p+1)(\ell+2) \quad , \quad \text{if } j=0 \\ 2\dim[p,\ell-p]_{\mathrm{SU}(3)} & = (p+1)(\ell-p+1)(\ell+2) \quad , \quad \text{if } j\neq 0\,. \end{cases} \qquad (7.101)$$

In summary, the complete eigenvalue spectrum of equation (7.96) is (7.100) with the eigenfunctions $\mathcal{Y}_{\ell,p,n}$ described above and with the quantum numbers ranging as in (7.98). The eigenvalues (7.100) have multiplicity (7.101), and have been given in terms of $D=4$ scalars. The massive KK graviton spectra about $D=11$ AdS_4 solutions in this sector are obtained by fixing the $D=4$ scalars to the corresponding vevs, as we will see next.

Graviton masses at the solutions

Using the previous results for the M-theory and type IIB configurations, as well as [24, 93, 143], we can write down the KK graviton spectra about the AdS_4 solutions of the ten and eleven-dimensional supergravities that uplift from critical points with at least SU(3) symmetry of the three $D=4$ $\mathcal{N}=8$ gauged supergravities in figure 3.1.

In M-theory, the spectrum above the AdS_4 solutions with at least G_2 symmetry and at least $\mathrm{SU}(4)_c$ symmetry can be obtained by particularising (7.78) and (7.93), respectively, to the scalar vevs given in table 3.2. We have

Solution	Mass	Degeneracy
$\mathcal{N}=8\ ,\ \mathrm{SO}(8)$	$L^2M_n^2=\frac{1}{4}n(n+6)$	$d_n=D_{n,8}$
$\mathcal{N}=2\ ,\ \mathrm{U}(3)_c$	$L^2M_{n,p,\ell,r}^2=\frac{1}{2}n(n+6)-\frac{1}{3}\ell(\ell+4)-\frac{1}{9}(\ell-2p)^2+\frac{1}{18}[3(n-2r)+4(\ell-2p)]^2$	$d_{n,p,\ell,r}=\frac{1}{2}(p+1)(\ell-p+1)(\ell+2)$
$\mathcal{N}=1\ ,\ \mathrm{G}_2$	$L^2M_{n,k}^2=\frac{5}{8}n(n+6)-\frac{5}{12}k(k+5)$	$d_{n,k}=D_{k,7}$
$\mathcal{N}=0\ ,\ \mathrm{SO}(7)_v$	$L^2M_{n,k}^2=\frac{3}{4}n(n+6)-\frac{3}{5}k(k+5)$	$d_{n,k}=D_{k,7}$
$\mathcal{N}=0\ ,\ \mathrm{SO}(7)_c$	$L^2M_n^2=\frac{3}{10}n(n+6)$	$d_n=D_{n,8}$
$\mathcal{N}=0\ ,\ \mathrm{SU}(4)_c$	$L^2M_{n,r}^2=\frac{3}{8}n(n+6)-\frac{3}{16}(n-2r)^2$	$d_{n,r}=\frac{1}{12}(n+3)(r+1)(r+2)(n-r+1)(n-r+2)$

Table 7.4: The KK graviton spectra of AdS_4 solutions of $D=11$ supergravity that uplift from critical points of $D=4$ $\mathcal{N}=8$ SO(8)-gauged supergravity with at least SU(3) symmetry. See (7.81) for the notation $D_{k,N}$. The quantum numbers range as in (7.102).

brought these results to table 7.4. In order to exhaust the KK graviton spectra of AdS_4 solutions of $D=11$ supergravity that uplift from critical points of $D=4$ $\mathcal{N}=8$ SO(8)-gauged supergravity with at least SU(3) symmetry, the table also includes the spectrum [24] about the $\mathcal{N}=8$ Freund-Rubin solution [92] and the spectrum [93] about the $\mathrm{SU}(3)\times\mathrm{U}(1)_c$–invariant AdS_4 solution [31, 77]. The latter is given as in [143], with $\ell_{\text{here}}=p_{\text{there}}+q_{\text{there}}$. The corresponding multiplicites are also given in the table, and the quantum numbers range as

$$n=0,1,2,\ldots\,,\quad r,\,k=0,1,\ldots,n\,,\quad \ell=p,\ldots,p+r\,,\quad p=0,1,\ldots,n-r\,. \tag{7.102}$$

The only quantum number that is free to range unrestricted over the non-negative integers is n, all the others being bound by it. This is consistent with the interpretation of n as the SO(8) KK level in (6.37). At fixed KK level n, the degeneracy of the $\mathcal{N}=8$ SO(8)-symmetric spectrum is broken into representations of the isometry group of the internal metric. This may be larger than the symmetry of each solution, as the fluxes will further break the isometry to the actual symmetry quoted in the table. Similarly, the eigenfunctions corresponding to each solution are simply the S^7 spherical harmonics branched out into the representations of the relevant group.

For convenience, table 7.5 imports from [143] the KK graviton spectra of AdS_4 solutions of massive IIA supergravity that uplift from critical points of $D=4$ $\mathcal{N}=8$ dyonic ISO(7)-gauged supergravity with at least SU(3) symmetry. The table includes the squared masses in units of the corresponding AdS radius L, as well as the multiplicites. In this case, the quantum numbers'

Solution	Mass	Degeneracy
$\mathcal{N}=2$, $\mathrm{U}(3)_v$	$L^2M^2_{k,\ell,p}=\frac{2}{3}k(k+5)-\frac{1}{3}\ell(\ell+4)+\frac{1}{9}(\ell-2p)^2$	$d_{k,\ell,p}=\frac{1}{2}(p+1)(\ell-p+1)(\ell+2)$
$\mathcal{N}=1$, G_2	$L^2M^2_k=\frac{5}{12}k(k+5)$	$d_k=D_{k,7}$
$\mathcal{N}=1$, $\mathrm{SU}(3)$	$L^2M^2_{k,\ell,p}=\frac{5}{6}k(k+5)-\frac{5}{12}\ell(\ell+4)-\frac{5}{36}(\ell-2p)^2$	$d_{k,\ell,p}=\frac{1}{2}(p+1)(\ell-p+1)(\ell+2)$
$\mathcal{N}=0$, $\mathrm{SO}(7)_v$	$L^2M^2_k=\frac{2}{5}k(k+5)$	$d_k=D_{k,7}$
$\mathcal{N}=0$, $\mathrm{SO}(6)_v$	$L^2M^2_{k,\ell}=k(k+5)-\frac{3}{4}\ell(\ell+4)$	$d_\ell=D_{\ell,6}$
$\mathcal{N}=0$, G_2	$L^2M^2_k=\frac{1}{2}k(k+5)$	$d_k=D_{k,7}$

Table 7.5: The KK graviton spectra of AdS_4 solutions of massive IIA supergravity that uplift from critical points of $D=4$ $\mathcal{N}=8$ dyonic ISO(7)-gauged supergravity with at least SU(3) symmetry, taken from [143]. See (7.81) for the notation $D_{k,N}$. The quantum numbers range as in (7.103).

ranges are

$$k=0,1,2,\dots\,,\qquad \ell=0,1,\dots,k\,,\qquad p=0,1,\dots,\ell\,,\tag{7.103}$$

with $k_{\text{here}}=n_{\text{in [143]}}$. Again, k is the only quantum number that is unrestricted. For this reason, k can be interpreted in this case as the SO(7) KK level in (6.37). The eigenfunctions are now the S^6 spherical harmonics split into representations of the internal isometry group. This again may be larger than the symmetry of each solution given in table 7.5 because the fluxes may further break the isometry to the actual symmetry of the full solution.

Finally, we turn to the spectrum of gravitons corresponding to the type IIB AdS_4 S-fold solutions that uplift from critical points with at least SU(3) symmetry [156] of $D=4$ $\mathcal{N}=8$ supergravity with $\big(\mathrm{SO}(6)\times\mathrm{SO}(1,1)\big)\ltimes\mathbb{R}^{12}$ gauging. These are found by bringing the corresponding vevs, collected in our conventions in table 7.3, to equation (7.100). The results are summarised in table 7.6. The KK graviton spectra are sensitive to the period T of the S-folded S^1. The eigenfunctions are products of S^5 harmonics, possibly branched out into $\mathrm{SU}(3)\times\mathrm{U}(1)_\tau$ representations, and $\mathrm{U}(1)_\eta$ harmonics. This $\mathrm{U}(1)_\eta$ is broken by the IIB fluxes.

The masses and degeneracies in tables 7.4–7.6 perfectly match the one obtained through the algebraic route discussed in chapter 6 for every solution. In the following, the $\mathrm{SL}(8,\mathbb{R})$ formulation of section 6.1 will prove particularly useful to formulate the universal properties identified in the spectra.

Solution	Mass	Degeneracy
$\mathcal{N}=1$, SU(3)	$L^2M^2_{\ell,p,n}=\frac{5}{6}\ell(\ell+4)-\frac{5}{36}(\ell-2p)^2+\frac{5\pi^2}{T^2}n^2$	$d_{\ell,p,n}$
$\mathcal{N}=0$, $\mathrm{SO}(6)_v$	$L^2M^2_{\ell,n}=\frac{3}{4}\ell(\ell+4)+\frac{6\pi^2}{T^2}n^2$	$d_{\ell,n}=(2-\delta_{n0})\,D_{\ell,6}$
$\mathcal{N}=0$, SU(3)	$L^2M^2_{\ell,n}=\frac{3}{4}\ell(\ell+4)+\frac{6\pi^2}{T^2}n^2$	$d_{\ell,n}=(2-\delta_{n0})\,D_{\ell,6}$

Table 7.6: The KK graviton spectra of AdS_4 S-fold solutions of type IIB supergravity that uplift from critical points of $D=4$ $\mathcal{N}=8$ $\big(\mathrm{SO}(6)\times\mathrm{SO}(1,1)\big)\ltimes\mathbb{R}^{12}$-gauged supergravity with at least SU(3) symmetry. See (7.81) for the notation $D_{k,N}$ and (7.101) for $d_{\ell,p,n}$. The quantum numbers range as in (7.98).

7.3.2 Universality in SU(3)-invariant sector

When regarded as vacua of their corresponding $D=4$ $\mathcal{N}=8$ gauged supergravities, the AdS solutions under consideration with at least SU(3) symmetry tend to exhibit the same mass spectrum of scalars, vectors and fermions within their $D=4$ supergravities. This is the case for all these solutions, except for the two $\mathcal{N}=0$, SU(3)-invariant critical points of ISO(7) supergravity and the $\mathcal{N}=0$, SU(3)-invariant critical locus of $\big(\mathrm{SO}(6)\times\mathrm{SO}(1,1)\big)\ltimes\mathbb{R}^{12}$ supergravity, as shown in table 7.7. The question that we would like to address in this section is whether this situation persists for higher KK modes. The spectrum of gravitons computed for these solutions and recorded in tables 7.4–7.6 shows that this universality is indeed lost at higher KK levels: the KK gravitons do have completely different masses for all the solutions considered.

However, as we will now show, universality is still maintained, though in a milder form that is not apparent from the individual mass values. It turns out that certain sums of KK graviton masses weighted with their multiplicities do remain universal. This is the case at least for solutions in the same or different $\mathcal{N}=8$ gaugings with the same symmetry and whose spectra within the $D=4$ supergravity are the same. Specifically, if two AdS_4 solutions of $D=11$ supergravity or type II supergravity uplift from critical points with the same supersymmetry $\mathcal{N}\leq 8$, the same symmetry $G\supset\mathrm{SU}(3)$ (possibly embedded differently into the gauge group) and the same spectrum within the $D=4$ $\mathcal{N}=8$ gauged supergravities, then there exist infinitely many discrete combinations $L^2\,\mathrm{tr}\,M^2_{(n)}$, $n=1,2,3,\ldots$, of graviton masses weighted with their multiplicities that are the same for both solutions. This statement was first observed for the $\mathcal{N}=2$ $\mathrm{SU}(3)\times\mathrm{U}(1)$-invariant solutions in [143]. Here we will extend that result to all other solutions with at least SU(3) symmetry

(Super)symmetry	SO(8)	ISO(7)	$\big(\mathrm{SO}(6)\times\mathrm{SO}(1,1)\big)\ltimes\mathbb{R}^{12}$	same spectrum?
$\mathcal{N}=8$, SO(8)	✓	×	×	–
$\mathcal{N}=2$, U(3)	✓	✓	×	✓
$\mathcal{N}=1$, G_2	✓	✓	×	✓
$\mathcal{N}=1$, SU(3)	×	✓	✓	✓
$\mathcal{N}=0$, SO(7)	✓	✓	×	✓
$\mathcal{N}=0$, SO(6)	✓	✓	✓	✓
$\mathcal{N}=0$, G_2	×	✓	×	–
$\mathcal{N}=0$, SU(3)	×	✓	✓	×

Table 7.7: Possible residual (super)symmetries, regardless of their $\mathrm{E}_{7(7)}$ embedding, of AdS vacua in the SU(3)-invariant sector of the three different gaugings that we consider.

in the gaugings of figure 3.1, summarised in table 7.7. With the hindsight gained in chapter 6, the notation $L^2\,\mathrm{tr}\,M^2_{(n)}$, relates to the fact that the combinations in question correspond to traces of the (infinite-dimensional) KK graviton mass matrix at fixed KK level n.

More concretely, for the M-theory solutions we define $L^2\,\mathrm{tr}\,M^2_{(n)}$ to be the sum of the squared masses in units of the corresponding AdS radius L, weighted with the corresponding multiplicity as given in table 7.4. The sum is taken at fixed KK level n and over all other quantum numbers ranging as in (7.102). For example, using this prescription, one obtains for the $\mathcal{N}=8$ SO(8) solution [143],

$$L^2\,\mathrm{tr}\,M^2_{(n)} = L^2 M_n^2\, d_n = 14\, D_{n-1,10}\,. \tag{7.104}$$

In the last step, we have made use of the definition (??) as a shorthand for the resulting 8th degree polynomial in n. Similarly, for the $\mathcal{N}=2$ $\mathrm{SU}(3)\times\mathrm{U}(1)_c$ solution, we have [143]

$$L^2\,\mathrm{tr}\,M^2_{(n)} = L^2 \sum_{r=0}^{n}\sum_{p=0}^{n-r}\sum_{\ell=p}^{p+r} M^2_{n,p,\ell,r}\, d_{n,p,\ell,r} = \tfrac{56}{3}\, D_{n-1,10}\,. \tag{7.105}$$

Proceeding similarly, we compute the quantities $L^2\,\mathrm{tr}\,M^2_{(n)}$, $n=1,2,\ldots$, for the KK graviton spectra summarised in table 7.4 for $D=11$ AdS_4 solutions that uplift from critical points of $D=4$ $\mathcal{N}=8$ SO(8) supergravity with at least SU(3) symmetry. We obtain:

$$
\begin{aligned}
\mathcal{N}=8\ ,\ \mathrm{SO}(8) &\quad : \quad L^2\,\mathrm{tr}\,M^2_{(n)} = 14\,D_{n-1,10} \quad , \\
\mathcal{N}=2\ ,\ \mathrm{SU}(3)\times\mathrm{U}(1)_c &\quad : \quad L^2\,\mathrm{tr}\,M^2_{(n)} = \tfrac{56}{3}\,D_{n-1,10} \quad , \\
\mathcal{N}=1\ ,\ \mathrm{G}_2 &\quad : \quad L^2\,\mathrm{tr}\,M^2_{(n)} = \tfrac{35}{2}\,D_{n-1,10} \quad , \\
\mathcal{N}=0\ ,\ \mathrm{SO}(7)_v &\quad : \quad L^2\,\mathrm{tr}\,M^2_{(n)} = \tfrac{84}{5}\,D_{n-1,10} \quad , \\
\mathcal{N}=0\ ,\ \mathrm{SO}(7)_c &\quad : \quad L^2\,\mathrm{tr}\,M^2_{(n)} = \tfrac{84}{5}\,D_{n-1,10} \quad , \\
\mathcal{N}=0\ ,\ \mathrm{SU}(4)_c &\quad : \quad L^2\,\mathrm{tr}\,M^2_{(n)} = \tfrac{39}{2}\,D_{n-1,10} \quad .
\end{aligned}
\tag{7.106}
$$

In particular, the two SO(7)-invariant solutions have their residual symmetry embedded differently into the SO(8) gauge group as $\mathrm{SO}(7)_v$ and $\mathrm{SO}(7)_c$. They have the same mass spectrum within $D=4$ $\mathcal{N}=8$ SO(8) supergravity, according to table 7.7. Their KK graviton spectra are different, though, according to table 7.4. But as can be seen from equation (7.106), the quantity $L^2\,\mathrm{tr}\,M^2_{(n)}$ is the same for both solutions for all n.

The quantities $L^2\,\mathrm{tr}\,M^2_{(k)}$ for the KK gravitons of massive IIA solutions with at least SU(3) symmetry that uplift from critical points of dyonic ISO(7) supergravity were computed similarly, for $k=1,2,\ldots$, in [143]:

$$
\begin{aligned}
\mathcal{N}=2\ ,\ \mathrm{SU}(3)\times\mathrm{U}(1)_v &\quad : \quad L^2\ \mathrm{tr}\,M^2_{(k)} = \tfrac{56}{3}\,D_{k-1,9} \quad , \\
\mathcal{N}=1\ ,\ \mathrm{G}_2 &\quad : \quad L^2\ \mathrm{tr}\,M^2_{(k)} = \tfrac{35}{2}\,D_{k-1,9} \quad , \\
\mathcal{N}=1\ ,\ \mathrm{SU}(3) &\quad : \quad L^2\ \mathrm{tr}\,M^2_{(k)} = \tfrac{65}{3}\,D_{k-1,9} \quad , \\
\mathcal{N}=0\ ,\ \mathrm{SO}(7)_v &\quad : \quad L^2\ \mathrm{tr}\,M^2_{(k)} = \tfrac{84}{5}\,D_{k-1,9} \quad , \\
\mathcal{N}=0\ ,\ \mathrm{SO}(6)_v &\quad : \quad L^2\ \mathrm{tr}\,M^2_{(k)} = \tfrac{39}{2}\,D_{k-1,9} \quad , \\
\mathcal{N}=0\ ,\ \mathrm{G}_2 &\quad : \quad L^2\ \mathrm{tr}\,M^2_{(k)} = 21\,D_{k-1,9} \quad .
\end{aligned}
\tag{7.107}
$$

Here, we have again made use of the notation $D_{k,N}$ defined in (7.81) as a shorthand for the degree-7 polynomial in k that apparears in the r.h.s.'s. Now, recall from (6.37) that k and n can respectively be regarded as the KK levels in massive IIA and $D=11$. At first KK level, the quantities $L^2\ \mathrm{tr}\,M^2_{n=1}$ in (7.106) and $L^2\ \mathrm{tr}\,M^2_{k=1}$ in (7.107) can be checked to match, by virtue of the first relation in (7.82), for solutions with the same symmetry group regardless of the embedding of the latter within the corresponding gauge group. For example, for the $D=11$ $\mathrm{SU}(3)\times\mathrm{U}(1)_c$ solution [31] and the massive IIA $\mathrm{SU}(3)\times\mathrm{U}(1)_v$ solution [26], $[L^2\ \mathrm{tr}\,M^2_{(1)}]_{\mathrm{11D}} = [L^2\ \mathrm{tr}\,M^2_{(1)}]_{\mathrm{IIA}} = \frac{56}{3}$, at $n = k = 1$, as already noted in [143]. Inspection of (7.106) and (7.107) confirms that similar matches occur at KK level one, $n = k = 1$, for the $D = 11$ and massive IIA solutions with common (super)symmetry $\mathcal{N} = 1$, G_2, and $\mathcal{N} = 0$, SO(7), and $\mathcal{N} = 0$, SU(4) $\sim$ SO(6).

Further, there is still matching at higher KK levels $n > 1$ in $D = 11$ and $k > 1$ in massive IIA, provided a prescription is adopted to relate n and k. This prescription is precisely (6.14), so that the $D = 11$ KK level n formally contains all IIA KK levels $k = 0, 1, \ldots, n$. Using this, it follows from (7.106) and (7.107) that

$$\sum_{k=0}^{n} [L^2 \operatorname{tr} M^2_{(k)}]_{\mathrm{IIA}} = [L^2 \operatorname{tr} M^2_{(n)}]_{\mathrm{11D}} \,, \qquad n = 0, 1, 2, \ldots \,, \tag{7.108}$$

for all the solutions that we are considering with the same symmetry and supersymmetry in massive IIA and $D = 11$. Here, $L^2 \operatorname{tr} M^2_{(0)} \equiv 0$ corresponds to the massless graviton, for both the $D = 11$ and type IIA cases, as well as for the IIB cases below. These sums are related to the mass matrices in section 6.1, as we will discuss momentarily. To see this, it is convenient to further add different SO(8) levels in (7.108) à la (6.13). It immediately follows that

$$\sum_{s=0}^{[\frac{m}{2}]} \sum_{k=0}^{m-2s} [L^2 \operatorname{tr} M^2_{(k)}]_{\mathrm{IIA}} = \sum_{s=0}^{[\frac{m}{2}]} [L^2 \operatorname{tr} M^2_{(m-2s)}]_{\mathrm{11D}} \,, \qquad m = 0, 1, 2, \ldots \,, \tag{7.109}$$

again for all solutions with the same (super)symmetry. The sums in (7.109) obviously run over repeated number of states, both in IIA and in $D = 11$. In (7.108), there are no repeated $D = 11$ states on the r.h.s., but the sum in the l.h.s. does run as well over repeated states in IIA. These overcounting issues can be avoided by subtracting two adjacent KK levels as in (6.15) or, using the identity (7.83),

$$[L^2 \operatorname{tr} M^2_{(n)}]_{\mathrm{IIA}} = [L^2 \operatorname{tr} M^2_{(n)}]_{\mathrm{11D}} - [L^2 \operatorname{tr} M^2_{(n-1)}]_{\mathrm{11D}} \,, \qquad n = 1, 2, \ldots \,, \tag{7.110}$$

for solutions with the same (super)symmetry. This relation was already shown to hold in [143] for the $\mathcal{N} = 2$ SU(3) $\times$ U(1) invariant solutions. Here, we have extended this result to all other AdS solutions in the SU(3)-invariant sectors of SO(8) and ISO(7) gauged supergravities with the same symmetry and supersymmetry.

The situation is similar, though slightly different, for the type IIB AdS_4 S-fold solutions that uplift from $D = 4$ $\mathcal{N} = 8$ $\big(\mathrm{SO}(6) \times \mathrm{SO}(1,1)\big) \ltimes \mathbb{R}^{12}$-gauged supergravity. According to table 7.7, this supergravity also has critical points with the same symmetry $G \supset \mathrm{SU}(3)$ and supersymmetry as other critical points of the SO(8) and ISO(7) gauging: $\mathcal{N} = 0$ SO(6), $\mathcal{N} = 1$ SU(3) and $\mathcal{N} = 0$ SU(3). The former two have the same spectrum within their corresponding $D = 4$ supergravities, while the latter does not. For this reason, we will only be interested in the former two vacua. Both for the $\mathcal{N} = 1$ SU(3) and the $\mathcal{N} = 0$ SO(6) $\sim$ SU(4) solutions there are combinations, $[L^2 \operatorname{tr} \tilde{M}^2_{(n)}]_{\mathrm{IIB}}$, of the eigenvalues in table 7.6 that match the quantities $[L^2 \operatorname{tr} M^2_{(k)}]_{\mathrm{IIA}}$ and $[L^2 \operatorname{tr} M^2_{(n)}]_{\mathrm{11D}}$ for the solutions with the same symmetry for a certain choice of the period T. The tilde in $[L^2 \operatorname{tr} \tilde{M}^2_{(n)}]_{\mathrm{IIB}}$ is taken to signify that, in this case, the combinations also involve negative weightings. More concretely, consider the following quantities for the type IIB solutions with the quantum numbers fixed as indicated:

$$
\begin{aligned}
\mathcal{N} = 1,\ \mathrm{SU}(3) \quad &: \quad \operatorname{tr} \tilde{M}^2_{(1)} \equiv \Big[\sum_{p=0}^{\ell} M^2_{\ell,p,n} d_{\ell,p,n}\Big]\Big|_{\substack{\ell=1,\\ n=0}} \\
&\qquad\qquad - \big[M^2_{\ell,p,n} d_{\ell,p,n}\big]\Big|_{\substack{\ell=p=0,\\ n=-1}} - \big[M^2_{\ell,p,n} d_{\ell,p,n}\big]\Big|_{\substack{\ell=p=0,\\ n=+1}} \,, \\
\mathcal{N} = 0,\ \mathrm{SO}(6)_v \quad &: \quad \operatorname{tr} \tilde{M}^2_{(1)} \equiv \big[M^2_{\ell,n} d_{\ell,n}\big]\Big|_{\substack{\ell=1,\\ n=0}} \\
&\qquad\qquad - \big[M^2_{\ell,n} d_{\ell,n}\big]\Big|_{\substack{\ell=0,\\ n=-1}} - \big[M^2_{\ell,n} d_{\ell,n}\big]\Big|_{\substack{\ell=0,\\ n=+1}} \,.
\end{aligned}
\tag{7.111}
$$

These quantities involve sums of mass eigenvalues, weighted with their degeneracies as given in table 7.6, and affected by a $+$ or a $-$ sign depending on whether $n = 0$ or $n \neq 0$. Plugging in the expressions given in the table, the quantity $L^2 \operatorname{tr} \tilde{M}^2_{(1)}$ for the $\mathcal{N} = 1$, SU(3) solution evaluates to $\frac{65}{3}$ if $T = 2\pi$, matching the quantity $L^2 \operatorname{tr} M^2_{(1)}$ for its counterpart type IIA solution at KK level $k = 1$, given in (7.107). Similarly, $L^2 \operatorname{tr} \tilde{M}^2_{(1)}$ for the $\mathcal{N} = 0$, $\mathrm{SO}(6)_v$ solution evaluated using the expressions given in table 7.6 gives $\frac{39}{2}$ for $T = 2\pi$. This again matches the quantity $L^2 \operatorname{tr} M^2_{(1)}$ at KK level $k = 1$ given in (7.107) for the $\mathcal{N} = 0$, $\mathrm{SO}(6)_v$ solution of massive IIA. It also matches $L^2 \operatorname{tr} M^2_{(1)}$ at KK level $n = 1$ given in (7.106) for the $D = 11$ $\mathcal{N} = 0$, $\mathrm{SU}(4)_c$ solution. Although it is not as clear cut in the type IIB case, it will be argued that the states that enter the sums in (7.111) also belong to KK level $m = 1$ in an SL(8)-covariant sense. The formal analytic continuation $n' = in$, with $i^2 = -1$, removes the minus signs in (7.111). Under this analytic continuation, relations similar to (7.108) and (7.109) relate these formal sums at higher KK levels for these type IIB solutions to their $D = 11$ and type IIA counterparts.

The covariant mass matrix perspective

As discussed above, the graviton states that enter the sums defining the universal traces correspond to the decompositions of $\mathrm{SL}(8,\mathbb{R})$ representations in (6.13) and (6.14) for the solutions in M-theory and massive type IIA. The trace of the $\mathrm{SL}(8,\mathbb{R})$ level $m = 1$ graviton mass matrix (6.10),

$$\mathrm{tr}\, M^2_{(1)} = -g^2 \mathcal{M}^{MN}\, \Theta_M{}^A{}_B\, \Theta_N{}^B{}_A\,, \tag{7.112}$$

thus reproduces the KK level-one traces discussed in (7.106) and (7.107). Particularising (7.112) to each specific critical point with at least SU(3) symmetry of the SO(8) and ISO(7) gaugings, making use of the relevant embedding tensors, and again trading g^2 for L^2, all the r.h.s.'s of (7.106) and (7.107) with $n = 1$ and $k = 1$ are reproduced. For example, using the appropriate embedding tensors and vevs, we find that (7.112) evaluates to $\frac{56}{3}$, both for the $\mathrm{SU}(3) \times \mathrm{U}(1)_c$ point of the SO(8) gauging and for the $\mathrm{SU}(3) \times \mathrm{U}(1)_v$ point of the ISO(7) gauging, once that g^2 is replaced with the relevant L^2. The trace relation (7.109) is a direct consequence of (7.112) and the overcounting feature mentioned in section 6.1.

Interestingly, the mass matrix (6.9)-(6.12) also reproduces the KK graviton spectrum of the SO(4) solution of section 7.2 and the ones in table 7.6 for type IIB S-folds with period $T = 2\pi$ that uplift from vacua of the $\big(\mathrm{SO}(6)\times\mathrm{SO}(1,1)\big)\ltimes\mathbb{R}^{12}$ gauging, *provided* the $\mathrm{U}(1)_\eta$ quantum number n is analytically continued as $n' = in$, with $i^2 = -1$. The origin of this analytic continuation can be put down to the fact that the SL(8)-covariant graviton mass matrix formula (6.9)-(6.12) actually sees the compactified $\mathrm{U}(1)_\eta$ as the non-compact $\mathrm{SO}(1,1)$ factor in the $D = 4$ gauge group $\big(\mathrm{SO}(6) \times \mathrm{SO}(1,1)\big) \ltimes \mathbb{R}^{12}$. From a IIB perspective, this factor is associated to a hyperboloid uplift [28]. In any case, the (analytically continued) spectra of the type IIB S-folds can be also organised in $\mathrm{SL}(8,\mathbb{R})$ KK levels $m = 0, 1, \ldots$ through the branching

$$[m,0,0,0,0,0,0]_{\mathrm{SL}(8)} \xrightarrow{\mathrm{SO}(6)\times\mathrm{SO}(1,1)} \sum_{s=0}^{\left[\frac{m}{2}\right]} \sum_{\ell=0}^{m-2s} \sum_{p=0}^{m-2s-\ell} [\ell,0,0]_{m-2s-\ell-2p}\,. \tag{7.113}$$

This approach contains redundant states that can be projected out as discussed below (6.13). It would be interesting to determine if (6.9)-(6.12) can be modified in such a way that the KK graviton spectra of vacua of the SO(8) and ISO(7) gauging are still obtained, and the physical spectra of the compactified S-folds is recovered as well.

The mass matrices for the lower-spin fields, (6.48) and (6.51), rendered by ExFT allow to extend these considerations to fields other than massive gravitons. Table 7.7 shows that at lowest level $n = k = 0$ the spectra for all modes tend to coincide, and we have checked that whenever this happens, the milder universality of sums like the ones relevant for the gravitons remains higher up in the tower modulo some provisos relating different gaugings that will be discussed in future work.

Coda

Conclusion and outlook

This thesis has addressed the relevance of the Kaluza-Klein problem in string theory, focusing on new methods applied to consistent truncations and Kaluza-Klein spectroscopy. For the dimensional reductions, we have seen in chapter 2 that, in line with the recently proved [44] conjecture of [32], for every $\mathcal{N}=2$ AdS_4 solution of M-theory, there exists a consistent truncation to minimal $\mathcal{N}=2$ gauged supergravity in $D=4$ such that the lower-dimensional supergravity fields enter the eleven-dimensional configuration coupled to the G-structure tensors which control the undeformed geometry. For the general class of solutions in [17], consistency has been checked in all detail in section 2.3, including Einstein's equations and supersymmetry variations. See appendix A for further details.

These supersymmetric AdS configurations have a clear holographic significance, and their construction has sometimes been anticipated out of field theory intuition based on RG flows. This was precisely the case for the cubic deformation of ABJM, whose IR fixed point is dual to the GMPS solution reviewed in section 2.2. The operator spectrum of this strongly coupled SCFT was studied in chapter 5 by means of the dual KK towers. The analysis of the spectrum of massive gravitons and $\mathcal{N}=2$ supermultiplets around this solution has exhibited some of its non-trivial global properties, as the fact that its metric cannot be isometrically embedded in $\mathbb{R}^8$.

Besides supersymmetry, the other main resource to study consistent truncations and spectroscopy is U-duality. This has been the main focus of chapter 3, where the notions of tensor and duality hierarchies have been introduced and used to study both four-dimensional maximal supergravity and its gaugings, and how these can be oxidised into higher-dimensional theories. Chapters 6 and 7 have then followed this route with in introduction of Exceptional Field Theory, which reformulates the full higher-dimensional theories in a duality-covariant way. Apart from the insight that this reformulation has granted on string duality itself, from the practical side it has sourced major breakthroughs in the construction of consistent truncations via Scherk-Schwarz factorisations, and the more recent obtention of KK spectra on warped inhomogeneous compactifications with fluxes even on configurations with little or none preserved (super)symmetry or even non-geometrically patched. This last topic was until very recently unapproachable, and some of the works during the graduate research that has culminated in this dissertation can be counted among the first to develop and employ this approach [C, E–G].

The methods discussed in this thesis will doubtlessly provide new tools to address a variety of problems currently under inspection by the high-energy physics community. Some of these problems lie in the framework of the Swampland Conjectures [189–191] (see [192, 193] for recent reviews). Among this complex net of conjectures, there are a few that have been already directly tackled using the new KK exceptional machinery. As mentioned in section 7.2, the absence of a free CFT point in the $\mathcal{N} = 2$ conformal manifold dual to the S-fold solutions can be seen from the fact that at no point in this locus all operators carry rational conformal dimensions (see *e.g.* (7.31) for the complete spectrum at levels $(\ell, n) = (0, n)$, and (7.36) for the gravitons at levels $(\ell, n) = (1, n)$). Moreover, from the Zamolodchikov metric (7.7) and the range of the scalars in (7.9), it follows that this conformal manifold is non-compact. Both of these facts are in tension with the CFT distance conjecture [194], which claims that every conformal manifold in string theory should be compact and always contain a free point. Other ideas that have been fruitfully informed by spectroscopy results are the conjectured absence of scale separation in string theory [195], and the non-susy AdS conjecture [196]. Both of them can be understood as inducing bounds on the spectra of string theory configurations. The first can be stated as saying that for every configuration of string theory there are modes in the KK spectrum such that

$$LM \sim \mathcal{O}(1)\,.$$

This is indeed the case for all the solutions addressed in this dissertation.

Regarding the stability of non-supersymmetric AdS vacua, one of the first uses of the ExFT tools showed that the KK spectrum of scalar modes up in the tower can contain tachyonic modes even if there are none in the lower-dimensional truncation [135]. However, there are also non-supersymmetric solutions in massive type IIA and type IIB uplifting from the maximal supergravities in figure 3.1 whose perturbative spectrum is completely free from instabilities [138, 141]. The lack of tachyonic modes does not guarantee the absence of non-perturbative decay channels such as bubble nucleations [197, 198] or brane-jet instabilities [174, 199]. However, some of these channels have also been discarded for a few of them [141, 174].

Some of our results can also find application beyond high-energy physics. The consistent truncations studied in Part I allow for holographic understanding of systems which can be of interest in theoretical condensed matter (CMT) via the AdS/CMT correspondence (see *e.g.* [200–203]). Some novel directions in this context include Janus solutions describing interfaces in CFT [178, 204–206], and Q-lattice and susy-Q [207, 208] constructions breaking Lorentz invariance. The latter have been employed to construct RG flows of boomerang type [209, 210] such that the field theories in the UV and IR coincide, posing a conundrum about the counting of degrees of freedom from a Wilsonian perspective. Another interesting holographic application of these consistent truncations related to counting of degrees of freedom in RG flows are the flows across dimensions constructed in [60, 211, 212].

There are a number of questions left open by our results. In order to keep these conclusions as concise as possible, let me focus on only two of them. Regarding the $\mathcal{N}=2$ configuration of chapters 2 and 5, the fact that the GMPS vacuum is only known as a numerical solution and that only its spin-2 spectrum has been addressed is clearly unsatisfactory and deserves further attention. A humble step in this direction would be obtaining the precise way in which the space invaders arrange themselves to from $\mathcal{N}=2$ supermultiplets. A most promising tool to achieve this are the fugacities briefly mentioned in appendix E, as they provide an organised way of keeping track of which states need to be subtracted for multiplet completion or (super)Higgsing.

The second point concerns the universality properties discussed in section 7.3. Despite already observed in a sizeable number of examples so as to regard it as a robust phenomenon, it completely lacks understanding from *both* the supergravity and the field theory sides. It would therefore be desirable to extend this analysis to solutions of other (possibly non-maximal) gaugings, and figure out what distinguishes the solutions in table 7.7 which exhibit universality from the ones which do not, such as the $N=0$ SU(3)-invariant points of dyonic ISO(7) [74] and $\big(\mathrm{SO}(6)\times\mathrm{SO}(1,1)\big)\ltimes\mathbb{R}^{12}$ supergravities

[156], who do not have the same spectrum within the $D = 4$ supergravities nor the universal traces higher up in the graviton KK tower.

From both these questions, it is manifest that a natural direction for the near future is how to extend the results that apply to maximal supergravity to non-maximal theories. This has already been done for a class of half-maximal $D = 3$ theories that can be embedded into half-maximal supergravity in six dimensions [G], where an entire family of non-supersymmetric AdS_3 solutions were found which are completely stable at the perturbative level. These methods are also being extended to other dimensions and supergravities with varying amounts of supersymmetry, such as the $\mathfrak{f}(4)$ solution [213–215] in $D = 6$ [J]. Another interesting direction to pursue in the near future would be the inclusion of α' corrections in the computation of the KK spectra, which would be relevant to further assess stability issues or the properties of the conformal manifolds dual to the S-fold familes of section 7.2.

In the longer term, one might hope that these exceptional formulations shed light in how to achieve a non-perturbative definition of M-theory. Some proposals in this direction have appeared in [216–222].

Conclusiones y perspectiva

Esta tesis ha abordado la relevancia del problema de Kaluza-Klein en teoría de cuerdas, centrándose en nuevos métodos aplicados a truncamientos consistentes y espectroscopía de Kaluza-Klein. Para las reducciones dimensionales, hemos visto en el capítulo 2 que, de acuerdo con la recientemente probada [44] conjetura de [32], para cada solución $\mathcal{N} = 2$ AdS$_4$ de la teoría M, existe un truncamiento consistente a supergravedad gauge $\mathcal{N} = 2$ mínima en $D = 4$ tal que los campos de la supergravedad en dimensión inferior entran en la configuración once-dimensional acoplados a los tensores de G-estructura que controlan la geometría no deformada. Para la clase general de soluciones en [17], la consistencia se ha verificado con todo detalle en la sección 2.3, incluidas las ecuaciones de Einstein y las variaciones de supersimetría. Véase el apéndice A para más detalles.

Estas configuraciones AdS supersimétricas tienen una clara relevancia en holografía, y su construcción ha sido en algunos casos anticipada por la intuición de teoría de campos basada en flujos de renormalización. Este fue precisamente el caso para la deformación cúbica de ABJM, cuyo punto fijo en el infrarrojo es dual a la solución GMPS recapitulada en la sección 2.2. El espectro de operadores de esta SCFT fuertemente acoplada se ha estudiado en el capítulo 5 mediante las torres KK duales. El análisis del espectro de gravitones masivos y supermultipletes $\mathcal{N} = 2$ en torno a esta solución ha sacado a descubierto algunas de sus propiedades globales no triviales, como el hecho de que la métrica no puede ser embebida isométricamente en $\mathbb{R}^8$.

Aparte de supersimetría, el otro recurso fundamental para estudiar truncamientos consistentes y espectroscopía es U-dualidad. Este ha sido el foco del capítulo 3, donde las nociones de jerarquías tensoriales y jerarquías de dualidades se han introducido y han sido utilizadas para estudiar tanto supergravidad maximal en $D = 4$ como sus posibles *gaugings*, y cómo estos se pueden oxidar a teorías con más dimensiones. Los capítulos 6 y 7 siguen esta ruta con la introducción de *Exceptional Field Theory*, que es una reformulación de las teorías con dimensiones extra al completo de una guisa covariante bajo dualidades. Además de haber ahondado nuestro entendimiento de las dualidades en teoría de cuerdas per se, desde un punto de vista práctico, esta reformulación ha traído consigo grandes avances en la construcción de truncamientos consistentes via la factorización de Scherk-Schwarz, y más recientemente la obtención de espectros KK sobre soluciones inhomogéneas, con *warping* y flujos, e incluso en los casos en los que poca o ninguna (super)simetría es preservada o la solución está construida mediante parcheamientos no-geométricos. Esto último era impensable hasta hace muy poco, y algunos de los trabajos durante la investigación predoctoral que ha culminado en esta disertación se cuentan entre los primeros en desarrollar y emplear este acercamiento [C, E–G].

Los métodos discutidos en en esta tesis van sin lugar a dudas a proporcionar nuevas herramientas con que afrontar multitud de problemas actualmente en el punto de mira de la comunidad de física de altas energías. Algunos de estos problemas se circunscriben al campo de las Conjeturas de Ciénaga [189–191] (véanse también las reseñas [192, 193]). Dentro de esta compleja red de conjeturas, algunas han sido ya directamente abordadas usando la nueva maquinaria excepcional de espectroscopía KK. Como se mencionó en la sección 7.2, la ausencia de una CFT libre dentro de la variedad conforme $\mathcal{N} = 2$ dual a las soluciones S-fold se sigue del hecho de que no hay ningún punto dentro de este locus tal que todos los operadores tengan dimensiones conformes racionales (véanse, *e.g.* (7.31) para el espectro completo a niveles $(\ell, n) = (0, n)$, y (7.36) para el espectro de gravitones a niveles $(\ell, n) = (1, n)$). Además, de la métrica de Zamolodchikov (7.7) y el rango de valores de los escalares en (7.9), se sigue también que esta variedad conforme es no compacta. Estos dos hechos están en tensión con la *CFT distance conjecture* [194], que afirma que toda variedad conforme en teoría de cuerdas debe ser compacta y siempre contener un punto libre. Otras ideas que han sido fructíferamente informadas por resultados en espectroscopía son la conjeturada ausencia de separación de escalas en teoría de cuerdas [195], y la *non-susy AdS conjecture* [196]. Ambas conjeturas inducen límites en los espectros de las compactificaciones de teoría de cuerdas. La primera puede ser formulada como la afirmación de que para toda configuración de teoría de cuerdas existen modos en el espectro KK que cumplen

$$LM \sim \mathcal{O}(1)\,.$$

Esto es desde luego cierto en todas las soluciones estudiadas en esta disertación.

En cuanto a la estabilidad de vacíos AdS no supersimétricos, uno de los primeros ejemplos de uso de las herramientas de ExFT mostró que el espectro de modos escalares en las torres KK puede contener modos taquiónicos incluso si estos no aparecen en el truncamiento dimensional [135]. Sin embargo, también existen otras soluciones no supersimétricas en tipo IIA masiva y tipo IIB que originan de las supergravedades maximales en la figura 3.1 cuyos espectros perturbativos están completamente libres de inestabilidades [138, 141]. La ausencia de modos taquiónicos no garantiza la ausencia de canales de decaimiento no perturbativos como la formación de burbujas [197, 198] o las inestabilidades brane-jet [174, 199]. No obstante, algunos de estos canales también han sido descartados para algunas de estas soluciones [141, 174].

La aplicación de algunos de nuestros resultados puede trascender la física de altas energías. Los truncamientos consistentes estudiados en la Parte I permiten comprender holográficamente sistemas que pueden ser de interés en física teórica de la materia condensada (CMT) via la correspondencia AdS/CMT (véase *e.g.* [200–203]). Algunas nuevas líneas en este contexto incluyen las soluciones de tipo Jano que describen interfaces en una CFT [178, 204–206], y construcciones de tipo *Q-lattice* y *susy-Q* [207, 208] que rompen la invarianza Lorentz. Estas últimas han sido empleadas en la construcción de flujos de renormalización de tipo boomerang [209, 210] tales que las teorías de campos en el UV y el IR coinciden, generando un enigma sobre el conteo de grados de libertad desde una perspectiva Wilsoniana. Otra aplicación holográfica interesante de estos truncamientos consistentes también relacionada con el conteo de grados de libertad en flujos de renormalización son los flujos entre dimensiones de [60, 211, 212].

Nuestros resultados dejan algunas cuestiones abiertas. Con objeto de mantener estas conclusiones concisas, me centraré en sólo dos de ellas. Respecto a las configuraciones $\mathcal{N} = 2$ de los capítulos 2 y 5, el hecho de que la solución GMPS sólo se conozca numéricamente y sólo su espectro de modos de espín-2 haya podido ser abordado hasta el momento es claramente insatisfactorio y merece atención futura. Un humilde paso en esta dirección sería obtener de manera precisa el modo en que los *space invaders* se organizan para formar supermultipletes $\mathcal{N} = 2$. Una herramienta prometedora para este propósito son las fugacidades mencionadas brevemente en el apéndice E, ya que proporcionan una manera organizada de llevar la cuenta de qué estados necesitan ser empleados en formar multipletes of participar en el (super)Higgsing.

El segundo punto se refiere a las propiedades de universalidad discutidas en la sección 7.3. A pesar de que han sido observadas en un buen número de ejemplos y pueden por tanto ser consideradas como un fenómeno robusto, carecen completamente de comprensión tanto desde el punto de vista gravitatorio como desde la teoría de campos. Sería por ende deseable extender nuestro análisis a soluciones de otros gaugings (posiblemente no maximales), y desentrañar qué distingue a las soluciones en la tabla 7.7 que exhiben universalidad de las que no, como es el caso de los puntos $N = 0$ SU(3)-invariantes de las supergravedades gauge con gaugings ISO(7) diónico [74] y $\big(\mathrm{SO}(6) \times \mathrm{SO}(1,1)\big) \ltimes \mathbb{R}^{12}$ [156], que no comparten el mismo espectro ni dentro de la supergravedad $D = 4$ ni las trazas arriba en la torre KK.

Ambas cuestiones sugieren que una dirección natural para el futuro próximo es extender nuestros resultados para supergravedades maximales a teorías con menor supersimetría. Esto se ha llevado a cabo ya para una clase de teorías semi-maximales en $D = 3$ que pueden ser embebidas en supergravedad semi-maximal en seis dimensiones [G], lo que ha llevado al descubrimiento de una familia entera de soluciones AdS_3 no supersimétricas completamente estables a nivel perturbativo. Estos métodos también se están extendiendo a otras dimensiones y supergravedades con distinta cantidad de supersimetría, como la solución $\mathfrak{f}(4)$ [213–215] en $D = 6$ [J]. Otra dirección interesante para emprender en el futuro próximo es la inclusión de correcciones α' en el cálculo es espectros KK, que podría ser relevante para escudriñar más a fondo las cuestiones de estabilidad o las propiedades de las variedades conformes duales a las familias de soluciones de tipo S-fold de las sección 7.2.

A largo plazo, cabe esperar que estas reformulaciones excepcionales arrojen luz en cómo obtener una definición no perturbativa de Teoría M. Algunas propuestas en esta dirección incluyen [216–222].

Appendices

Appendix A

Consistency of the truncation in chapter 2

A.1 Consistency of Equations of motion

Assuming that the background geometry (2.10) satisfies the $D = 11$ field equations (I.1) and imposing their $D = 4$ counterparts (2.34), the KK ansatz (2.37) also solves the $D = 11$ field equations provided the unknown forms X, Y on the background geometry obey the restrictions (2.40)–(2.44). Equation (2.42) must in turn yield the $D = 4$ Einstein equation. Let us derive these equations and show that X and Y given in (2.45) solve them.

In order to do this, it is convenient to split the hatted form $\hat{F}_{(4)}$ in (2.39) into a background contribution, $F_{(4)}$ in (2.10), plus a $D = 4$ graviphoton contribution using $i_\xi E_1 = \|\xi\|$:

$$\hat{F}_{(4)} = F_{(4)} - \frac{g}{4}\bar{A} \wedge i_\xi F_{(4)} . \tag{A.1}$$

The unknown forms X and Y can be similarly split. For calculation purposes, however, it is more convenient to sweep the $\|\xi\|$ factors under the rug and write

$$X = \hat{e} \wedge i_{\hat{e}^*} X + \tilde{X} , \qquad Y = \hat{e} \wedge i_{\hat{e}^*} Y + \tilde{Y} , \qquad \text{with} \quad i_{\hat{e}^*}\tilde{X} = i_{\hat{e}^*}\tilde{Y} = 0 , \tag{A.2}$$

where $\hat{e} \equiv d\psi + \mathcal{A} - g\bar{A}$ and $\hat{e}^*$ is the dual vector such that $i_{\hat{e}^*}\hat{e} = 1$. We thus have

$$dX = (d\mathcal{A} - g\bar{F}) \wedge i_{\hat{e}^*} X - \hat{e} \wedge d\, i_{\hat{e}^*} X + d\tilde{X} , \tag{A.3}$$

and similarly for dY. With these definitions, it is now straightforward to see that $G_{(4)}$ in (2.37) obeys

$$\begin{aligned} dG_{(4)} = & -\frac{g}{4}\bar{F} \wedge i_\xi F_{(4)} - \frac{g}{4}\bar{A} \wedge d\, i_\xi F_{(4)} \\ & - g\bar{F} \wedge \Big[(d\mathcal{A} - g\bar{F}) \wedge i_{\hat{e}^*} X - \hat{e} \wedge d\, i_{\hat{e}^*} X + d\tilde{X}\Big] \\ & - g \star_4 \bar{F} \wedge \Big[(d\mathcal{A} - g\bar{F}) \wedge i_{\hat{e}^*} Y - \hat{e} \wedge d\, i_{\hat{e}^*} Y + d\tilde{Y}\Big] , \end{aligned} \tag{A.4}$$

on the $D = 4$ field equations (2.34) for $\bar{F}$. Imposing $dG_{(4)} = 0$ and requiring that the terms linear and quadratic in $\bar{F}$ and $\star_4\bar{F}$ separately vanish, we arrive at (2.40). These equations imply $X = \tilde{X}$, $Y = \tilde{Y}$, which we set henceforth.

We next move on to the four-form equation of motion. We fix the orientation such that $\mathrm{vol}_{11} = e^{11\Delta}\ \mathrm{vol}_4 \wedge \mathrm{vol}_7$, with $\mathrm{vol}_4 = e^0 \wedge e^1 \wedge e^2 \wedge e^3$ and [17]

$$\mathrm{vol}_7 = -e^4 \wedge \cdots \wedge e^{10} = -E_1 \wedge E_2 \wedge E_3 \wedge \mathrm{vol}(g_{\mathrm{SU}(2)}) , \tag{A.5}$$

in terms of the frame introduced in footnote (2) with g_4 taking the rôle of g_{AdS_4}. In the following, the Hodge operators $\star_{11}$, $\star_4$, $\star_7$ are understood to be associated to the volume forms corresponding to g_{11}, g_4 and g_7, with g_4 the metric in (2.37) and g_7 as in the vacuum solution. With these conventions, using the torsion conditions (2.19)–(2.21) and the $D = 4$ field equations (2.34) of the graviphoton, we compute

$$\begin{aligned} d \star_{11} G_{(4)} = & \ \mathrm{vol}_4 \wedge d\,(e^{3\Delta} \star_7 F_{(4)}) - g\bar{F} \wedge \Big(\frac{m}{4}\|\xi\| e^{3\Delta}\, \mathrm{vol}(g_{\mathrm{SU}(2)}) \wedge E_2 \wedge E_3\Big) \\ & - \frac{g}{4} \star_4 \bar{F} \wedge \big[(d\mathcal{A} - g\bar{F}) e^{3\Delta} \wedge i_\xi \star_7 X - \hat{e} \wedge d\,(e^{3\Delta} \wedge i_\xi \star_7 X)\big] \\ & + \frac{g}{4}\bar{F} \wedge \big[(d\mathcal{A} - g\bar{F}) e^{3\Delta} \wedge i_\xi \star_7 Y - \hat{e} \wedge d\,(e^{3\Delta} \wedge i_\xi \star_7 Y)\big] . \end{aligned} \tag{A.6}$$

We also find

$$\begin{aligned} G_{(4)} \wedge G_{(4)} = & \ 2m\, \mathrm{vol}_4 \wedge F_{(4)} - 2g\hat{F}_{(4)} \wedge (\bar{F} \wedge X + \star_4\bar{F} \wedge Y) \\ & + 2g^2 \bar{F} \wedge \star_4\bar{F} \wedge X \wedge Y + g^2 \bar{F} \wedge \bar{F} \wedge (X \wedge X - Y \wedge Y) . \end{aligned} \tag{A.7}$$

Putting (A.6) and (A.7) together, we obtain the set of equations in (2.41).

Finally, we deal with the Einstein equation. In a frame $\{\tilde{e}^A\}$ for the metric in (2.37), $g_{11} = \eta_{AB}\tilde{e}^A \otimes \tilde{e}^B$, we obtain the following components of the Ricci tensor:

$$\begin{aligned} \tilde{\mathrm{Ric}}_{\alpha\beta} &= e^{-2\Delta}\left\{\mathrm{Ric}_{\alpha\beta} - \frac{g^2}{32}\|\xi\|^2 \bar{F}_{\alpha\gamma}\bar{F}_\beta{}^\gamma - 9(\partial_a\Delta\partial^a\Delta + \nabla_a\nabla^a\Delta)\eta_{\alpha\beta}\right\} , \\ \tilde{\mathrm{Ric}}_{\alpha b} &= e^{-2\Delta}\left\{-\frac{g}{8}\|\xi\|\delta_{8b}\nabla_\gamma\bar{F}_\alpha{}^\gamma\right\} , \end{aligned}$$

$$\tilde{\text{Ric}}_{ab} = e^{-2\Delta}\Big\{\text{Ric}_{ab} + \frac{g^2}{64}\|\xi\|^2\delta_{8a}\delta_{8b}\bar{F}_{\gamma\delta}\bar{F}^{\gamma\delta} + 9\big[\partial_a\Delta\partial_b\Delta - \nabla_a\nabla_b\Delta - (\partial_c\Delta\partial^c\Delta - \nabla_c\nabla^c\Delta)\delta_{ab}\big]\Big\}\,, \tag{A.8}$$

where we have split the global indices $A = (\alpha, a)$ with $\alpha = 0, \ldots, 3$ and $a = 4, \ldots, 10$. In these expressions, $\text{Ric}_{\alpha\beta}$ and Ric_{ab} are the external and internal Ricci tensors in tangent space. In the same frame, the components of the four-form in (2.37) can be read off to be

$$\begin{aligned} G_{\alpha\beta\gamma\delta} &= me^{-4\Delta}\epsilon_{\alpha\beta\gamma\delta}\,, \qquad G_{abcd} = e^{-4\Delta}F_{abcd}\,, \\ G_{\alpha\beta ab} &= -ge^{-4\Delta}\big[\bar{F}_{\alpha\beta}X_{ab} + \tfrac{1}{2}\epsilon_{\alpha\beta\gamma\delta}\bar{F}^{\gamma\delta}Y_{ab}\big], \end{aligned} \tag{A.9}$$

with $\epsilon_{0123} = 1$. The tangent space components of the right-hand-side of the Einstein equation follow from the trace-reversed stress-energy tensor

$$T_{AB} \equiv \frac{1}{12}\left(G_{ACDE}G_B{}^{CDE} - \frac{1}{12}\eta_{AB}G^2\right)\,, \tag{A.10}$$

where $T = T_{AB}\tilde{e}^A \otimes \tilde{e}^B$, and read

$$\begin{aligned} e^{8\Delta}T_{\alpha\beta} &= -\frac{1}{3}m^2\eta_{\alpha\beta} + \frac{g^2}{4}(X^2+Y^2)\bar{F}_{\alpha\gamma}\bar{F}_\beta{}^\gamma - \frac{g^2}{24}\eta_{\alpha\beta}\bar{F}^2(X^2+2Y^2) \\ &\quad - \frac{g^2}{4}\bar{F}_{\gamma(\alpha}\epsilon_{\beta)}{}^{\gamma\mu\nu}\bar{F}_{\mu\nu}\,X_{cd}Y^{cd} - \frac{g^2}{24}\eta_{\alpha\beta}\epsilon_{\mu\nu\rho\sigma}\bar{F}^{\mu\nu}\bar{F}^{\rho\sigma}X_{cd}Y^{cd}\,, \\ e^{8\Delta}T_{\alpha b} &= 0\,, \\ e^{8\Delta}T_{ab} &= \frac{1}{2}\left[F_{acde}F_b{}^{cde} - \frac{1}{12}\eta_{ab}F^2\right] \\ &\quad + \frac{g^2}{24}\bar{F}^2\left[6(X_{ac}X_b{}^c - Y_{ac}Y_b{}^c) - \eta_{ab}(X^2-Y^2)\right] \\ &\quad + \frac{g^2}{24}\epsilon_{\mu\nu\rho\sigma}\bar{F}^{\mu\nu}\bar{F}^{\rho\sigma}\left[3(X_{ac}Y_b{}^c + Y_{ac}X_b{}^c) - \eta_{ab}\,X_{cd}Y^{cd}\right] + \frac{1}{2}m^2\,\eta_{ab}\,. \end{aligned} \tag{A.11}$$

Equating (A.8) and (A.11) we obtain equations (2.42)–(2.44) of the main text.

We have thus shown that the system of equations (2.40)–(2.44) is equivalent to the $D = 11$ Bianchi identities and equations of motion (I.1) evaluated on the KK ansatz (2.37), when the $D = 4$ graviphoton's field equations in (2.34) are imposed. Let us now verify that X and Y given in (2.45) solve these equations and that, for this choice, (2.42) reduces to the $D = 4$ Einstein equation written in (2.34). The contribution in (2.40) that is linear in $\bar{F}$, combined with the fact that $\tilde{X} = X$, implies

$$dX = -\frac{1}{4} i_\xi F_{(4)} = -\frac{1}{4} d\left(e^{3\Delta}\sqrt{1-\|\xi\|^2} J_1\right) , \tag{A.12}$$

where we have used (2.22) to compute the inner product with ξ. Thus,

$$X = -\frac{1}{4} e^{3\Delta}\sqrt{1-\|\xi\|^2}\, J_1 + \delta , \tag{A.13}$$

for a closed two-form δ. As for Y, we see from the torsion condition (2.19) that a natural ansatz for it that is free from legs along E^1 and is closed (in fact, exact), is

$$Y = k\, e^{3\Delta} \left(J_3 - \|\xi\| E_2 \wedge E_3\right) , \tag{A.14}$$

for some constant k. The forms X, Y in (A.13), (A.14) solve, for all closed δ and k, the conditions (2.40) coming from the $D = 11$ Bianchi identity.

The four-form equations of motion, (2.41), fix δ and k. First, the seven-dimensional Hodge duals of (A.13), (A.14) need to be worked out. We get

$$\begin{aligned} i_\xi \star_7 X &= \frac{\|\xi\|}{4} e^{3\Delta}\sqrt{1-\|\xi\|^2} J_1 \wedge E_2 \wedge E_3 + i_\xi \star_7 \delta , \\ i_\xi \star_7 Y &= -k\|\xi\| e^{3\Delta}\left(E_2 \wedge E_3 - \|\xi\| J_3\right) \wedge J_3 . \end{aligned} \tag{A.15}$$

Using (A.13)–(A.15), and (2.23) for $d\mathcal{A}$, the set of equations (2.41) becomes, after some rearrangement,

$$\begin{aligned} e^{6\Delta}\left\{\frac{k\|\xi\|}{4}(1+4k) J_3 \wedge E_2 \wedge E_3 + \left[\frac{\|\xi\|^2}{32}(1+4k) + \frac{1}{2}\left(k^2 - \frac{1}{16}\right)\right] J_1 \wedge J_1\right\} \\ +\frac{1}{2}\delta \wedge \left(\delta - \frac{1}{2} e^{3\Delta}\sqrt{1-\|\xi\|^2} J_1\right) = 0 , \end{aligned} \tag{A.16}$$

$$\begin{aligned} \frac{1}{4}\left(k + \frac{1}{4}\right) e^{6\Delta}\|\xi\|\sqrt{1-\|\xi\|^2} J_1 \wedge E_2 \wedge E_3 \\ +e^{3\Delta}\left[\frac{1}{4} i_\xi \star_7 \delta + k(J_3 - \|\xi\| E_2 \wedge E_3) \wedge \delta\right] = 0 , \end{aligned} \tag{A.17}$$

$$\begin{aligned} m e^{3\Delta}\left[-\|\xi\|\left(\frac{k}{2} + \frac{1}{8}\right) + \frac{1}{\|\xi\|}\left(k + \frac{1}{4}\right)\right] J_3 \wedge J_3 \wedge E_2 \wedge E_3 - \delta \wedge \hat{F}_{(4)} \\ -\frac{1}{8}\left(k + \frac{1}{4}\right) \hat{e} \wedge d\left[e^{6\Delta}(1-\|\xi\|^2) J_1 \wedge J_1\right] = 0 , \end{aligned} \tag{A.18}$$

$$\frac{m}{3\|\xi\|^2} i_\xi \star_7 \delta \wedge \left[J_3 + \left(3\|\xi\| - \frac{4}{\|\xi\|} \right) E_2 \wedge E_3 \right] - \frac{1}{4} \hat{e} \wedge d\left(e^{3\Delta} i_\xi \star_7 \delta \right) = 0 \ . \tag{A.19}$$

It is now easy to see that all these equations are satisfied for the (very possibly, unique) choice

$$\delta = 0 \ , \qquad k = -\tfrac{1}{4} \ . \tag{A.20}$$

The expressions (2.45) for X and Y that we brought to the main text correspond to (A.13), (A.14) with (A.20). At this point we have shown that X and Y thus defined solve the equations (2.40), (2.41) implied by the Bianchi identity and equation of motion for the $D = 11$ four-form. Let us see that these are also compatible with the restrictions (2.42), (2.44) implied by the $D = 11$ Einstein equation. These equations can be further simplified by noting the following relation between m, Δ and the AdS_4 cosmological constant:

$$9(\partial_a \Delta \partial^a \Delta + \nabla_a \nabla^a \Delta) - \tfrac{1}{3} e^{-6\Delta} m^2 = -12 \ . \tag{A.21}$$

Next, reading off the tangent space components of X, Y in (2.45), we can compute the following contractions

$$\begin{aligned} X_{ac} Y_b{}^c &= -\frac{1}{16} \sqrt{1 - \|\xi\|^2} e^{6\Delta} \left[\delta^6_a \delta^5_b - \delta^7_a \delta^4_b + \delta^4_a \delta^7_b - \delta^5_a \delta^6_b \right] \ , \\ X_{ac} X^{bc} &= \frac{1}{16} (1 - \|\xi\|^2) e^{6\Delta} \left[\delta^4_a \delta^b_4 + \delta^5_a \delta^b_5 + \delta^6_a \delta^b_6 + \delta^7_a \delta^b_7 \right] \ , \\ Y_{ac} Y^{bc} &= \frac{1}{16} e^{6\Delta} \left[\delta^4_a \delta^b_4 + \delta^5_a \delta^b_5 + \delta^6_a \delta^b_6 + \delta^7_a \delta^b_7 + \|\xi\|^2 (\delta^9_a \delta^b_9 + \delta^{10}_a \delta^b_{10}) \right] \ . \end{aligned} \tag{A.22}$$

Using these expressions, and assuming that the undeformed internal Einstein equations hold, we find that the internal components (2.44) of the Einstein equation vanish automatically for all values of the graviphoton $\bar{F}$. Similarly, the external components (2.42) of the $D = 11$ Einstein equation become

$$\mathrm{Ric}_{\alpha\beta} + 12\, \eta_{\alpha\beta} = \frac{g^2}{8} \left(\bar{F}_{\alpha\gamma} \bar{F}_\beta{}^\gamma - \frac{1}{4} \eta_{\alpha\beta} \bar{F}^2 \right) \ . \tag{A.23}$$

This coincides with the Einstein equation that derives from the $D = 4$ $\mathcal{N} = 2$ gauged supergravity Lagrangian after a rescaling,

$$\bar{g}_4 = 4 g^{-2} g_4 \ , \tag{A.24}$$

of the four-dimensional metric.

A.2 Consistency of Supersymmetry variations

As discussed in the main text, for the ansatz (2.46) the internal components of the $D = 11$ gravitino variation (I.3) under supersymmetry identically vanish on the KK ansatz (2.37), and the external components reduce to the supersymmetry variations for the $D = 4$ $\mathcal{N} = 2$ gravitino, (2.36).

Let us first address the internal components. Using the gamma matrix decomposition (1.7) and the $G_{(4)}$ components (A.9), some calculation allows us to write

$$\begin{aligned}\delta\Psi_a = \delta^0\Psi_a - g\,e^{-\Delta/2}\Big\{ &\bar{F}_{\beta\gamma}(\rho^{\beta\gamma}\otimes\mathbb{1})\Big[-\frac{1}{8}k_a\bar{\psi}_i\otimes\chi_i - \frac{1}{8}k_a(\bar{\psi}_i)^c\otimes\chi_i^c \\ &+\frac{e^{-3\Delta}}{48}X_{de}\bar{\psi}_i\otimes{\gamma_a}^{de}\chi_i - \frac{e^{-3\Delta}}{48}X_{de}(\bar{\psi}_i)^c\otimes{\gamma_a}^{de}\chi_i^c \\ &-\frac{e^{-3\Delta}}{12}X_{ae}\bar{\psi}_i\otimes\gamma^e\chi_i + \frac{e^{-3\Delta}}{12}X_{ae}(\bar{\psi}_i)^c\otimes\gamma^e\chi_i^c\Big] \\ &+\bar{F}^*_{\beta\gamma}(\rho^{\beta\gamma}\otimes\mathbb{1})\Big[\frac{e^{-3\Delta}}{48}Y_{de}\bar{\psi}_i\otimes{\gamma_a}^{de}\chi_i - \frac{e^{-3\Delta}}{48}Y_{de}(\bar{\psi}_i)^c\otimes{\gamma_a}^{d}e\chi_i^c \\ &-\frac{e^{-3\Delta}}{12}Y_{ae}\bar{\psi}_i\otimes\gamma^e\chi_i + \frac{e^{-3\Delta}}{12}Y_{ae}(\bar{\psi}_i)^c\otimes\gamma^e\chi_i^c\Big]\Big\}\,, \end{aligned} \tag{A.25}$$

where we have defined $\bar{F}^*_{\delta\epsilon} \equiv \frac{1}{2}\epsilon_{\delta\epsilon\kappa\lambda}\bar{F}^{\kappa\lambda}$ and $k_a = \frac{1}{4}\xi_a = \frac{1}{4}\|\xi\|\delta_{a8}$. Here, $\delta^0\Psi_a$ is the tensor product of $\bar{\psi}_i$ with the left-hand-side of the first equation in (2.14), and thus vanishes. In the following, it is useful to note that

$$\epsilon_{\alpha\beta\gamma\delta}\rho^\delta = -i\rho_{\alpha\beta\gamma}\rho_5\,, \qquad \epsilon_{\alpha\beta\gamma\delta}\rho^{\gamma\delta} = -2i\rho_{\alpha\beta}\rho_5\,, \qquad \epsilon_{\alpha\beta\gamma\delta}\rho^{\beta\gamma\delta} = 6i\rho_\alpha\rho_5\,, \tag{A.26}$$

and

$${\rho_\alpha}^{\delta\epsilon} = \rho^{\delta\epsilon}\rho_\alpha - 2\rho^{[\delta}\delta_\alpha^{\epsilon]}\,. \tag{A.27}$$

Using these relations, equation (A.25) can be further simplified into

$$\begin{aligned}\delta\Psi_a = -g\,e^{-\Delta/2}\bar{F}_{\beta\gamma}(\rho^{\beta\gamma}\otimes\mathbb{1})\Big[&-\frac{1}{8}k_a\bar{\psi}_i\otimes\chi_i - \frac{1}{8}k_a(\bar{\psi}_i)^c\otimes\chi_i^c \\ &+\frac{e^{-3\Delta}}{48}X_{de}\bar{\psi}_i\otimes{\gamma_a}^{de}\chi_i - \frac{e^{-3\Delta}}{48}X_{de}(\bar{\psi}_i)^c\otimes{\gamma_a}^{de}\chi_i^c \\ &-\frac{e^{-3\Delta}}{12}X_{ae}\bar{\psi}_i\otimes\gamma^e\chi_i + \frac{e^{-3\Delta}}{12}X_{ae}(\bar{\psi}_i)^c\otimes\gamma^e\chi_i^c \\ &-\frac{ie^{-3\Delta}}{48}Y_{de}\bar{\psi}_i\otimes{\gamma_a}^{d}e\chi_i - \frac{ie^{-3\Delta}}{48}Y_{de}(\bar{\psi}_i)^c\otimes ?{\gamma_a}^{de}\chi_i^c \\ &+\frac{ie^{-3\Delta}}{12}Y_{ae}\bar{\psi}_i\otimes\gamma^e\chi_i + \frac{ie^{-3\Delta}}{12}Y_{ae}(\bar{\psi}_i)^c\otimes\gamma^e\chi_i^c\Big]\,. \end{aligned} \tag{A.28}$$

Acting with $P_\pm = \frac{1}{2}(\mathbb{1} \pm \rho_5) \otimes \mathbb{1}$, we get that $\delta\Psi_a = 0$ if, and only if, the following projection holds,

$$\Big(-6k_a + e^{-3\Delta}(X_{de} - iY_{de})({\gamma_a}^{de} - 4\delta_a^d \gamma^e)\Big)\chi_i = 0 \, , \tag{A.29}$$

independently for $i = 1, 2$. Introducing the explicit expressions (2.45) for X and Y, some algebra allows us to massage the relation (A.29), for $a = 8$, into (2.47) and, for $a \neq 8$, into (2.48) of the main text. These projections can be checked to be fully compatible with the SU(2)–structure that defines the background geometry, without giving independent restrictions on the Killing spinors χ_i. As an instance of how this works, the projector (2.47) gives rise to a bilinear

$$\begin{aligned} &\bar{\chi}^c_+ \Big[\|\xi\|(3\gamma^8 + i\gamma^{910}) + \sqrt{1-\|\xi\|^2}(\gamma^{46} - \gamma^{57}) - i(\gamma^{45} + \gamma^{67})\Big]\chi_- \\ &\quad = \|\xi\|\Big(3(-i\|\xi\|) + i\|\xi\|\Big) + \sqrt{1-\|\xi\|^2}\big(-2i\sqrt{1-\|\xi\|^2}\big) - i(-2) \, , \end{aligned} \tag{A.30}$$

with $\chi_\pm = \frac{1}{\sqrt{2}}(\chi_1 \pm i\chi_2)$, and where we have used (B.2), (B.3) of [17]. This vanishes identically.

Next, we turn to the external variations of the gravitino. Particularising (I.3) to external indices, employing the basis (1.7) for the Dirac matrices, and extensively using the underformed Killing spinor equations (2.14), we can write

$$\begin{aligned} \delta\Psi_\mu = e^{\Delta/2}\Big\{ &\nabla_\mu\bar{\psi}_i \otimes \chi_i - \rho_\mu\bar{\psi}_i \otimes \chi_i^c - \frac{g\|\xi\|}{16}\bar{F}_{\mu\beta}\rho^\beta\bar{\psi}_i \otimes \gamma^8\chi_i \\ &+ \frac{g}{4}\nabla_b k_c \bar{A}_\mu\bar{\psi}_i \otimes \gamma^{bc}\chi_i + \frac{g\|\xi\|}{4}\bar{A}_\mu\bar{\psi}_i \otimes \nabla_8\chi_i \\ &- \frac{g^2\|\xi\|^2}{128}\bar{A}_\mu\bar{F}_{\beta\gamma}\rho^{\beta\gamma}\bar{\psi}_i \otimes \chi_i - \frac{ge^{-3\Delta}}{48}\left(\bar{F}_{\delta\epsilon}X_{bc} + \bar{F}^*_{\delta\epsilon}Y_{bc}\right){\rho_\mu}^{\delta\epsilon}\bar{\psi}_i \otimes \gamma^{bc}\chi_i \\ &+ \frac{g^2\|\xi\|e^{-3\Delta}}{192}\bar{A}_\mu\left(\bar{F}_{\delta\epsilon}X_{bc} + \bar{F}^*_{\delta\epsilon}Y_{bc}\right)\rho^{\delta\epsilon}\bar{\psi}_i \otimes {\gamma_8}^{bc}\chi_i \\ &+ \frac{ge^{-3\Delta}}{12}\left(\bar{F}_{\mu\gamma}X_{de} + \bar{F}^*_{\mu\gamma}Y_{de}\right)\rho^\gamma\bar{\psi}_i \otimes \gamma^{de}\chi_i\Big\} + m.c. \end{aligned} \tag{A.31}$$

From (2.24) of [17] and $\mathcal{L}_\xi\chi = \nabla_\xi\chi + \frac{1}{4}\nabla_a\xi_b\gamma^{ab}\chi$ (see [223]), we find that $\mathcal{L}_\xi\chi_1 = -2\chi_2$ and $\mathcal{L}_\xi\chi_2 = 2\chi_1$, so that

$$\|\xi\|\nabla_8\chi_1 + \nabla_a k_b\gamma^{ab}\chi_1 = -2\chi_2 \, , \qquad \|\xi\|\nabla_8\chi_2 + \nabla_a k_b\gamma^{ab}\chi_2 = 2\chi_1 \, . \tag{A.32}$$

Bringing these relations to (A.31) and using the $D = 4$ Dirac matrix relations (A.26), (A.27) to get rid of the $\bar{F}^*_{\delta\epsilon}$ terms, we obtain

$$\begin{aligned}\delta\Psi_\mu = e^{\Delta/2}\Big\{&\nabla_\mu\bar{\psi}_i\otimes\chi_i - \rho_\mu(\bar{\psi}_i)^c\otimes\chi_i - \frac{g\|\xi\|}{16}\bar{F}_{\mu\beta}\rho^\beta\bar{\psi}_i\otimes\gamma^8\chi_i\\ &-\frac{ig}{2}\epsilon_{ij}\bar{A}_\mu\bar{\psi}_i\otimes\chi_j - \frac{ge^{-3\Delta}}{48}\bar{F}_{\delta\epsilon}\Big[X_{bc}\big(\rho^{\delta\epsilon}\rho_\mu + 2\rho^\delta e^\epsilon{}_\mu\big)\bar{\psi}_i\\ &+2iY_{bc}\big(\rho^{\delta\epsilon}\rho_\mu - \rho^\delta e^\epsilon{}_\mu\big)\bar{\psi}_i\Big]\otimes\gamma^{bc}\chi_i\Big\} + m.c.\,, \qquad (A.33)\end{aligned}$$

where $e^\epsilon{}_\mu$ are the frame components. We can now use the G-structure compatible projections (2.47), (2.48) to further simplify the result. Using them, (A.33) becomes

$$\begin{aligned}\delta\Psi_\mu = e^{\Delta/2}\Big\{&\nabla_\mu\bar{\psi}_i\otimes\chi_i - \rho_\mu(\bar{\psi}_i)^c\otimes\chi_i - \frac{ig}{2}\epsilon_{ij}\bar{A}_\mu\bar{\psi}_i\otimes\chi_j\\ &+\frac{ig}{16}\bar{F}_{\delta\epsilon}\,\rho^{\delta\epsilon}\rho_\mu\,\bar{\psi}_i\otimes\gamma^{45}\chi_i\Big\} + m.c. \qquad (A.34)\end{aligned}$$

At this point, we recognise one more projection, (2.49) of the main text, that may be imposed to relate the internal spinors χ_i to their charge conjugates χ_i^c. This projection is, again, fully compatible with the original Killing spinor equations (2.14) and does not constrain the background geometry any further. Using (2.49) along with $(\chi_i^c)^c = \chi_i$ and $(\rho_{(n)}\bar{\psi}_i)^c = \rho_{(n)}(\bar{\psi}_i)^c$, equation (A.34) finally yields

$$\delta\Psi_\mu = e^{\Delta/2}\Big\{\nabla_\mu\bar{\psi}_i - \rho_\mu(\bar{\psi}_i)^c + \tfrac{ig}{2}\epsilon_{ij}\bar{A}_\mu\psi_j^+ + \tfrac{g}{16}\bar{F}_{\delta\epsilon}\,\rho^{\delta\epsilon}\rho_\mu\,\epsilon_{ij}(\bar{\psi}_j)^c\Big\}\otimes\chi_i + m.c. \qquad (A.35)$$

If the external components Ψ_μ of the $D = 11$ gravitino and the $D = 4$ gravitini $\psi^+_{i\mu}$ are related as in equation (2.50) of the main text, then (A.35) reproduces the supersymmetry variations (2.36) for the gravitini of $D = 4$ $\mathcal{N} = 2$ supergravity, after the metric rescaling (A.24) is taken into account.

Appendix B

SU(3) subsector of maximal SO(8)-supergravity

In chapter 3 we focused on the truncation of the $D = 4$ $\mathcal{N} = 8$ SO(8)-gauged supergravity and the corresponding class of solutions in $D = 11$ supergravity to the subsector invariant under

$$\mathrm{SU}(3) \subset \mathrm{SO}(8) \subset \mathrm{E}_{7(7)}\,. \tag{B.1}$$

In the SL$(8, \mathbb{R})$ basis, the generators of this $\mathrm{E}_{7(7)}$ can be given as $t_A{}^B$ and t_{ABCD}, with $t_A{}^A = 0$ and $t_{ABCD} = t_{[ABCD]}$. In the fundamental representation broken into $\mathbf{28} \oplus \mathbf{28'}$ pieces, they take the form

$$\begin{aligned} [t_A{}^B]_{CD}{}^{EF} &= 4\Big(\delta^B_{[C}\delta^{EF}_{D]A} + \frac{1}{8}\delta^A_B\delta^{EF}_{CD}\Big)\,, \qquad & [t_A{}^B]^{CD}{}_{EF} &= -[t_A{}^B]_{EF}{}^{CD}\,, \\ [t_{ABCD}]_{EFGH} &= \tfrac{1}{12}\epsilon_{ABCDEFGH}\,, & [t_{ABCD}]^{EFGH} &= 2\delta^{EFGH}_{ABCD}\,. \end{aligned} \tag{B.2}$$

The hatted generators of chapter 6 are defined in exactly the same way.

The generators of the $\mathrm{SO}(8) \subset \mathrm{SL}(8, \mathbb{R}) \subset \mathrm{E}_{7(7)}$ subgroup can be taken only out of the $t_A{}^B$. They can be chosen as

$$T_{AB} \equiv 2\, t_{[A}{}^C\delta_{B]C}\,. \tag{B.3}$$

The generators of $\mathrm{SU}(3) \subset \mathrm{SO}(8)$ can then be taken to be $\tilde{\lambda}_{\tilde{\alpha}}$, $\tilde{\alpha} = 1, \ldots, 8$, defined as

$$\begin{aligned} &\tilde{\lambda}_1 = T_{14} - T_{23}\,, \qquad \tilde{\lambda}_2 = -T_{13} - T_{24}\,, \qquad \tilde{\lambda}_3 = T_{12} - T_{34}\,, \\ &\tilde{\lambda}_4 = T_{16} - T_{25}\,, \qquad \tilde{\lambda}_5 = -T_{15} - T_{26}\,, \qquad \tilde{\lambda}_6 = T_{36} - T_{45}\,, \\ &\quad \tilde{\lambda}_7 = -T_{35} - T_{46}\,, \qquad \tilde{\lambda}_8 = \tfrac{1}{\sqrt{3}}\big(T_{12} + T_{34} - 2T_{56}\big)\,. \end{aligned} \tag{B.4}$$

These generators indeed close into the SU(3) commutation relations

$$[\tilde{\lambda}_{\tilde{\alpha}}, \tilde{\lambda}_{\tilde{\beta}}] = 2 f_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}} \, \tilde{\lambda}_{\tilde{\gamma}} \, , \tag{B.5}$$

with $f_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}} = f_{[\tilde{\alpha}\tilde{\beta}\tilde{\gamma}]}$ Gell-Mann's structure constants,

$$\begin{gathered} f_{123} = 1 \, , \qquad f_{458} = f_{678} = \tfrac{\sqrt{3}}{2} \, , \\ f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \tfrac{1}{2} \, . \end{gathered} \tag{B.6}$$

Inside $E_{7(7)}$, the SU(3) generated by (B.4) commutes with $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2,1)$, with the first factor generated by

$$H_0 = -\tfrac{1}{2}\big(t_i{}^i - 3 t_a{}^a\big) \, , \qquad E_0 = 3 \, J^{(6)ij} \epsilon^{ab} \, t_{ijab} \, , \qquad F_0 = \tfrac{3}{2} \, J^{(6)ij} J^{(6)kh} \, t_{ijkh} \, , \tag{B.7}$$

and the second factor by

$$\begin{gathered} H_1 = -t_7{}^7 + t_8{}^8 \, , \qquad H_2 = J^{(6)i}_{j} \, t_i{}^j \, , \\ E_{11} = -\sqrt{2} \, \mathrm{Im}\, \Omega^{(6)ijk} \, t_{ijk8} \, , \quad E_{12} = -\sqrt{2} \, \mathrm{Re}\, \Omega^{(6)ijk} \, t_{ijk8} \, , \quad E_2 = -\sqrt{2} \, t_8{}^7 \, , \\ F_{11} = \sqrt{2} \, \mathrm{Re}\, \Omega^{(6)ijk} \, t_{ijk7} \, , \qquad F_{12} = -\sqrt{2} \, \mathrm{Im}\, \Omega^{(6)ijk} \, t_{ijk7} \, , \quad F_2 = -\sqrt{2} \, t_7{}^8 \, . \end{gathered} \tag{B.8}$$

These are the numerator groups in the scalar manifold (3.53). In (B.7) and (B.8) we have split the indices as $I = (i, a)$, with $i = 1, \ldots, 6$ in the fundamental of $\mathrm{SO}(6)_v$ and $a = 7, 8$, by effectively identifying the fundamental of $\mathrm{SL}(8,\mathbb{R})$ with the $\mathbf{8}_v$ of SO(8). We have employed the SU(3)–invariant Calabi-Yau $(1,1)$ and $(3,0)$ forms

$$J^{(6)} = e^{12} + e^{34} + e^{56} \, , \qquad \Omega^{(6)} = (e^1 + i e^2) \wedge (e^3 + i e^4) \wedge (e^5 + i e^6) \, , \tag{B.9}$$

on $\mathbb{R}^6 \subset \mathbb{R}^8$, with $e^{12} \equiv dx^1 \wedge dx^2$, etc, and x^i the $\mathbb{R}^6$ Cartesian coordinates. We have also introduced the Levi-Civita tensor ϵ_{ab} in the $\mathbb{R}^2 \subset \mathbb{R}^8$ plane spanned by the $7, 8$ directions. Indices i, j and a, b are raised and lowered with δ_{ij} and δ_{ab}. The generators (B.7) and (B.8) indeed commute with each other and respectively close into the $\mathrm{SL}(2,\mathbb{R})$,

$$[H_0, E_0] = 2 \, E_0 \, , \qquad [H_0, F_0] = -2 \, F_0 \, , \qquad [E_0, F_0] = H_0 \, , \tag{B.10}$$

and $\mathrm{SU}(2,1)$ commutation relations,

$$\begin{gathered} [H_1, \, H_2] = [H_2, \, E_2] = [H_2, \, F_2] = [F_{1i}, \, F_2] = [E_{1i}, \, E_2] = 0, \\ \begin{aligned} &[H_1, \, E_{1i}] = E_{1i}, && [H_2, \, E_{1i}] = -3\epsilon_{ij} E_{1j}, && [H_1, \, E_2] = 2E_2, \\ &[H_1, \, F_{1i}] = -E_{1i}, && [H_2, \, F_{1i}] = -3\epsilon_{ij} F_{1j}, && [H_1, \, F_2] = -2F_2, \\ &[E_{11}, \, E_{12}] = -\sqrt{2} E_2, && [F_{11}, \, F_{12}] = \sqrt{2} E_2, && [E_2, \, F_2] = 2H_1 \, , \\ &[E_{1i}, \, F_{1j}] = \delta_{ij} H_1 + \epsilon_{ij} H_2, && [E_{1i}, \, F_2] = \sqrt{2} \epsilon_{ij} F_{1j}, && [E_2, \, F_{1i}] = \sqrt{2} \epsilon_{ij} E_{1j}, \end{aligned} \end{gathered} \tag{B.11}$$

with, here and only here, $i = 1, 2$. The generators of the maximal compact subgroup of $\mathrm{SU}(2,1)$ are

$$\begin{aligned} &K_0 \equiv E_2 - F_2 - \frac{\sqrt{2}}{3} H_2 \,, && K_1 \equiv \tfrac{1}{\sqrt{8}}\big(E_{11} - F_{11}\big) \,, \\ &K_2 \equiv \tfrac{1}{\sqrt{8}}\big(E_{12} - F_{12}\big) \,, && K_3 \equiv -\tfrac{1}{4\sqrt{2}}\big(E_2 - F_2\big) - \tfrac{1}{4} H_2 \,, \end{aligned} \tag{B.12}$$

and close into the $\mathrm{SU}(2) \times \mathrm{U}(1)$ commutation relations

$$[K_0, K_x] = 0 \,, \qquad [K_x, K_y] = \epsilon_{xyz} K_z \,, \qquad x = 1, 2, 3 \,. \tag{B.13}$$

It is also interesting to note that the three different U(1)'s with which SU(3) commutes inside the SO(8) subgroups $\mathrm{SO}(6)_v$, $\mathrm{SU}(4)_c$ and $\mathrm{SU}(4)_s$ are respectively generated by

$$\mathrm{U}(1)_v \quad : \qquad -J^{(6)\,i}_{j}\, t_i{}^j \,, \tag{B.14}$$

$$\mathrm{U}(1)_c \quad : \qquad -J^{(6)\,i}_{j}\, t_i{}^j + 3\,\epsilon_b{}^a\, t_a{}^b \,, \tag{B.15}$$

$$\mathrm{U}(1)_s \quad : \qquad -\lambda\, J^{(6)\,i}_{j}\, t_i{}^j + 3\,\epsilon_b{}^a\, t_a{}^b \,, \quad \text{with}\, \lambda \in \mathbb{R} \,,\ \lambda \neq 1 \,. \tag{B.16}$$

With these details, the SU(3)–invariant bosonic field content and its interactions described in section 3.1.2 can be constructed from the parent $\mathcal{N} = 8$ supergravity. Per the analysis above, the SU(3)–invariant scalar manifold is (3.53). A coset representative is

$$\mathcal{V} = e^{-\chi E_0} e^{-\frac{1}{2}\varphi H_0} e^{\frac{1}{\sqrt{2}}(aE_2 - \zeta E_{11} - \tilde{\zeta} E_{12})} e^{-\phi H_1} \,, \tag{B.17}$$

and the quadratic scalar matrix that enters the bosonic Lagrangian is $\mathcal{M} = \mathcal{V}\mathcal{V}^{\mathrm{T}}$. The metric on (3.53) that determines the scalar kinetic terms in the Lagrangian (3.54) is then reproduced through $-\frac{1}{48} D\mathcal{M} \wedge *D\mathcal{M}^{-1}$. For reference, the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2,1)$ Killing vectors of this metric, normalised to obey the commutation relations (B.10), (B.11), are

$$k[H_0] = 2\partial_\varphi - 2\chi\partial_\chi \,, \qquad k[E_0] = \partial_\chi \,, \qquad k[F_0] = 2\chi\partial_\varphi + (e^{-2\varphi} - \chi^2)\partial_\chi \,, \tag{B.18}$$

and

$$
\begin{gathered}
k[H_1] = \partial_\phi - 2a\partial_a - \zeta\partial_\zeta - \tilde{\zeta}\partial_{\tilde{\zeta}}\,, \qquad k[H_2] = 3\tilde{\zeta}\partial_\zeta - 3\zeta\partial_{\tilde{\zeta}}\,, \\
k[E_{11}] = \tfrac{1}{\sqrt{2}}(\tilde{\zeta}\partial_a - 2\partial_\zeta)\,, \qquad k[E_{12}] = \tfrac{1}{\sqrt{2}}(\zeta\partial_a + 2\partial_{\tilde{\zeta}})\,, \qquad k[E_2] = \sqrt{2}\partial_a\,, \\
k[F_2] = \sqrt{2}\left(a\partial_\phi - e^{-4\phi}(Z^2 - Y^2)\partial_a - (a\zeta - e^{-2\phi}\tilde{\zeta}Y)\partial_\zeta - e^{-2\phi}(\tilde{\zeta}Z + \zeta Y)\partial_{\tilde{\zeta}}\right)\,, \\
k[F_{11}] = \tfrac{1}{\sqrt{2}}\left(-\zeta\partial_\phi + (a\zeta - e^{-2\phi}\tilde{\zeta}Y)\partial_a - \tfrac{1}{2}(4e^{-2\phi} - \zeta^2 + 3\tilde{\zeta}^2)\partial_\zeta + 2(a + \zeta\tilde{\zeta})\partial_{\tilde{\zeta}}\right)\,, \\
k[F_{12}] = \tfrac{1}{\sqrt{2}}\left(\tilde{\zeta}\partial_\phi - (a\tilde{\zeta} + e^{-2\phi}\zeta Y)\partial_a + 2(a - \zeta\tilde{\zeta})\partial_\zeta + \tfrac{1}{2}(4e^{-2\phi} + 3\zeta^2 - \tilde{\zeta}^2)\partial_{\tilde{\zeta}}\right)\,.
\end{gathered}
\tag{B.19}
$$

Moving on, we need to specify how the SU(3)–invariant tensor fields in (3.52) are embedded into their $\mathcal{N}=8$ counterparts. As detailed in (3.47), the restricted $\mathcal{N}=8$ tensor hierarchy contains $\mathbf{28'}$ electric vectors $\mathcal{A}^{AB}$, $\mathbf{28}$ magnetic vectors $\tilde{\mathcal{A}}_{AB}$, $\mathbf{63}$ two-forms $\mathcal{B}_A{}^B$ and $\mathbf{36}$ three-forms $\mathcal{C}^{AB}$, in representations of $\mathrm{SL}(8,\mathbb{R})$ [30]. In order to determine the embedding of the SU(3)–invariant vectors A^Λ, $\tilde{A}_\Lambda$, $\Lambda = 0,1$, into their $\mathcal{N}=8$ counterparts, we note that SU(3) commutes inside $\mathrm{SO}(8) \subset \mathrm{E}_{7(7)}$ with the $\mathrm{U}(1)^2$ generated, in the notation of (B.8), by $(E_2 - F_2)$ and H_2 or, equivalently, by K^0 and K^3 defined in (B.12). These are the Cartan generators of the maximal compact subgroup $\mathrm{SU}(2)\times\mathrm{U}(1)$ of the hypermultiplet scalar manifold. Splitting again the $\mathcal{N}=8$ index as below (B.8), $A=(i,a)$, and fixing the normalisations for convenience we have the following embedding into the $\mathcal{N}=8$ vectors,

$$
\mathcal{A}^{ij} = A^1 J^{(6)ij}\,, \qquad \mathcal{A}^{ab} = \epsilon^{ab} A^0\,, \qquad \tilde{\mathcal{A}}_{ij} = \tfrac{1}{3}\tilde{A}_1 J_{(6)ij}\,, \qquad \tilde{\mathcal{A}}_{ab} = \tilde{A}_0\,\epsilon_{ab}\,. \tag{B.20}
$$

Similarly, for the two-form potentials we define

$$
\mathcal{B}_i{}^j = -\tfrac{1}{12}\,B_a{}^a\,\delta_i{}^j + \tfrac{1}{3}\,B^2\,J^{(6)j}_i\,, \qquad \mathcal{B}_a{}^b = \tfrac{1}{2}\,B_a{}^b - \tfrac{1}{2}\,B^0\,\epsilon_a{}^b\,, \tag{B.21}
$$

and for the three-form potentials,

$$
\mathcal{C}^{ij} = C^1\,\delta^{ij}\,, \qquad \mathcal{C}^{ab} = C^{ab}\,. \tag{B.22}
$$

Additionally, the gauge covariant derivative acting on the scalars reduces to $D = d + \frac{1}{\sqrt{2}}g(k[E_2] - k[F_2])\,A^0 - g\,k[H_2]\,A^1$, and this in turn reproduces (3.55) upon use of the relevant Killing vectors in (B.19).

Appendix C

Geometric structures on S^7

There are various sets of intrinsic coordinates that prove useful to understand the symmetries of the different M-theory configurations in section 3.2, each of them adapted to different geometric structures on S^7. The expressions below have been used to particularise the general SU(3)-invariant consistent embedding formulae (3.119)-(3.122) to the further subsectors and AdS_4 solutions discussed in section 3.2.2.

C.1 S^7 as the join of S^1 and a Sasaki-Einstein S^5

The first set of coordinates solves the constraint (3.106) by splitting μ^A, $A = 1, \ldots, 8$, as

$$\mu^i = \cos\alpha\, \tilde{\mu}^i\,, \quad i = 1, \ldots, 6\,, \qquad \mu^7 = \sin\alpha \cos\psi\,, \qquad \mu^8 = \sin\alpha \sin\psi\,, \tag{C.1}$$

with $0 \leq \alpha \leq \pi/2$, $0 \leq \psi < 2\pi$, and $\tilde{\mu}^i$, $i = 1, \ldots, 6$, defining in turn an S^5, *i.e.* subject to the constraint $\delta_{ij}\tilde{\mu}^i\tilde{\mu}^j = 1$. The intrinsic coordinates (C.1) are adapted to the topological description of S^7 as the join of S^5 and S^1, for which the round, Einstein, SO(8)–invariant metric,

$$ds^2(S^7) = \delta_{AB}\, d\mu^A d\mu^B\,, \tag{C.2}$$

on S^7 displays only a manifest $\mathrm{SO}(6)_v \times \mathrm{SO}(2)$ symmetry,

$$ds^2(S^7) = d\alpha^2 + \cos^2\alpha\, ds^2(S^5) + \sin^2\alpha\, d\psi^2\,, \tag{C.3}$$

with $ds^2(S^5) = \delta_{ij}\, d\tilde{\mu}^i d\tilde{\mu}^j$ the round, Einstein metric on S^5 normalised so that the Ricci tensor equals four times the metric. This S^5 comes naturally equipped with the Sasaki-Einstein structure $(\boldsymbol{\eta}^{(5)}, \boldsymbol{J}^{(5)}, \boldsymbol{\Omega}^{(5)})$ endowed upon it from the Calabi-Yau forms $J^{(6)}$, $\Omega^{(6)}$, (B.9), on the $\mathbb{R}^6$ factor of $\mathbb{R}^8 = \mathbb{R}^6 \times \mathbb{R}^2$ in which S^5 is embedded,

$$\boldsymbol{\eta}^{(5)} = J^{(6)}_{ij}\, \tilde\mu^i d\tilde\mu^j\,, \qquad \boldsymbol{J}^{(5)} = \tfrac{1}{2} J^{(6)}_{ij}\, d\tilde\mu^i \wedge d\tilde\mu^j\,, \qquad \boldsymbol{\Omega}^{(5)} = \tfrac{1}{2}\, \Omega^{(6)}_{ijk}\, \tilde\mu^i d\tilde\mu^j \wedge d\tilde\mu^k\,. \tag{C.4}$$

These satisfy

$$\boldsymbol{J}^{(5)} \wedge \boldsymbol{\Omega}^{(5)} = 0\,, \qquad \tfrac{1}{2}\, \boldsymbol{J}^{(5)} \wedge \boldsymbol{J}^{(5)} \wedge \boldsymbol{\eta}^{(5)} = \tfrac{1}{4} \boldsymbol{\Omega}^{(5)} \wedge \bar{\boldsymbol{\Omega}}^{(5)} \wedge \boldsymbol{\eta}^{(5)} = \mathrm{vol}(S^5)\,, \tag{C.5}$$

and

$$d\boldsymbol{\eta}^{(5)} = 2\boldsymbol{J}^{(5)}\,, \qquad d\boldsymbol{\Omega}^{(5)} = 3i\boldsymbol{\eta}^{(5)} \wedge \boldsymbol{\Omega}^{(5)}\,. \tag{C.6}$$

It is also useful to relate the Calabi-Yau forms $J^{(6)}$ and $\Omega^{(6)}$ written in terms of constrained $\mathbb{R}^8$ coordinates $\mu^A = (\mu^i, \mu^a)$, $i = 1, \ldots, , 6$, $a = 7, 8$, to the intrinsic S^7 coordinate α in (C.1) and Sasaki-Einstein forms (C.4):

$$\begin{aligned} J^{(6)}_{ij}\, \mu^i d\mu^j &= \cos^2\!\alpha\, \boldsymbol{\eta}^{(5)}\,, \\ \tfrac{1}{2}\, J^{(6)}_{ij} d\mu^i \wedge d\mu^j &= \cos^2\!\alpha\, \boldsymbol{J}^{(5)} - \sin\alpha \cos\alpha\, d\alpha \wedge \boldsymbol{\eta}^{(5)}\,, \\ \tfrac{1}{2}\, \Omega^{(6)}_{ijk} \mu^i d\mu^j \wedge d\mu^k &= \cos^3\!\alpha\, \boldsymbol{\Omega}^{(5)}\,, \\ \tfrac{1}{6}\, \Omega^{(6)}_{ijk} d\mu^i \wedge d\mu^j \wedge d\mu^k &= i \cos^3\!\alpha\, \boldsymbol{\Omega}^{(5)} \wedge \boldsymbol{\eta}^{(5)} - \sin\alpha \cos^2\!\alpha\, d\alpha \wedge \boldsymbol{\Omega}^{(5)}\,. \end{aligned} \tag{C.7}$$

The round metric $ds^2(S^5)$ in (C.3) naturally adapts itself to the Sasaki-Einstein structure (C.4) when written as

$$ds^2(S^5) = ds^2(\mathbb{CP}^2) + (d\tau + \sigma)^2\,, \tag{C.8}$$

with $ds^2(\mathbb{CP}^2)$ the Fubini-Study metric on the complex projective plane, normalised so that the Ricci tensor equals six times the metric, $0 \le \tau < 2\pi$ an angle on the S^5 Hopf fiber, and σ a one-form on $\mathbb{CP}^2$ such that $d\sigma = 2\boldsymbol{J}^{(4)}$ with $\boldsymbol{J}^{(4)}$ the Kähler form on $\mathbb{CP}^2$, so that $\boldsymbol{\eta}^{(5)} \equiv d\tau + \sigma$ and $\boldsymbol{J}^{(5)} \equiv \boldsymbol{J}^{(4)}$. For completeness, we note that $ds^2(\mathbb{CP}^2)$ can be written in terms of complex projective coordinates ξ^i, $i = 1, 2$, as

$$ds^2(\mathbb{CP}^2) = \frac{d\bar\xi_i\, d\xi^i}{1 + \bar\xi_k \xi^k} - \frac{(\bar\xi_i d\xi^i)(\xi^j d\bar\xi_j)}{(1 + \bar\xi_k \xi^k)^2}\,, \tag{C.9}$$

by introducing complex coordinates on $\mathbb{R}^6 = \mathbb{C}^3$ through

$$\tilde\mu^1 + i\tilde\mu^2 = \tfrac{1}{\sqrt{1+\bar\xi_i\xi^i}}\, e^{i\tau} \xi^1\,, \quad \tilde\mu^3 + i\tilde\mu^4 = \tfrac{1}{\sqrt{1+\bar\xi_i\xi^i}}\, e^{i\tau} \xi^2\,, \quad \tilde\mu^5 + i\tilde\mu^6 = \tfrac{1}{\sqrt{1+\bar\xi_i\xi^i}}\, e^{i\tau}\,. \tag{C.10}$$

In these coordinates, the one-form σ in (C.8) reads

$$\sigma = \frac{i}{2}\, \frac{\xi^i d\bar\xi_i - \bar\xi_i d\xi^i}{1 + \bar\xi_k \xi^k}\,. \tag{C.11}$$

C.2 S^7 as a homogeneous Sasaki-Einstein space

A second set of intrinsic coordinates on S^7 can be chosen that adapt themselves to its two natural, homogeneous seven-dimensional Sasaki-Einstein structures. These descend on S^7 from the Calabi-Yau forms $J^{(8)}_\pm$, $\Omega^{(8)}_\pm$ on $\mathbb{R}^8$,

$$\begin{aligned} J^{(8)}_\pm &= J^{(6)} \pm e^{78} = e^{12} + e^{34} + e^{56} \pm e^{78} \,, \\ \Omega^{(8)}_\pm &= \Omega^{(6)} \wedge (e^7 \pm i e^8) = (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6) \wedge (e^7 \pm ie^8), \end{aligned} \tag{C.12}$$

that are invariant under $\mathrm{SU}(4)_c$ for the + sign and $\mathrm{SU}(4)_s$ for the − sign. In terms of the constrained coordinates μ^A, $A = 1, \ldots, 8$, that define S^7 as the locus (3.106) in $\mathbb{R}^8$, the Sasaki-Einstein structure forms induced on S^7 are

$$\begin{aligned} \boldsymbol{\eta}^{(7)}_\pm &= J^{(8)}_{\pm AB}\, \mu^A d\mu^B \,, \qquad \boldsymbol{J}^{(7)}_\pm = \tfrac{1}{2} J^{(8)}_{\pm AB}\, d\mu^A \wedge d\mu^B \,, \\ \boldsymbol{\Omega}^{(7)}_\pm &= \tfrac{1}{6}\, \Omega^{(8)}_{\pm ABCD}\, \mu^A d\mu^B \wedge d\mu^C \wedge d\mu^D \,. \end{aligned} \tag{C.13}$$

These are subject to

$$\boldsymbol{J}^{(7)}_\pm \wedge \boldsymbol{\Omega}^{(7)}_\pm = 0 \,, \qquad \boldsymbol{J}^{(7)}_\pm \wedge \boldsymbol{J}^{(7)}_\pm \wedge \boldsymbol{J}^{(7)}_\pm \wedge \boldsymbol{\eta}^{(7)}_\pm = \tfrac{3i}{4}\, \boldsymbol{\Omega}^{(7)}_\pm \wedge \bar{\boldsymbol{\Omega}}^{(7)}_\pm \wedge \boldsymbol{\eta}^{(7)}_\pm = \mp 6\, \mathrm{vol}(S^7) \,, \tag{C.14}$$

and

$$d\boldsymbol{\eta}^{(7)}_\pm = 2 \boldsymbol{J}^{(7)}_\pm \,, \qquad d\boldsymbol{\Omega}^{(7)}_\pm = 4i \boldsymbol{\eta}^{(7)}_\pm \wedge \boldsymbol{\Omega}^{(7)}_\pm \,. \tag{C.15}$$

The seven-dimensional Sasaki-Einstein structure (C.13) is related to its five-dimensional counterpart (C.4) and the angles (C.1) through

$$\begin{aligned} \boldsymbol{\eta}^{(7)}_\pm &= \cos^2\!\alpha\, \boldsymbol{\eta}^{(5)} \pm \sin^2\!\alpha\, d\psi \,, \\ \boldsymbol{J}^{(7)}_\pm &= \cos^2\!\alpha\, \boldsymbol{J}^{(5)} \pm \sin\alpha \cos\alpha\, d\alpha \wedge (d\psi \mp \boldsymbol{\eta}^{(5)}) \,, \\ \boldsymbol{\Omega}^{(7)}_\pm &= e^{\pm i\psi} \cos^2\!\alpha\, \left[d\alpha \pm i \cos\!\alpha \sin\!\alpha (d\psi \mp \boldsymbol{\eta}^{(5)}) \right] \wedge \boldsymbol{\Omega}^{(5)} \,. \end{aligned} \tag{C.16}$$

The round metric on S^7 adapted to seven-dimensional Sasaki-Einstein structure reads, similarly to (C.8),

$$ds^2(S^7) = ds^2(\mathbb{CP}^3_\pm) + \left(d\psi_\pm + \sigma_\pm\right)^2 , \tag{C.17}$$

where $ds^2(\mathbb{CP}^3_\pm)$ is the Fubini-Study metric, normalised so that the Ricci tensor equals eight times the metric. The $\pm$ refers to two different embeddings of $\mathbb{CP}^3$ into S^7, with isometry group $\mathrm{SU}(4)_c \subset \mathrm{SO}(8)$ for the + sign and $\mathrm{SU}(4)_s \subset \mathrm{SO}(8)$ for the − sign. The angles $\psi_\pm$ have period 2π and the one-forms $\sigma_\pm$ in (C.17) obey $d\sigma_\pm = 2\boldsymbol{J}^{(7)}_\pm$ so that $\boldsymbol{\eta}^{(7)}_\pm \equiv d\psi_\pm + \sigma_\pm$. It is also useful to make manifest the $\mathbb{CP}^2$ that resides inside $\mathbb{CP}^3_\pm$, which is equipped with the complex projective coordinates ξ^i, $i = 1, 2$, that appear in (C.10) and the metric (C.9). This can be achieved by writing

$$\mu^1 + i\mu^2 = \frac{\cos\alpha}{\sqrt{1+\bar{\xi}_i\xi^i}}\, e^{i(\psi_\pm+\tau_\pm)}\,\xi^1\,, \qquad \mu^3 + i\mu^4 = \frac{\cos\alpha}{\sqrt{1+\bar{\xi}_i\xi^i}}\, e^{i(\psi_\pm+\tau_\pm)}\,\xi^2\,,$$
$$\mu^5 + i\mu^6 = \frac{\cos\alpha}{\sqrt{1+\bar{\xi}_i\xi^i}}\, e^{i(\psi_\pm+\tau_\pm)}\,, \qquad \mu^7 + i\mu^8 = \sin\alpha\, e^{\pm i\psi_\pm}\,, \tag{C.18}$$

where $\tau_\pm$ are angles of period 2π. The metrics $ds^2(\mathbb{CP}^3_\pm)$ and one-forms $\sigma_\pm$ inside the round S^7 metric (C.17) can be written in terms of the coordinates (C.18) as

$$ds^2(\mathbb{CP}^3_\pm) = d\alpha^2 + \cos^2\alpha\, ds^2(\mathbb{CP}^2) + \cos^2\alpha \sin^2\alpha\,(d\tau_\pm + \sigma)^2\,, \tag{C.19}$$

and

$$\sigma_\pm = \cos^2\alpha\,(d\tau_\pm + \sigma)\,, \tag{C.20}$$

with $ds^2(\mathbb{CP}^2)$ and σ respectively given by (C.9) and (C.11). The round S^7 metrics (C.3) with (C.8) and (C.17) with (C.19) are of course diffeomorphic: they are brought into each other by the change of coordinates

$$\psi = \pm\psi_\pm\,, \qquad \tau = \tau_\pm + \psi_\pm\,. \tag{C.21}$$

C.3 S^7 as the sine-cone over a nearly-Kähler S^6

A third and final set of intrinsic angles on S^7 is better suited to describe the solutions with at least G_2 symmetry. First split the μ^A, $A = 1, \ldots, 8$, as $\mu^A = (\mu^{\mathtt{I}}, \mu^8)$, with $\mathtt{I} = 1, \ldots, 7$, and then let

$$\mu^{\mathtt{I}} = \sin\beta\,\tilde{\nu}^{\mathtt{I}}\,, \qquad \mu^8 = \cos\beta\,, \tag{C.22}$$

where $0 \leq \beta \leq \pi$, and $\tilde{\nu}^{\mathtt{I}}$, $\mathtt{I} = 1, \ldots, 7$, define an S^6 through the constraint $\delta_{\mathtt{IJ}}\tilde{\nu}^{\mathtt{I}}\tilde{\nu}^{\mathtt{J}} = 1$. In these coordinates, the round metric (C.2) takes on the local sine-cone form

$$ds^2(S^7) = d\beta^2 + \sin^2\beta\, ds^2(S^6)\,, \tag{C.23}$$

where $ds^2(S^6) = \delta_{\mathtt{IJ}}\, d\tilde{\nu}^{\mathtt{I}} d\tilde{\nu}^{\mathtt{J}}$ is the round, Einstein metric on S^6 normalised so that the Ricci tensor equals five times the metric. This S^6 is naturally endowed with the homogeneous nearly-Kähler structure[1] $(\mathcal{J}, \Omega)$ inherited

[1] The typography we use for the nearly-Kähler forms on S^6 differentiates them from the Calabi-Yau forms (B.9) on $\mathbb{R}^6$. For that reason, we omit labels $^{(6)}$ for the former. Similarly, we omit labels $^{(7)}$ for the associative and co-associative forms on $\mathbb{R}^7$.

from the closed associative and co-associative forms,

$$\begin{aligned} \psi &= e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}\,, \\ \tilde{\psi} &= e^{1234} + e^{1256} + e^{3456} + e^{1367} + e^{1457} + e^{2357} - e^{2467}\,, \end{aligned} \tag{C.24}$$

on the $\mathbb{R}^7$ factor of $\mathbb{R}^8 = \mathbb{R}^7 \times \mathbb{R}$ in which S^6 is embedded:

$$\begin{aligned} \mathcal{J} &= \tfrac{1}{2}\,\psi_{\mathtt{IJK}}\,\tilde{\nu}^{\mathtt{I}} d\tilde{\nu}^{\mathtt{J}} \wedge d\tilde{\nu}^{\mathtt{K}}\,, \\ \Omega &= \tfrac{1}{6}\Big(\psi_{\mathtt{JKL}} - i\,\tilde{\psi}_{\mathtt{IJKL}}\,\tilde{\nu}^{\mathtt{I}}\Big)\, d\tilde{\nu}^{\mathtt{J}} \wedge d\tilde{\nu}^{\mathtt{K}} \wedge d\tilde{\nu}^{\mathtt{L}}\,. \end{aligned} \tag{C.25}$$

The nearly-Kähler forms are subject to

$$\mathcal{J} \wedge \Omega = 0\,, \qquad \Omega \wedge \bar{\Omega} = -\tfrac{4i}{3}\mathcal{J} \wedge \mathcal{J} \wedge \mathcal{J} = -8i\,\mathrm{vol}(S^6)\,, \tag{C.26}$$

and

$$d\mathcal{J} = 3\,\mathrm{Re}\,\Omega\,, \qquad d\,\mathrm{Im}\,\Omega = -2\,\mathcal{J} \wedge \mathcal{J}\ . \tag{C.27}$$

It is also useful to note the following relations between the associative and co-associative forms ψ, $\tilde{\psi}$ written in constrained $\mathbb{R}^8$ coordinates $\mu^A = (\mu^{\mathtt{I}}, \mu^8)$, the S^7 coordinate β in (C.22), and the nearly-Kähler forms (C.25):

$$\begin{aligned} \tfrac{1}{2}\psi_{\mathtt{IJK}}\mu^{\mathtt{I}} d\mu^{\mathtt{J}} \wedge d\mu^{\mathtt{K}} \wedge d\mu^8 &= -\sin^4\!\beta\, \mathcal{J} \wedge d\beta\,, \\ \tfrac{1}{6}\psi_{\mathtt{IJK}} d\mu^{\mathtt{I}} \wedge d\mu^{\mathtt{J}} \wedge d\mu^{\mathtt{K}} &= \sin^3\!\beta\,\mathrm{Re}\,\Omega + \sin^2\!\beta\,\cos\beta\,\mathcal{J} \wedge d\beta\,, \\ \tfrac{1}{6}\tilde{\psi}_{\mathtt{IJKL}}\mu^{\mathtt{I}} d\mu^{\mathtt{J}} \wedge d\mu^{\mathtt{K}} \wedge d\mu^{\mathtt{L}} &= -\sin^4\!\beta\,\mathrm{Im}\,\Omega\,, \\ \tfrac{1}{24}\tilde{\psi}_{\mathtt{IJKL}} d\mu^{\mathtt{I}} \wedge d\mu^{\mathtt{J}} \wedge d\mu^{\mathtt{K}} \wedge d\mu^{\mathtt{L}} &= \tfrac{1}{2}\sin^4\!\beta\,\mathcal{J} \wedge \mathcal{J} + \sin^3\!\beta\,\cos\beta\,\mathrm{Im}\,\Omega \wedge d\beta\,. \end{aligned} \tag{C.28}$$

Finally, the following relations hold between the associative and co-associative forms on $\mathbb{R}^8 = \mathbb{R}^7 \times \mathbb{R}$ and the Calabi-Yau forms $\mathbb{R}^8 = \mathbb{R}^6 \times \mathbb{R}^2$:

$$\begin{aligned} \tfrac{1}{2}\psi_{\mathtt{IJK}}\mu^{\mathtt{I}} d\mu^{\mathtt{J}} \wedge d\mu^{\mathtt{K}} &= J^{(6)}_{ij}\mu^i\, d\mu^j \wedge d\mu^7 + \tfrac{1}{2}\big(J^{(6)}_{jk}\,\mu^7 + \mathrm{Re}\,\Omega^{(6)}_{ijk}\,\mu^i\big) d\mu^j \wedge d\mu^k\,, \\ \tfrac{1}{6}\psi_{\mathtt{IJK}} d\mu^{\mathtt{I}} \wedge d\mu^{\mathtt{J}} \wedge d\mu^{\mathtt{K}} &= \tfrac{1}{6}\mathrm{Re}\,\Omega^{(6)}_{ijk}\, d\mu^i \wedge d\mu^j \wedge d\mu^k + \tfrac{1}{2}J^{(6)}_{ij} d\mu^i \wedge d\mu^j \wedge d\mu^7\,, \\ \tfrac{1}{6}\tilde{\psi}_{\mathtt{IJKL}}\mu^{\mathtt{I}} d\mu^{\mathtt{J}} \wedge d\mu^{\mathtt{K}} \wedge d\mu^{\mathtt{L}} &= \tfrac{1}{2}J^{(6)}_{ij}\,J^{(6)}_{kl}\,\mu^i\, d\mu^j \wedge d\mu^k \wedge d\mu^l \\ &\quad + \tfrac{1}{2}\,\mathrm{Im}\,\Omega^{(6)}_{ijk}\,\big(\mu^i\, d\mu^7 - \tfrac{1}{3}\,\mu^7 d\mu^i\big) \wedge d\mu^j \wedge d\mu^k\,. \end{aligned} \tag{C.29}$$

These expressions come handy to derive the G_2–invariant consistent uplifting formulae (3.137)-(3.139) from the general expressions (3.119)-(3.122). They are also useful to rewrite the solutions (3.147)-(3.151) with at least G_2 symmetry in the form (D.6)-(D.12), in order to verify that they satisfy the equations of motion.

Appendix D

Checks on $D = 11$ field equations in the SU(3) sector

In this appendix we provide a detailed account of the consistency of the minimal $\mathcal{N} = 2$ truncation and the AdS_4 solutions presented in section 3.2.2.

D.1 Consistency of the minimal $\mathcal{N} = 2$ truncation

We have explicitly verified at the level of the $D = 4$ field equations that the restrictions (3.93)-(3.98) define a consistent truncation of the SU(3)–invariant theory (3.54) to minimal $\mathcal{N} = 2$ gauged supergravity (2.35). In turn, the consistency of the $D = 11$ embedding of the entire SU(3) sector described in section 3.2.2 guarantees the consistency of the new uplift of minimal $\mathcal{N} = 2$ supergravity given in (3.142)-(3.145). We have nevertheless checked consistency explicitly at the level of the Bianchi identity and the equation of motion of the $D = 11$ four-form $\hat{F}_{(4)} = d\hat{A}_{(3)}$,

$$d\hat{F}_{(4)} = 0 \,, \qquad d\,\hat{*}\,\hat{F}_{(4)} + \tfrac{1}{2}\hat{F}_{(4)} \wedge \hat{F}_{(4)} = 0 \,. \tag{D.1}$$

The configuration (3.142), (3.145) does solve the $D = 11$ field equations (D.1) provided the Bianchi identity and the Maxwell equation for the $D = 4$ graviphoton,

$$d\bar{F} = 0 \,, \qquad d\,\bar{*}\,\bar{F} = 0 \,, \tag{D.2}$$

are imposed.

It is straightforward to see that the $D = 11$ Bianchi identity is satisfied. Hitting (3.145) with the differential operator we obtain, after using (D.1) and the algebraic and differential conditions for the local five-dimensional

Sasaki-Einstein structure (3.141) (that is, (C.5), (C.6) written for the primed forms $\boldsymbol{\eta}'$, $\boldsymbol{J}'$ and $\boldsymbol{\Omega}'$),

$$\begin{aligned} d\hat{F}_{(4)} = \tfrac{g^{-3}}{\sqrt{3}} \Bigg[& \frac{\cos^2\alpha(7 - 10\cos 2\alpha + \cos 4\alpha)}{\left(1 + 2\sin^2\alpha\right)^2} d\alpha \wedge \left(\tfrac{g}{2} \bar{F} \wedge \mathrm{Re}\,\boldsymbol{\Omega}' + 3D\psi' \wedge \mathrm{Im}\,\boldsymbol{\Omega}' \wedge \boldsymbol{\eta}' \right) \\ & + 6\,\partial_\alpha \left(\frac{\sin\alpha\cos^3\alpha}{1 + 2\sin^2\alpha} \right) d\alpha \wedge D\psi' \wedge \boldsymbol{\eta}' \wedge \mathrm{Im}\,\boldsymbol{\Omega}' + g \frac{3\sin\alpha\cos^3\alpha}{1 + 2\sin^2\alpha} \bar{F} \wedge \boldsymbol{\eta}' \wedge \mathrm{Im}\,\boldsymbol{\Omega}' \Bigg] \\ & + \frac{g^{-2}}{2\sqrt{3}} \left[2\,\partial_\alpha \left(\frac{\sin\alpha\cos^3\alpha}{1 + 2\sin^2\alpha} \right) d\alpha \wedge \bar{F} \wedge \mathrm{Re}\,\boldsymbol{\Omega}' - \frac{6\sin\alpha\cos^3\alpha}{1 + 2\sin^2\alpha} \bar{F} \wedge \mathrm{Im}\,\boldsymbol{\Omega}' \wedge \boldsymbol{\eta}' \right] . \end{aligned} \tag{D.3}$$

Terms with the same form dependence cancel each other, thus leading to $d\hat{F}_{(4)} = 0$.

Moving on to the equation of motion, we find it useful for the calculation to introduce the obvious frame that can be read off from (3.142),

$$\begin{aligned} \hat{e}^\alpha &= \frac{\left(1 + 2\sin^2\alpha\right)^{1/3}}{2^{1/3}\sqrt{3}}\, \bar{e}^\alpha , && \text{with } \bar{e}^\alpha \text{ a vierbein for } d\bar{s}_4^2 , \\ \hat{e}^p &= \frac{2^{1/6}\cos\alpha}{g\left(1 + 2\sin^2\alpha\right)^{1/6}}\, e^p , && \text{with } e^p \text{ a vierbein for } ds^2(\mathbb{CP}^2) , \\ \hat{e}^8 &= \frac{2^{1/6}\left(1 + 2\sin^2\alpha\right)^{1/3}}{\sqrt{3}\,g}\, d\alpha , \\ \hat{e}^9 &= \frac{2^{1/6}\sqrt{3}\sin\alpha\cos\alpha\left(1 + 2\sin^2\alpha\right)^{1/3}}{g\left(1 + 8\sin^4\alpha\right)^{1/2}}\, \boldsymbol{\eta}' , \\ \hat{e}^{10} &= \frac{\left(1 + 8\sin^4\alpha\right)^{1/2}}{2^{1/3}\sqrt{3}\,g\left(1 + 2\sin^2\alpha\right)^{2/3}} \left(D\psi' - \frac{3\cos^2\alpha}{1 + 8\sin^4\alpha} \boldsymbol{\eta}' \right) , \end{aligned} \tag{D.4}$$

with $\alpha = 0, 1, 2, 3$ and $p = 4, 5, 6, 7$. Using this frame, the Hodge dual of $\hat{F}_{(4)}$ reads

$$\begin{aligned} \hat{*}\, \hat{F}_{(4)} = & -\frac{3^{\frac{3}{2}} g^{-3} \cos^4\alpha}{(1 + 2\sin^2\alpha)^2}\, \hat{e}^{8910} \wedge \boldsymbol{J}' \wedge \boldsymbol{J}' \\ & - 2^{-\frac{1}{6}} \cdot 3^{-\frac{3}{2}}\, g^{-1} \left(1 + 2\sin^2\alpha\right)^{2/3} \cos^2\alpha\; \overline{\mathrm{vol}_4} \wedge \hat{e}^8 \wedge \mathrm{Im}\,\boldsymbol{\Omega}' \\ & + \frac{2^{-\frac{7}{6}} \cdot 3^{-\frac{3}{2}}\, g^{-1} \cos^2\alpha\,(7 - 10\cos 2\alpha + \cos 4\alpha)}{\left(1 + 2\sin^2\alpha\right)^{1/3} \left(1 + 8\sin^4\alpha\right)^{1/2}}\, \overline{\mathrm{vol}_4} \wedge \hat{e}^9 \wedge \mathrm{Re}\,\boldsymbol{\Omega}' \end{aligned}$$

$$
- \frac{3^{-\frac{3}{2}} \cdot 2^{-\frac{5}{3}} \, g^{-1} \cos^3\!\alpha \, (7 - 10\cos 2\alpha + \cos 4\alpha)}{\sin\alpha \, \big(1 + 2\sin^2\alpha\big)^{4/3} \big(1 + 8\sin^4\alpha\big)^{1/2}} \, \overline{\mathrm{vol}_4} \wedge \hat{e}^{10} \wedge \, \mathrm{Re}\,\boldsymbol{\Omega}'
$$
$$
+ \frac{g^{-2}\sin\alpha\cos^3\!\alpha}{\sqrt{3}\,\big(1+2\sin^2\alpha\big)} \, \bar{*}\,\bar{F} \wedge \hat{e}^{8910} \wedge \, \mathrm{Re}\,\boldsymbol{\Omega}' - \frac{g^{-2}\cos^2\!\alpha}{2\sqrt{3}} \, \bar{F} \wedge \hat{e}^{8910} \wedge \boldsymbol{J}'
$$
$$
+ \frac{2^{-\frac{5}{3}} \cdot 3^{-\frac{1}{2}} \, g^{-4} \big(1 + 8\sin^4\alpha\big)^{1/2} \cos^4\!\alpha}{\big(1+2\sin^2\alpha\big)^{4/3}} \, \bar{F} \wedge \hat{e}^{10} \wedge \boldsymbol{J}' \wedge \boldsymbol{J}' \, , \qquad \text{(D.5)}
$$

where $\hat{e}^{8910} = \hat{e}^8 \wedge \hat{e}^9 \wedge \hat{e}^{10}$. Computing the differential of (D.5) with the help of the Sasaki-Einstein conditions satisfied by $\boldsymbol{\eta}'$, $\boldsymbol{J}'$ and $\boldsymbol{\Omega}'$, as well as $\hat{F}_{(4)} \wedge \hat{F}_{(4)}$ from (3.145) and putting everything together, we find that the $D = 11$ equation of motion in (D.1) is indeed satisfied on the $D = 4$ field equations (D.2).

D.2 $D = 11$ field equations on the AdS$_4$ solutions

The AdS$_4$ solutions that we brought to section 3.2.2 are obtained from the consistent uplifting formulae (3.119)-(3.122) by turning off the relevant tensor hierarchy fields, fixing the $D = 4$ scalars to the vevs recorded in table 3.2, and fixing the $\mathbb{R}^8$ embedding coordinates μ^A, $A = 1, \ldots, 8$, in terms of various sets of intrinsic angles on S^7 discussed in appendix C. The particular choice of intrinsic coordinates for each solution was made on a case-by-case basis, as specific sets of coordinates are more suitable than others to highlight the specific symmetry of a solution. While this is obviously the best approach for the sake of presentation, it is definitely inconvenient to check the $D = 11$ equations of motion, as one would also need to proceed on a case-by-case basis for each solution.

In order to check that the $D = 11$ equations of motion hold it is more convenient to proceed differently. Firstly, leave the $D = 4$ scalars as temporarily unfixed constants, and make a choice of intrinsic S^7 coordinates (regardless of whether they would be well adapted to specific sectors). For this purpose, we have chosen the intrinsic coordinates (C.1). The $D = 11$ metric and four-form then get expanded in terms of the global five-dimensional Sasaki-Einstein structure $\boldsymbol{\eta}^{(5)}$, $\boldsymbol{J}^{(5)}$, $\boldsymbol{\Omega}^{(5)}$ specified in appendix C, with coefficients that depend on the $D = 4$ scalars along with the S^7 angles α and ψ. Secondly, plug these expressions into the $D = 11$ field equations (D.1) and obtain, with the help of the Sasaki-Einstein relations (C.5), (C.6), the set of equations that the coefficients must obey for the $D = 11$ equations to hold. Finally, verify that these equations are satisfied when the $D = 4$ scalars are fixed to the critical points recorded in table 3.2.

Proceeding this way, we find that the $D = 11$ metric (3.119) can be written in terms of the intrinsic angles (C.1) as

$$\begin{aligned} d\hat{s}_{11}^2 &= \Delta^{-1} ds_4^2 + ds_7^2 \, , \\ ds_7^2 &= -2(G_1\, d\alpha + G_2\, d\psi)\, \boldsymbol{\eta}^{(5)} + (G_3 + G_4)\, (\boldsymbol{\eta}^{(5)})^2 + G_4\, ds^2(\mathbb{CP}^2) \\ &\quad + G_5\, d\alpha^2 + 2\, G_6\, d\alpha\, d\psi + G_7\, d\psi^2 \, , \end{aligned} \tag{D.6}$$

where both the warp factor,

$$\Delta^{-1} \equiv e^{-\varphi} X^{1/3} \Delta_1^{2/3} \, , \tag{D.7}$$

given by Δ_1 in (3.118) with (C.1), and the coefficients of the internal metric ds_7^2 depend on the S^7 angles α, ψ and the $D = 4$ scalars:

$$\begin{aligned} G_1 &= \tfrac{\Delta^2}{g^2} \Big[-\tfrac{1}{2} e^{-2\phi} \sin\alpha\, \cos^3\alpha\, (X - Y) \big(2a e^{4\phi} \cos 2\psi - \sin 2\psi (-Y^2 - Z^2 + e^{4\phi}) \big) \Big] \, , \\ G_2 &= \tfrac{\Delta^2}{g^2} \Big[e^{-2\phi} \sin^2\alpha\, \cos^2\alpha\, (X - Y) \big(a e^{4\phi} \sin 2\psi + \sin^2\psi (Y^2 + Z^2) + e^{4\phi} \cos^2\psi \big) \Big] \, , \\ G_3 &= \tfrac{\Delta^2}{g^2} \Big[Y \cos^4\alpha\, (Y - X) \Big] \, , \\ G_4 &= \tfrac{\Delta^2}{g^2} \Big[X^2 \sin^2\alpha\, \cos^2\alpha\, e^{-2(\varphi+\phi)} \big(a e^{4\phi} \sin 2\psi + \sin^2\psi\, (Y^2 + Z^2) + e^{4\phi} \cos^2\psi \big) \\ &\quad + XY \cos^4\alpha \Big] \, , \\ G_5 &= \tfrac{\Delta^2}{g^2} \Big\{ XY \sin^2\alpha\, \cos^2\alpha - \tfrac{1}{64} \sin^2 2\alpha \big(e^{2\phi} (\zeta^2 + \tilde{\zeta}^2) + 4 \big) \big(e^{2\phi} (\zeta^2 + \tilde{\zeta}^2) - 4 e^{2\varphi} \chi^2 \big) \\ &\quad + X^2 \sin^4\alpha\, e^{-2(\varphi+\phi)} \big(a e^{4\phi} \sin 2\psi + \sin^2\psi\, (Y^2 + Z^2) + e^{4\phi} \cos^2\psi \big) \\ &\quad + e^{-4\phi} \cos^2\alpha \Big[-2a e^{4\phi} \sin\psi\, \cos\psi + \cos^2\psi\, (Y^2 + Z^2) + e^{4\phi} \sin^2\psi \Big] \\ &\qquad \times \Big[\sin^2\alpha \big(a e^{4\phi} \sin 2\psi + \sin^2\psi\, (Y^2 + Z^2) + e^{4\phi} \cos^2\psi \big) + \cos^2\alpha\, e^{2(\varphi+\phi)} \Big] \Big\} \, , \\ G_6 &= \tfrac{\Delta^2}{g^2} \Big[e^{-4\phi} \sin\alpha\, \cos\alpha \big(-a e^{4\phi} \cos 2\psi + \sin\psi\, \cos\psi\, (-Y^2 - Z^2 + e^{4\phi}) \big) \Big] \\ &\quad \times \Big[\sin^2\alpha\, \big(a e^{4\phi} \sin 2\psi + \sin^2\psi\, \big(Y^2 + Z^2 \big) + e^{4\phi} \cos^2\psi \big) + \cos^2\alpha\, e^{2(\varphi+\phi)} \Big] \, , \\ G_7 &= \tfrac{\Delta^2}{g^2} \Big[e^{-4\phi} \sin^2\alpha \big(a e^{4\phi} \sin 2\psi + \sin^2\psi\, (Y^2 + Z^2) + e^{4\phi} \cos^2\psi \big) \Big] \\ &\quad \times \Big[\sin^2\alpha\, \big(a e^{4\phi} \sin 2\psi + \sin^2\psi\, \big(Y^2 + Z^2 \big) + e^{4\phi} \cos^2\psi \big) + \cos^2\alpha\, e^{2(\varphi+\phi)} \Big] \, . \end{aligned} \tag{D.8}$$

Turning off the $D = 4$ tensor hierarchy fields (except for the local three-form $C_{\text{FR}} \equiv C^1 = C^{77} = C^{88}$ whose rôle is merely to serve as a local potential for the Freund-Rubin term) in the three form (3.121), its pull-back on S^7 induced by (C.1) reads

$$\begin{aligned}\hat{A}_{(3)} &= C_{\rm FR} + L_1\, d\alpha\wedge d\psi\wedge \boldsymbol{\eta}^{(5)} + (L_2\, d\alpha + L_3\, d\psi)\wedge \boldsymbol{J}^{(5)} \\ &\quad + (L_4\, d\alpha + L_5\, d\psi)\wedge {\rm Re}\,\boldsymbol{\Omega}^{(5)} + (L_6\, d\alpha + L_7\, d\psi)\wedge {\rm Im}\,\boldsymbol{\Omega}^{(5)} \\ &\quad + \big(L_8\,{\rm Im}\,\boldsymbol{\Omega}^{(5)} + L_9\,{\rm Re}\,\boldsymbol{\Omega}^{(5)} + L_{10}\,\boldsymbol{J}^{(5)}\big)\wedge\boldsymbol{\eta}^{(5)}\ . \end{aligned} \tag{D.9}$$

The coefficients here are given by

$$\begin{aligned}
L_1 &= \tfrac{\Delta^3}{16g}\,\chi\sin\alpha\,\cos^2\!\alpha\, e^{-\varphi-4\phi}\Big[e^{4\phi}\cos^2\!\psi \\
&\quad + \sin\alpha\sin 2\alpha\,(X-Y)e^{2(\varphi+\phi)}\left(e^{2\varphi}\chi^2 - Y + 1\right)\left(ae^{4\phi}\sin 2\psi + \sin^2\!\psi\,\left(Y^2+Z^2\right)\right) \\
&\quad - 2\left(X^2\sin^2\!\alpha\,\left(ae^{4\phi}\sin 2\psi + \sin^2\!\psi\,\left(Y^2+Z^2\right) + e^{4\phi}\cos^2\!\psi\right) + Y^2\cos^2\!\alpha\, e^{2(\varphi+\phi)}\right) \\
&\quad \times\left(\cos\alpha\, e^{2(\varphi+\phi)} + \sin\alpha\tan\alpha\,\left(\sin^2\!\psi\,\left(Y^2+Z^2\right) + Ze^{2\phi}\sin 2\psi + e^{4\phi}\cos^2\!\psi\right)\right)\Big]\,, \\
L_2 &= \tfrac{\Delta^3}{g^3}\Big[-\chi e^{-\varphi-4\phi}X\sin\alpha\cos^3\!\alpha\left(\sin\psi\cos\psi\,\left(-Y^2-Z^2+e^{4\phi}\right) - Ze^{2\phi}\cos 2\psi\right)\Big] \\
&\quad \times\Big[X\sin^2\!\alpha\,\left(ae^{4\phi}\sin 2\psi + \sin^2\!\psi\,\left(Y^2+Z^2\right) + e^{4\phi}\cos^2\!\psi\right) + Y\cos^2\!\alpha\, e^{2(\varphi+\phi)}\Big]\,, \\
L_3 &= -\frac{\tan\alpha\sin\psi\,\left(Y^2+Z^2+2Ze^{2\phi}\cot\psi\, + e^{4\phi}\cot^2\!\psi\right)}{Ze^{2\phi}\cos 2\psi - \sin\psi\cos\psi\,\left(-Y^2-Z^2+e^{4\phi}\right)}\ L_2\,, \\
L_4 &= \tfrac{\Delta^3}{2g^3}\,X\cos^2\!\alpha\, e^{-3\varphi-2\phi}\Big[X\sin^2\!\alpha\left(\zeta e^{2\phi}\cos\psi + \sin\psi\,(\tilde\zeta Y + \zeta Z)\right) \\
&\quad + e^{2\varphi}\cos^2\!\alpha\left(\tilde\zeta e^{2\phi}\sin\psi + \cos\psi\,(\zeta Y - \tilde\zeta Z)\right)\Big] \\
&\quad \times\Big[X\sin^2\!\alpha\left(ae^{4\phi}\sin 2\psi + \sin^2\!\psi\,\left(Y^2+Z^2\right) + e^{4\phi}\cos^2\!\psi\right) + Y\cos^2\!\alpha\, e^{2(\varphi+\phi)}\Big] \\
L_5 &= -\frac{e^{2\phi}\left(\tilde\zeta e^{2\phi}\cos\psi + \sin\psi\,(\tilde\zeta Z - \zeta Y)\right)}{\chi\left(\sin 2\psi\,(-Y^2-Z^2+e^{4\phi}) - 2Ze^{2\phi}\cos 2\psi\right)}\ L_2\,, \\
L_6 &= \frac{2}{\sin 2\alpha}\left(e^{-2\varphi}X\sin^2\!\alpha + \frac{\cos^2\!\alpha\,\left(\cos\psi\,(\tilde\zeta Y + \zeta Z) - \zeta e^{2\phi}\sin\psi\right)}{\tilde\zeta e^{2\phi}\cos\psi + \sin\psi\,(\tilde\zeta Z - \zeta Y)}\right)\ L_5\,, \\
L_7 &= -\frac{\zeta e^{2\phi}\cos\psi + \tilde\zeta Y\sin\psi + \zeta Z\sin\psi}{\tilde\zeta e^{2\phi}\cos\psi - \zeta Y\sin\psi + \tilde\zeta Z\sin\psi}\ L_5\,, \\
L_8 &= -e^{-2\varphi}X\ L_7\,, \\
L_9 &= -e^{-2\varphi}X\ L_5\,, \\
L_{10} &= \tfrac{\Delta^3}{g^3}\Big(-e^{\varphi}\chi Y\cos^2\!\alpha\Big)\Big[XY\cos^4\!\alpha \\
&\quad + X^2\sin^2\!\alpha\,\cos^2\!\alpha\, e^{-2(\varphi+\phi)}\left(ae^{4\phi}\sin 2\psi + \sin^2\!\psi\,\left(Y^2+Z^2\right) + e^{4\phi}\cos^2\!\psi\right)\Big]\,.
\end{aligned} \tag{D.10}$$

Finally, the $D = 11$ four-form $\hat{F}_{(4)} = d\hat{A}_{(3)}$ is

$$\begin{aligned}\hat{F}_{(4)} &= U\,\text{vol}_4 + d\alpha \wedge d\psi \wedge \big(f_1\, \boldsymbol{J}^{(5)} + f_2 \,\text{Re}\,\boldsymbol{\Omega}^{(5)} + f_3 \,\text{Im}\,\boldsymbol{\Omega}^{(5)}\big) \\ &\quad + \big(f_4\, d\alpha + f_5\, d\psi\big) \wedge \text{Re}\,\boldsymbol{\Omega}^{(5)} \wedge \boldsymbol{\eta}^{(5)} + \big(f_6\, d\alpha + f_7\, d\psi\big) \wedge \text{Im}\,\boldsymbol{\Omega}^{(5)} \wedge \boldsymbol{\eta}^{(5)} \\ &\quad + \big(f_8\, d\alpha + f_9\, d\psi\big) \wedge \boldsymbol{J}^{(5)} \wedge \boldsymbol{\eta}^{(5)} + f_{10}\, \boldsymbol{J}^{(5)} \wedge \boldsymbol{J}^{(5)}\,, \end{aligned} \tag{D.11}$$

where the Freund Rubin term is given by $U\text{vol}_4 = H^1_{(4)}\mu_i\mu^i + H^{ab}_{(4)}\mu_a\mu_b$ evaluated on (C.1) and on the $D = 4$ dualisation conditions (3.69). The functional coefficients in (D.11) can be written in terms of the coefficients (D.10) of the three form (D.9) as

$$\begin{aligned} f_1 &= 2L_1 + \partial_\alpha L_3 - \partial_\psi L_2\,, & f_6 &= 3L_4 + \partial_\alpha L_8\,, \\ f_2 &= \partial_\alpha L_5 - \partial_\psi L_4\,, & f_7 &= 3L_5 + \partial_\psi L_8\,, \\ f_3 &= \partial_\alpha L_7 - \partial_\psi L_6\,, & f_8 &= \partial_\alpha L_{10}\,, \\ f_4 &= -3L_6 + \partial_\alpha L_9\,, & f_9 &= \partial_\psi L_{10}\,, \\ f_5 &= -3L_7 + \partial_\psi L_9\,, & f_{10} &= 2L_{10}\,. \end{aligned}$$

The Bianchi identity $d\hat{F}_{(4)} = 0$ amounts to the following relations:

$$\begin{gathered} 3f_3 + \partial_\alpha f_5 - \partial_\psi f_4 = 0\,, \qquad -3f_2 + \partial_\alpha f_7 - \partial_\psi f_6 = 0\,, \\ \partial_\alpha f_{10} - 2f_8 = 0\,, \qquad \partial_\alpha f_9 - \partial_\psi f_8 = 0\,, \qquad \partial_\psi f_{10} - 2f_9 = 0\,. \end{gathered} \tag{D.12}$$

Of course, these conditions are automatically satisfied by construction for all values of the $D = 4$ scalars upon using (D.12).

We next compute the Hodge dual of the $\hat{F}_{(4)}$ given in (D.11) with respect to the $D = 11$ metric (D.6). We obtain

$$\begin{aligned} \hat{*}\,\hat{F}_{(4)} &= \Delta^2\,\text{vol}_7 + \Delta^{-2}\,\text{vol}_4 \wedge \\ &\quad \Big[(p_1\, d\alpha + p_2\, d\psi + p_3\, \boldsymbol{\eta}^{(5)}) \wedge \boldsymbol{J}^{(5)} + (p_4\, d\alpha + p_5\, d\psi + p_6\, \boldsymbol{\eta}^{(5)}) \wedge \text{Re}\,\boldsymbol{\Omega}^{(5)} \\ &\quad + (p_7\, d\alpha + p_8\, d\psi + p_9\, \boldsymbol{\eta}^{(5)}) \wedge \text{Im}\,\boldsymbol{\Omega}^{(5)} + p_{10}\ d\alpha \wedge d\psi \wedge \boldsymbol{\eta}^{(5)}\Big]\,, \end{aligned} \tag{D.13}$$

with coefficients

$$\begin{aligned} p_1 &= \tfrac{\Delta^{-2}}{G_V}\Big[f_1G_1 - f_9G_5 + f_8G_6\Big]\,, & p_6 &= \tfrac{\Delta^{-2}}{G_V}\Big[f_5G_1 - f_4G_2 - f_2G_3 - f_2G_4\Big]\,, \\ p_2 &= \tfrac{\Delta^{-2}}{G_V}\Big[f_1G_2 - f_9G_6 + f_8G_7\Big]\,, & p_7 &= \tfrac{\Delta^{-2}}{G_V}\Big[f_3G_1 - f_7G_5 + f_6G_6\Big]\,, \\ p_3 &= \tfrac{\Delta^{-2}}{G_V}\Big[f_9G_1 - f_8G_2 - f_1G_3 - f_1G_4\Big]\,, & p_8 &= \tfrac{\Delta^{-2}}{G_V}\Big[f_3G_2 - f_7G_6 + f_6G_7\Big]\,, \\ p_4 &= \tfrac{\Delta^{-2}}{G_V}\Big[f_2G_1 - f_5G_5 + f_4G_6\Big]\,, & p_9 &= \tfrac{\Delta^{-2}}{G_V}\Big[f_7G_1 - f_6G_2 - f_3G_3 - f_3G_4\Big]\,, \\ p_5 &= \tfrac{\Delta^{-2}}{G_V}\Big[f_2G_2 - f_5G_6 + f_4G_7\Big]\,, & p_{10} &= -2\tfrac{G_V G_4^2}{\Delta^2} f_{10}\,. \end{aligned} \tag{D.14}$$

Here,

$$G_V = \sqrt{-G_7 G_1^2 + 2G_2 G_6 G_1 - G_3 G_6^2 - G_4 G_6^2 - G_2^2 G_5 + G_3 G_5 G_7 + G_4 G_5 G_7} \tag{D.15}$$

is related to the volume element corresponding to the internal metric ds_7^2 in (D.6). With these definitions, the equation of motion in (D.1) for the $D = 11$ four-form becomes equivalent to the following conditions:

$$\begin{aligned}
U f_1 + \partial_\alpha p_2 - \partial_\psi p_1 + 2p_{10} &= 0\,, & U f_6 + \partial_\alpha p_9 + 3p_4 &= 0\,,\\
U f_2 + \partial_\alpha p_5 - \partial_\psi p_4 &= 0\,, & U f_7 + \partial_\psi p_9 + 3p_5 &= 0\,,\\
U f_3 + \partial_\alpha p_8 - \partial_\psi p_7 &= 0\,, & U f_8 + \partial_\alpha p_3 &= 0\,,\\
U f_4 + \partial_\alpha p_6 - 3p_7 &= 0\,, & U f_9 + \partial_\psi p_3 &= 0\,,\\
U f_5 + \partial_\psi p_6 - 3p_8 &= 0\,, & U f_{10} + 2p_3 &= 0\,.
\end{aligned} \tag{D.16}$$

We have verified that equations (D.16) hold when the $D = 4$ scalars are evaluated at any of the critical points collected in table 3.2. We have also checked that all the metric and four-forms for the explicit AdS$_4$ solutions written in section 3.2.2 can be brought to the form (D.6)-(D.12), with the help of the relations given in appendix C. Thus, the explicit AdS$_4$ configurations of section 3.2.2 do indeed solve the $D = 11$ field equations (D.1).

Appendix E

Group theory compendium

E.1 Structure of the KK spectra from $\mathcal{N} = 8$

The KK spectra of the solutions of M-theory, massive IIA and type IIB which include a topological S^7, S^6, or $S^5 \times S^1$ can be respectively organised in terms of quantum numbers of SO(8), SO(7) or $\mathrm{SO}(6)_v \times \mathrm{SO}(2)$ following (6.37). We denote these groups here as G_{round} and note that it is not necessary that the equations of motion enjoy a solution preserving that much symmetry, but only that such a point exists in the scalar manifold (*e.g.* at the scalar origin in an adequate parametrisation).

The individual states of definite spin in these spectra come in representations of the symmetry group $G \subset G_{\text{round}}$ of the actual solution. The modes lying at the bottom of the KK towers correspond to the linearisation of the $D = 4$ $\mathcal{N} = 8$ gauged supergravity fields. In particular, the G representations present at this level are obtained by branching the $D = 4$ $\mathcal{N} = 8$ fields under $\mathrm{SU}(8) \supset \mathrm{SO}(8) \supset G_{\text{round}} \supset G$. In this appendix we record the intermediate G_{round} representations that appear in this branching.

In general, the algebraic structure of the spectrum at higher KK levels is obtained by the following two-step process. Firstly, tensor representations of the gauged supergravity fields (tables E.1 (left), E.2 (left), and E.3 (upper) below) with the symmetric-traceless representation in (6.37) that the massive gravitons furnish and remove Goldstones and Goldstini. The resulting representation content is summarised for convenience in tables E.1 (right), E.2 (right), and E.3 (lower) below. Secondly, branch under $G \subset G_{\text{round}}$. This algorithm has already been applied in different instances *e.g.* [105, 134, 140], and tables E.1 and E.2 are taken from [24] and [134].

E.1.1 SO(8) towers

In the M-theory case, the spectra are labelled by a single independent KK level n ranging as

$$n = 0, 1, 2, \ldots \tag{E.1}$$

spin	SO(8) irrep	SO(8) Dynkin labels
2	$\mathbf{1}$	$[0,0,0,0]$
$\frac{3}{2}$	$\mathbf{8}_s$	$[0,0,0,1]$
1	$\mathbf{28}$	$[0,1,0,0]$
$\frac{1}{2}$	$\mathbf{56}_s$	$[1,0,1,0]$
0^+	$\mathbf{35}_v$	$[2,0,0,0]$
0^-	$\mathbf{35}_c$	$[0,0,2,0]$

spin	SO(8) Dynkin labels
2	$[n,0,0,0]$
$\frac{3}{2}$	$[n,0,0,1] \oplus [n-1,0,1,0]$
1	$[n,1,0,0] \oplus [n-1,0,1,1] \oplus [n-2,1,0,0]$
$\frac{1}{2}$	$[n+1,0,1,0] \oplus [n-1,1,1,0] \oplus [n-2,1,0,1] \oplus [n-2,0,0,1]$
0^+	$[n+2,0,0,0] \oplus [n-2,2,0,0] \oplus [n-2,0,0,0]$
0^-	$[n,0,2,0] \oplus [n-2,0,0,2]$

Table E.1: States in SO(8) representations at KK levels $n = 0$ (left) and $n = 1, 2, \ldots$ (right) in the KK towers for AdS$_4$ solutions of M-theory that uplift from $D = 4$ $\mathcal{N} = 8$ electrically SO(8)-gauged supergravity. Representations with negative Dynkin labels are absent and need to be crossed out. Taken from [24].

E.1.2 SO(7) towers

In the massive type IIA case, the spectra are labelled by a single independent KK level k ranging as

$$k = 0, 1, 2, \ldots \tag{E.2}$$

spin	SO(7) irrep	SO(7) Dynkin labels
2	$\mathbf{1}$	$[0,0,0]$
$\frac{3}{2}$	$\mathbf{8}$	$[0,0,1]$
1	$\mathbf{21}+\mathbf{7}$	$[0,1,0]+[1,0,0]$
$\frac{1}{2}$	$\mathbf{48}+\mathbf{8}$	$[1,0,1]+[0,0,1]$
0^+	$\mathbf{27}+\mathbf{1}$	$[2,0,0]+[0,0,0]$
0^-	$\mathbf{35}$	$[0,0,2]$

spin	SO(7) Dynkin labels
2	$[k,0,0]$
$\frac{3}{2}$	$[k,0,1]+[k-1,0,1]$
1	$[k,1,0]+[k-1,0,2]+[k-2,1,0]+[k+1,0,0]+[k-1,1,0]+[k-1,0,0]$
$\frac{1}{2}$	$[k+1,0,1]+[k-1,1,1]+[k-2,1,1]+[k-2,0,1]+[k,0,1]+[k-1,0,1]$
0^+	$[k+2,0,0]+[k,0,0]+[k-2,2,0]+[k-2,0,0]$
0^-	$[k,0,2]+[k-1,1,0]+[k-2,0,2]$

Table E.2: States in SO(7) representations at KK level $k = 0$ (left) and $k = 1, 2, \ldots$ (right) that compose the KK towers for AdS$_4$ solutions of massive IIA that uplift from ISO(7) supergravity. Representations with negative Dynkin labels are absent. The tables exclude some 0^+ states of $D = 4$ supergravity that are always Higgsed away. Taken from [134].

E.1.3 SO(6)$_v$ × SO(2) towers

In the type IIB case, the spectra are labelled by two independent KK levels ℓ and n ranging as

$$\ell = 0, 1, 2 \ldots, \qquad n = 0, \pm 1, \pm 2, \ldots . \tag{E.3}$$

spin	SO(6)$_v$×SO(2) irrep	SO(6)$_v$×SO(2) Dynkin labels
2	$\mathbf{1}_0$	$[0,0,0]_0$
$\frac{3}{2}$	$\mathbf{4}_1+\bar{\mathbf{4}}_{-1}$	$[1,0,0]_1+[0,0,1]_{-1}$
1	$\mathbf{15}_0+\mathbf{1}_0+\mathbf{6}_2+\mathbf{6}_{-2}$	$[1,0,1]_0+[0,0,0]_0+[0,1,0]_2+[0,1,0]_{-2}$
$\frac{1}{2}$	$\bar{\mathbf{20}}_{-1}+\mathbf{20}_1+\mathbf{4}_1+\mathbf{4}_{-3}+\bar{\mathbf{4}}_3+\bar{\mathbf{4}}_{-1}$	$[1,1,0]_{-1}+[0,1,1]_1+[1,0,0]_1+[1,0,0]_{-3}+[0,0,1]_3+[0,0,1]_{-1}$
0^+	$\mathbf{20}'_0+\mathbf{1}_4+\mathbf{1}_0+\mathbf{1}_{-4}$	$[0,2,0]_0+[0,0,0]_4+[0,0,0]_0+[0,0,0]_{-4}$
0^-	$\mathbf{15}_0+\mathbf{10}_{-2}+\bar{\mathbf{10}}_2$	$[1,0,1]_0+[2,0,0]_{-2}+[0,0,2]_2$

spin	SO(6)$_v$×SO(2) Dynkin labels
2	$[0,\ell,0]_{2n}$
$\frac{3}{2}$	$[1,\ell,0]_{2n+1}+[0,\ell-1,1]_{2n+1}+[0,\ell,1]_{2n-1}+[1,\ell-1,0]_{2n-1}$
1	$[1,\ell,1]_{2n}+[2,\ell-1,0]_{2n}+[0,\ell-1,2]_{2n}+[1,\ell-2,1]_{2n}+[0,\ell,0]_{2n}$ $+[0,\ell+1,0]_{2n+2}+[1,\ell-1,1]_{2n+2}+[0,\ell-1,0]_{2n+2}+[0,\ell+1,0]_{2n-2}+[1,\ell-1,1]_{2n-2}+[0,\ell-1,0]_{2n+2}$
$\frac{1}{2}$	$[1,\ell+1,0]_{2n-1}+[0,\ell,1]_{2n-1}+[2,\ell-1,1]_{2n-1}+[1,\ell-1,0]_{2n-1}+[1,\ell-2,2]_{2n-1}+[0,\ell-2,1]_{2n-1}$ $+[0,\ell+1,1]_{2n+1}+[1,\ell,0]_{2n+1}+[1,\ell-1,2]_{2n+1}+[0,\ell-1,1]_{2n+1}+[2,\ell-2,1]_{2n+1}+[1,\ell-2,0]_{2n+1}$ $+[1,\ell,0]_{2n-3}+[0,\ell-1,1]_{2n-3}+[0,\ell,1]_{2n+3}+[1,\ell-1,0]_{2n+3}$
0^+	$[0,\ell+2,0]_{2n}+[1,\ell,1]_{2n}+[0,\ell,0]_{2n}+[2,\ell-2,2]_{2n}+[1,\ell-2,1]_{2n}+[0,\ell-2,0]_{2n}+[0,\ell,0]_{2n+4}+[0,\ell,0]_{2n-4}$
0^-	$[2,\ell,0]_{2n-2}+[1,\ell-1,1]_{2n-2}+[0,\ell-2,2]_{2n-2}+[0,\ell,2]_{2n+2}+[1,\ell-1,1]_{2n+2}+[2,\ell-2,0]_{2n+2}$

Table E.3: States in SO(6)$_v$×SO(2) representations at KK levels $(\ell, n) = (0,0)$ (above) and $\ell = 0,1,2,\ldots$, $n \in \mathbb{Z}$ (below) in the KK towers for AdS$_4$ solutions of type IIB that uplift from $D = 4$ $\mathcal{N} = 8$ $[\mathrm{SO}(6)\times\mathrm{SO}(1,1)]\ltimes\mathbb{R}^{12}$–gauged supergravity. SO(6)$_v$ representations are given in terms of SU(4) Dynkin labels, and SO(2) charges are given as subscripts. Representations with negative Dynkin labels are absent and need to be crossed out. The tables exclude some 0^+ states of $D = 4$ supergravity that are always Higgsed away.

E.2 Embedding SU(3)×U(1)$_p$ into SO(8)

The internal bosonic symmetry group SU(3)×U(1)$_p$, with $p = 2$ for CPW and $p = 3$ for GMPS, is embedded into SO(8) via

$$\mathrm{SO}(8) \supset \mathrm{SO}(6)_v \times \mathrm{SO}(2) \supset \big[\mathrm{SU}(3)\times\mathrm{U}(1)\big]\times\mathrm{SO}(2) \supset \mathrm{SU}(3)\times\mathrm{U}(1)_p\,. \tag{E.4}$$

Under the first two steps in the branching (E.4), the three basic irreps of SO(8) split as

$$\begin{aligned}
\mathbf{8}_v &\longrightarrow \mathbf{6}_0+\mathbf{1}_1+\mathbf{1}_{-1} \longrightarrow \mathbf{3}_{(-\frac{2}{3},\,0)}+\overline{\mathbf{3}}_{(\frac{2}{3},\,0)}+\mathbf{1}_{(0,\,+1)}+\mathbf{1}_{(0,\,-1)}\,,\\
\mathbf{8}_s &\longrightarrow \mathbf{4}_{\frac{1}{2}}+\overline{\mathbf{4}}_{-\frac{1}{2}} \longrightarrow \mathbf{3}_{(\frac{1}{3},\,\frac{1}{2})}+\overline{\mathbf{3}}_{(-\frac{1}{3},\,-\frac{1}{2})}+\mathbf{1}_{(-1,\,\frac{1}{2})}+\mathbf{1}_{(+1,\,-\frac{1}{2})},\\
\mathbf{8}_c &\longrightarrow \mathbf{4}_{-\frac{1}{2}}+\overline{\mathbf{4}}_{\frac{1}{2}} \longrightarrow \mathbf{3}_{(\frac{1}{3},\,-\frac{1}{2})}+\overline{\mathbf{3}}_{(-\frac{1}{3},\,\frac{1}{2})}+\mathbf{1}_{(-1,\,-\frac{1}{2})}+\mathbf{1}_{(+1,\,\frac{1}{2})}.
\end{aligned} \tag{E.5}$$

The IR R-symmetry group U(1)$_p$ is the combination of the U(1) that commutes with SU(3) inside SO(6)$_v$ and the SO(2) that commutes with SO(6)$_v$ inside SO(8) which leads to the allocation of R-charges (5.8) for $p = 3$ and (5.7) for $p = 2$. Assigning the transverse M2-brane coordinates to the $\mathbf{8}_v$, we thus require that, under the third and final step in the branching (E.4),

$$\mathbf{8}_v \longrightarrow \mathbf{3}_{R_1} + \overline{\mathbf{3}}_{-R_1} + \mathbf{1}_{R_2} + \mathbf{1}_{-R_2}\,, \tag{E.6}$$

with

$$R_1 = \frac{2p-2}{3p}\,, \qquad R_2 = \frac{2}{p}\,. \tag{E.7}$$

For completeness, we note that

$$\begin{aligned}
\mathbf{8}_s &\longrightarrow \mathbf{3}_{\frac{1}{2}(-R_1+R_2)} + \overline{\mathbf{3}}_{\frac{1}{2}(R_1-R_2)} + \mathbf{1}_{\frac{1}{2}(3R_1+R_2)} + \mathbf{1}_{-\frac{1}{2}(3R_1+R_2)}\,,\\
\mathbf{8}_c &\longrightarrow \mathbf{3}_{-\frac{1}{2}(R_1+R_2)} + \overline{\mathbf{3}}_{\frac{1}{2}(R_1+R_2)} + \mathbf{1}_{\frac{1}{2}(3R_1-R_2)} + \mathbf{1}_{\frac{1}{2}(-3R_1+R_2)}\,.
\end{aligned} \tag{E.8}$$

Taking tensor products and (anti)symmetrisations of (E.6), (E.8), an arduous calculation allows us to find the branching under $\mathrm{SU}(3)\times\mathrm{U}(1)_p$ of the SO(8) representations (5.26) that characterise the KK spectrum at the $\mathcal{N}=8$ point. We obtain[1]

$$\begin{aligned}
G_n = {}& [n,0,0,0]\\
&\xrightarrow{\mathrm{SU}(3)\times\mathrm{U}(1)_\mathrm{R}} \bigoplus_{\ell=0}^{n}\bigoplus_{t=0}^{n-\ell}\bigoplus_{p=0}^{\ell}[p,\ \ell-p]_{-R_1(\ell-2p)+R_2(n-\ell-2t)}\,,
\end{aligned} \tag{E.9}$$

$$\begin{aligned}
\mathcal{G}_n = {}& [n,0,0,1]\oplus[n-1,0,1,0]\\
&\xrightarrow{\mathrm{SU}(3)\times\mathrm{U}(1)_\mathrm{R}}\\
&\bigoplus_{\ell=0}^{n}\bigoplus_{t=0}^{n-\ell}\bigoplus_{p=0}^{\ell}\bigoplus_{k=0}^{1}\bigoplus_{a=0}^{1-k}\bigoplus_{b=0}^{k}[p+1-k-a,\ \ell-p+k-b]_{\substack{-R_1(\ell-2p-k-2a+2b+\frac{1}{2})\\ +R_2(n-\ell-2t-k+\frac{1}{2})}}\\
&\oplus\bigoplus_{\ell=0}^{n-1}\bigoplus_{t=0}^{n-1-\ell}\bigoplus_{p=0}^{\ell}\bigoplus_{k=0}^{1}\bigoplus_{a=0}^{1-k}\bigoplus_{b=0}^{k}[p+1-k-a,\ \ell-p+k-b]_{\substack{-R_1(\ell-2p-k-2a+2b+\frac{1}{2})\\ +R_2(n-\ell-2t+k-\frac{3}{2})}}\,,
\end{aligned} \tag{E.10}$$

$$\begin{aligned}
V_n = {}& [n,1,0,0]\oplus[n-1,0,1,1]\oplus[n-2,1,0,0]\\
&\xrightarrow{\mathrm{SU}(3)\times\mathrm{U}(1)_\mathrm{R}}
\end{aligned}$$

[1] In (A.6)–(A.11) we have renamed $\mathrm{SU}(3)\times\mathrm{U}(1)_p$ as $\mathrm{SU}(3)\times\mathrm{U}(1)_\mathrm{R}$ in order to avoid confusion with the Dynkin label p.

$$\bigoplus_{\ell=0}^{n}\bigoplus_{t=0}^{n-\ell}\bigoplus_{p=0}^{\ell}\bigoplus_{a,b=0}^{1}[p+a,\ \ell-p+b]_{\substack{-R_1(\ell-2p+2a-2b)\\+R_2(n-\ell-2t)}}$$

$$\oplus\bigoplus_{\ell=0}^{n}\bigoplus_{t=0}^{n-\ell}\bigoplus_{p=0}^{\ell+1}\bigoplus_{k=0}^{1}[p,\ \ell-p+1]_{\substack{-R_1(\ell-2p+1)\\+R_2(n-\ell-2t-2k+1)}}\oplus\bigoplus_{\ell=0}^{n}[0,0]_{R_2(n-2\ell)}$$

$$\oplus\bigoplus_{\ell=0}^{n-1}\bigoplus_{t=0}^{n-1-\ell}\bigoplus_{p=0}^{\ell}\bigoplus_{a,b=0}^{1}\bigoplus_{c=0}^{2-a-b}\bigoplus_{d=0}^{a+b}[p+c,\ \ell-p+d]_{\substack{-R_1(\ell-2p+3a+3b+2c-2d-3)\\+R_2(n-\ell-2t+a-b-1)}}$$

$$\oplus\bigoplus_{\ell=0}^{n-1}\bigoplus_{t=0}^{n-1-\ell}\bigoplus_{p=0}^{\ell+1}[p,\ \ell-p+1]_{\substack{-R_1(\ell-2p+1)\\+R_2(n-\ell-2t-1)}}$$

$$\oplus\bigoplus_{\ell=0}^{n-2}\bigoplus_{t=0}^{n-2-\ell}\bigoplus_{p=0}^{\ell}\bigoplus_{a,b=0}^{1}[p+a,\ \ell-p+b]_{\substack{-R_1(\ell-2p+2a-2b)\\+R_2(n-\ell-2t-2)}}$$

$$\oplus\bigoplus_{\ell=0}^{n-2}\bigoplus_{t=0}^{n-2-\ell}\bigoplus_{p=0}^{\ell+1}\bigoplus_{k=0}^{1}[p,\ \ell-p+1]_{\substack{-R_1(\ell-2p+1)\\+R_2(n-\ell-2t-2k-1)}}\oplus\bigoplus_{\ell=0}^{n-2}[0,0]_{R_2(n-2\ell-2)}\,, \tag{E.11}$$

$$\mathcal{F}_n = [n+1,0,1,0]\oplus[n-1,1,1,0]\oplus[n-2,1,0,1]\oplus[n-2,0,0,1]$$

$$\xrightarrow{\mathrm{SU(3)\times U(1)_R}}$$

$$\bigoplus_{\ell=0}^{n+1}\bigoplus_{t=0}^{n+1-\ell}\bigoplus_{p=0}^{\ell}\bigoplus_{k=0}^{1}\bigoplus_{a=0}^{1-k}\bigoplus_{b=0}^{k}[p+1-k-a,\ \ell-p+k-b]_{\substack{-R_1(\ell-2p-k-2a+2b+\frac{1}{2})\\+R_2(n-\ell-2t+k+\frac{1}{2})}}$$

$$\oplus\bigoplus_{\ell=0}^{n-1}\bigoplus_{t=0}^{n-1-\ell}\bigoplus_{q=0}^{1}\bigoplus_{p=0}^{\ell}\bigoplus_{a=0}^{q+1}\bigoplus_{b=0}^{2-q}[p+a,\ \ell-p+b]_{\substack{-R_1(\ell-2p-3q+2a-2b+\frac{3}{2})\\+R_2(n-\ell-2t-q-\frac{1}{2})}}$$

$$\oplus\bigoplus_{\ell=0}^{n-1}\bigoplus_{t=0}^{n-1-\ell}\bigoplus_{k,q=0}^{1}\bigoplus_{p=0}^{\ell+1}\bigoplus_{a=0}^{q}\bigoplus_{b=0}^{1-q}[p+a,\ \ell+1-p+b]_{\substack{-R_1(\ell-2p-3q+2a-2b+\frac{5}{2})\\+R_2(n-\ell-2t-2k-q+\frac{1}{2})}}$$

$$\oplus\bigoplus_{\ell=0}^{n-1}\bigoplus_{q=0}^{1}\bigoplus_{a=0}^{q}\bigoplus_{b=0}^{1-q}[a,\ b]_{-R_1(\frac{3}{2}-3q+2a-2b)+R_2(n-2\ell-q-\frac{1}{2})}$$

$$\oplus\bigoplus_{\ell=0}^{n-2}\bigoplus_{t=0}^{n-2-\ell}\bigoplus_{q=0}^{1}\bigoplus_{p=0}^{\ell}\bigoplus_{a=0}^{q+1}\bigoplus_{b=0}^{2-q}[p+a,\ \ell-p+b]_{\substack{-R_1(\ell-2p-3q+2a-2b+\frac{3}{2})\\+R_2(n-\ell-2t+q-\frac{5}{2})}}$$

$$\oplus\bigoplus_{\ell=0}^{n-2}\bigoplus_{t=0}^{n-2-\ell}\bigoplus_{k,q=0}^{1}\bigoplus_{p=0}^{\ell+1}\bigoplus_{a=0}^{q}\bigoplus_{b=0}^{1-q}[p+a,\ \ell+1-p+b]_{\substack{-R_1(\ell-2p-3q+2a-2b+\frac{5}{2})\\+R_2(n-\ell-2t+2k+q-\frac{7}{2})}}$$

$$
\oplus \bigoplus_{\ell=0}^{n-2}\bigoplus_{q=0}^{1}\bigoplus_{a=0}^{q}\bigoplus_{b=0}^{1-q}[a,\ b]_{-R_1(\frac{3}{2}-3q+2a-2b)+R_2(n-2\ell+q-\frac{5}{2})}
$$

$$
\oplus \bigoplus_{\ell=0}^{n-2}\bigoplus_{t=0}^{n-2-\ell}\bigoplus_{p=0}^{\ell}\bigoplus_{k=0}^{1}\bigoplus_{a=0}^{1-k}\bigoplus_{b=0}^{k}[p+1-k-a,\ \ell-p+k-b]_{\substack{-R_1(\ell-2p-k-2a+2b+\frac{1}{2})\\ +R_2(n-\ell-2t-k-\frac{3}{2})}}\,, \tag{E.12}
$$

$$
\begin{aligned}
S_n^+ = \ & [n+2,0,0,0]\oplus[n-2,2,0,0]\oplus[n-2,0,0,0] \\
& \xrightarrow{\mathrm{SU(3)\times U(1)_R}} \\
& \bigoplus_{\ell=0}^{n+2}\bigoplus_{t=0}^{n+2-\ell}\bigoplus_{p=0}^{\ell}[p,\ \ell-p]_{-R_1(\ell-2p)+R_2(n-\ell-2t+2)} \\
& \oplus\bigoplus_{\ell}^{n-2}\bigoplus_{t=0}^{n-2-\ell}\bigoplus_{q=0}^{2}\bigoplus_{k=0}^{q}\bigoplus_{p=0}^{\ell+q}\bigoplus_{a,b=0}^{2-q}[p+a,\ \ell+q-p+b]_{\substack{-R_1(\ell+q-2p+2a-2b)\\ +R_2(n-\ell-2t+q-2k-2)}} \\
& \quad\oplus\bigoplus_{\ell}^{n-2}\bigoplus_{k=0}^{1}\bigoplus_{a,b=0}^{k}[a,\ b]_{\substack{-2R_1(a-b)\\ +R_2(n-2\ell-2)}}\oplus\bigoplus_{\ell}^{n-2}\bigoplus_{k,p=0}^{1}[p,\ 1-p]_{\substack{-R_1(1-2p)\\ +R_2(n-2\ell-2k-1)}} \\
& \oplus\bigoplus_{\ell=0}^{n-2}\bigoplus_{t=0}^{n-2-\ell}\bigoplus_{p=0}^{\ell}[p,\ \ell-p]_{-R_1(\ell-2p)+R_2(n-\ell-2t-2)}\,,
\end{aligned} \tag{E.13}
$$

$$
\begin{aligned}
S_n^- = \ & [n,0,2,0]\oplus[n-2,0,0,2] \\
& \xrightarrow{\mathrm{SU(3)\times U(1)_R}} \\
& \bigoplus_{\ell=0}^{n}\bigoplus_{t=0}^{n-\ell}\bigoplus_{k=0}^{2}\bigoplus_{p=0}^{\ell}\bigoplus_{a=0}^{2-k}\bigoplus_{b=0}^{k}[p+a,\ \ell-p+b]_{\substack{-R_1(\ell-2p+3k+2a-2b-3)\\ +R_2(n-\ell-2t+k-1)}} \\
& \oplus\bigoplus_{\ell=0}^{n-2}\bigoplus_{t=0}^{n-2-\ell}\bigoplus_{k=0}^{2}\bigoplus_{p=0}^{\ell}\bigoplus_{a=0}^{2-k}\bigoplus_{b=0}^{k}[p+a,\ \ell-p+b]_{\substack{-R_1(\ell-2p+3k+2a-2b-3)\\ +R_2(n-\ell-2t-k-1)}}\,.
\end{aligned} \tag{E.14}
$$

E.3 $\mathcal{N}=4$ supermultiplets

As explained in section 7.2.1, the algebraic structure of the KK spectrum across the holographic CM (7.9) is inherited from that at the $\mathcal{N}=4$ point. Thus, it is useful to collect some aspects of OSp(4|4) representation theory. More concretely, in this appendix we give the explicit state content of the multiplets present in the $\mathcal{N}=4$ spectrum. We also give some relevant shortening conditions and branching rules under (7.23) into $\mathcal{N}=2$ multiplets.

The general representation theory of OSp(4|4) has been laid out in [224]. The states that compose a given[2] MULT_4 representation of OSp(4|4) carry definite SO(4) R-charges. In our conventions, these are labelled with half-integer Dynkin labels $(\ell_1,\, \ell_2)$. Unfortunately, the generic expressions for the OSp(4|4) multiplet contents given in [224] do not work well for scalar superconformal primaries or low values of $(\ell_1,\, \ell_2)$, where many states are actually absent and need to be sieved out. These are the cases relevant to our analysis. Here, we will determine the state content of the OSp(4|4) long graviton multiplet (7.19) for all possible values of the Dynkin labels on a case-by-case basis. Only multiplets with integer $(\ell_1,\, \ell_2)$ enter the KK spectra of interest in this paper. Once we got down to business though, it only took a finite amount of additional pain to get the strictly half-integer cases as well. Similar remarks apply to the OSp(3|4) representation theory contained in [224]: see appendix B of [134] for complete listings.

It is useful to start by listing the possible Lorentz spins, $[s]=0,\frac{1}{2},1,\frac{3}{2},2$, and SO(4) Dynkin labels, $(\ell_1,\, \ell_2)$, that subsequent powers Q^p, $p=0,1,\ldots,8$, of the OSp(4|4) supercharge Q may have. The result is:

$$
\begin{aligned}
1 \;\&\; Q^8 \;&:\; [0] \qquad (0,0)\;,\\
Q \;\&\; Q^7 \;&:\; [\tfrac{1}{2}] \qquad \left(\tfrac{1}{2},\tfrac{1}{2}\right)\;,\\
Q^2 \;\&\; Q^6 \;&:\; \begin{cases} [1] & (1,0)+(0,1)\;,\\ [0] & (1,1)+(0,0)\;,\end{cases}\\
Q^3 \;\&\; Q^5 \;&:\; \begin{cases} [\tfrac{3}{2}] & \left(\tfrac{1}{2},\tfrac{1}{2}\right)\;,\\ [\tfrac{1}{2}] & \left(\tfrac{3}{2},\tfrac{1}{2}\right)+\left(\tfrac{1}{2},\tfrac{3}{2}\right)+\left(\tfrac{1}{2},\tfrac{1}{2}\right)\;,\end{cases}\\
Q^4 \;&:\; \begin{cases} [2] & (0,0)\;,\\ [1] & (1,1)+(1,0)+(0,1)\;,\\ [0] & (2,0)+(0,2)+(1,1)+(0,0)\;.\end{cases}
\end{aligned}
\tag{E.15}
$$

Together with the fact that the action with Q increases the conformal

[2] As in the main text, we use the acronym MULT_4 to refer to a generic multiplet of OSp(4|4). We specifically denote long and short graviton and short gravitino multiplets as LGRAV_4, SGRAV_4 and SGINO_4. These respectively correspond to the multiplets denoted in [224] as L (with $j_{\text{there}}=0$), A_2 and B_1. Graviton and gravitino OSp(4|4) multiplets have scalar, $s_0=0$, superconformal primaries, and gravitino multiplets are necessarilyshort.

dimension by $\frac{1}{2}$, the information summarised in (E.15) is the basic building block to find out the state content of the long graviton multiplets (7.19) of OSp(4|4). For a (scalar) superconformal primary with Lorentz and SO(4) spins $s_0 = 0$ and (ℓ_1, ℓ_2) and dimension E_0, the descendants have all possible Lorentz spins $[s]$ shown in (E.15), and lie in the SO(4) representations that result from tensoring row by row the representations listed in (E.15) with (ℓ_1, ℓ_2). Finally, the dimension of the p-th descendant is $E_0 + \frac{p}{2}$.

For easy reference, the outcome of this exercise for all (ℓ_1, ℓ_2), with $\ell_1 \geq \ell_2$ without loss of generality, is listed in tables E.4–E.13. The table entries show the spin and SO(4) charges, in the format $[s]^{(\ell_1', \ell_2')}$, of each possible state in the multiplet. The corresponding dimensions Δ are given next to each entry, and these are grouped as descendants of the superconformal primary at the top of each table. An entry of the form $[s]^{(\ell_1 \pm a, \ell_2 \pm b)}$ denotes four states in total (this differs from the convention adopted, in a different context, in the main text: see below (7.43)). Also, negative Dynkin labels are not allowed, and the corresponding states must be removed as they are actually absent. These spurious states only occur in tables E.10–E.13. Table E.13 is valid at face value for all $\ell_1, \ell_2 \geq 2$, with all entries therein present. The same table is also valid for $\ell_i = \frac{3}{2}$ for either or both $i = 1, 2$, but the states at level Q^4 with negative Dynkin labels need to be discarded. Similar comments apply to tables E.10–E.12. Tables E.4–E.12 involve fewer states compared to table E.13 and, without going through the constructive algorithm specified above, it is not obvious which states must be crossed out in table E.13 to recover tables E.4–E.12. Only tables E.4, E.7, E.9, E.10, E.12 and E.13 play a role in the KK spectra described in this paper. The remaining tables necessarily involve strictly half-integer SO(4) labels for the superconformal primary and are only included for completeness.

The dimensions E_0 and Dynkin labels (ℓ_1, ℓ_2) of the (superconformal primary of the) OSp(4|4) multiplets constructed with the above algorithm must respect the unitarity bound

$$E_0 \geq s_0 + \ell_1 + \ell_2 + 1 \,, \tag{E.16}$$

with $s_0 = 0$ for the graviton multiplets listed in the tables. The multiplets undergo shortening when the bound is saturated, in which case they split into short graviton and gravitino multiplets as:

$$\begin{aligned}&\mathrm{LGRAV}_4[\ell_1+\ell_2+1,\ell_1,\,\ell_2]\\&\qquad\to \mathrm{SGRAV}_4[\ell_1+\ell_2+1,\ell_1,\,\ell_2]+\mathrm{SGINO}_4[\ell_1+\ell_2+3,\ell_1+1,\,\ell_2+1]\,.\end{aligned} \tag{E.17}$$

See [224] for the state contents of these $\mathcal{N} = 4$ short multiplets for specific values of (ℓ_1, ℓ_2).

It is also useful to give the splitting of the above $\mathcal{N} = 4$ graviton multiplets under the supergroup embedding (7.23) into $\mathcal{N} = 2$ multiplets of definite $\mathrm{U}(1)_F$ flavour charge. The $\mathrm{U}(1)_R$ R-symmetry group of OSp(2|4) and $\mathrm{U}(1)_F$ are the subgroups of the SO(4) R-symmetry of OSp(4|4) specified in (7.10) and below that equation. Branching accordingly the SO(4) representations of the states in tables E.4–E.13, recombining them into OSp(2|4) multiplets using the $\mathcal{N} = 2$ tables of appendix A of [82], and keeping track of the flavour charges, we obtain

$$\begin{aligned}&\mathrm{LGRAV}_4\big[E_0,\ell_1,\ell_2\big] =\\&\bigoplus_{m_1=-\ell_1}^{\ell_1}\bigoplus_{m_2=-\ell_2}^{\ell_2}\Big\{\underline{\mathrm{LGRAV}}_2\big[E_0+1,y_{m_1m_2};f_{m_1m_2}\big]\\&\oplus \underline{\mathrm{LGINO}}_2\big[E_0+\tfrac{1}{2},y_{m_1m_2};f_{m_1m_2}+1\big]\oplus \underline{\mathrm{LGINO}}_2\big[E_0+\tfrac{1}{2},y_{m_1m_2};f_{m_1m_2}-1\big]\\&\oplus \mathrm{LGINO}_2\big[E_0+\tfrac{3}{2},y_{m_1m_2};f_{m_1m_2}+1\big]\oplus \mathrm{LGINO}_2\big[E_0+\tfrac{3}{2},y_{m_1m_2};f_{m_1m_2}-1\big]\\&\oplus \underline{\mathrm{LVEC}}_2\big[E_0,y_{m_1m_2};f_{m_1m_2}\big]\\&\oplus \mathrm{LVEC}_2\big[E_0+1,y_{m_1m_2};f_{m_1m_2}+2\big]\oplus \mathrm{LVEC}_2\big[E_0+1,y_{m_1m_2};f_{m_1m_2}\big]\\&\oplus \mathrm{LVEC}_2\big[E_0+1,y_{m_1m_2};f_{m_1m_2}-2\big]\oplus \mathrm{LVEC}_2\big[E_0+2,y_{m_1m_2};f_{m_1m_2}\big]\Big\}\,,\end{aligned} \tag{E.18}$$

with $y_{m_1m_2}$ and $f_{m_1m_2}$ given in (7.25). When E_0 saturates the $\mathcal{N} = 4$ unitarity bound (E.16), the $\mathcal{N} = 4$ multiplet on the l.h.s. of (E.18) becomes short as in (E.17), and the underlined $\mathcal{N} = 2$ multiplets on the r.h.s. at $(m_1 = -\ell_1, m_2 = -\ell_2)$ and $(m_1 = \ell_1, m_2 = \ell_2)$ shorten as well. The case of interest to this paper has, in particular, ℓ_1 and ℓ_2 further restricted to be equal, as in (7.22), in the short multiplets. With this further restriction, the relevant $\mathcal{N} = 2$ multiplets in (E.18) that undergo shortening are thus

$$\mathrm{LGRAV}_2\big[\ell+2+\epsilon,\pm\ell;0\big] \tag{E.19}$$

$$\rightarrow \text{SGRAV}_2\big[\ell+2,\pm\ell;0\big] \oplus \text{SGINO}_2\big[\ell+\tfrac{5}{2},\pm(\ell+1);0\big]\,,$$

$$\text{LGINO}_2\big[\ell+\tfrac{3}{2}+\epsilon,\pm\ell;+1\big] \tag{E.20}$$

$$\rightarrow \text{SGINO}_2\big[\ell+\tfrac{3}{2},\pm\ell;+1\big] \oplus \text{SVEC}_2\big[\ell+2,\pm(\ell+1);+1\big]\,,$$

$$\text{LGINO}_2\big[\ell+\tfrac{3}{2}+\epsilon,\pm\ell;-1\big] \tag{E.21}$$

$$\rightarrow \text{SGINO}_2\big[\ell+\tfrac{3}{2},\pm\ell;-1\big] \oplus \text{SVEC}_2\big[\ell+2,\pm(\ell+1);-1\big]\,,$$

$$\text{LVEC}_2\big[\ell+1+\epsilon,\pm\ell;0\big] \tag{E.22}$$

$$\rightarrow \text{SVEC}_2\big[\ell+1,\pm\ell;0\big] \oplus \text{HYP}_2\big[\ell+2,\pm(\ell+2);0\big]\,, \tag{E.23}$$

as in (7.26). Only the flavour-neutral short multiplets here make it to the list of protected multiplets in table 7.2. The short flavoured multiplets appear accidentally in the spectra of the SO(4) and $\text{SU}(2)_F$ points, joining other multiplets into SO(4) and $\text{SU}(2)_F$ representations. An extreme case of the shortening conditions occurs when the graviton becomes massless. In this case, we have the following splitting of a massless $\mathcal{N}=4$ graviton multiplet into $\mathcal{N}=2$ massless ones:

$$\text{MGRAV}_4[1,0,0] = \text{MGRAV}_2[2,0;0] \oplus \text{MGINO}_2[\tfrac{3}{2},0;\pm1] \oplus \text{MVEC}_2[1,0;0]\,. \tag{E.24}$$

We conclude with the observation that the multiplicities, the (superconformal primary) $\text{U}(1)_R$ charge, and the (overall) $\text{U}(1)_F$ flavour charge of the $\mathcal{N}=2$ multiplets that compose $\text{LGRAV}_4\big[E_0,\ell_1,\ell_2\big]$ according to (E.18) can be also retrieved in the following manner. Introducing fugacites u and x for $\text{U}(1)_R$ and $\text{U}(1)_F$, define for each multiplet on the r.h.s. of (E.18) the functions

$$\nu^{E_0+1}_{\text{LGRAV}_2} = \nu^{E_0}_{\text{LVEC}_2} = \nu^{E_0+2}_{\text{LVEC}_2} = \frac{\big[1-\big(ux\big)^{2\ell_1+1}\big]\big[1-\big(\frac{u}{x}\big)^{2\ell_2+1}\big]}{\big(ux\big)^{\ell_1}\big(\frac{u}{x}\big)^{\ell_2}\big(1-ux\big)\big(1-\frac{u}{x}\big)}\,,$$

$$\nu^{E_0+\frac{1}{2}}_{\text{LGINO}_2} = \nu^{E_0+\frac{3}{2}}_{\text{LGINO}_2} = \frac{(x+1)}{x}\frac{\big[1-\big(ux\big)^{2\ell_1+1}\big]\big[1-\big(\frac{u}{x}\big)^{2\ell_2+1}\big]}{\big(ux\big)^{\ell_1}\big(\frac{u}{x}\big)^{\ell_2}\big(1-ux\big)\big(1-\frac{u}{x}\big)}\,,$$

$$\nu^{E_0+1}_{\text{LVEC}_2} = \frac{(x^2+x+1)}{x}\frac{\big[1-\big(ux\big)^{2\ell_1+1}\big]\big[1-\big(\frac{u}{x}\big)^{2\ell_2+1}\big]}{\big(ux\big)^{\ell_1}\big(\frac{u}{x}\big)^{\ell_2}\big(1-ux\big)\big(1-\frac{u}{x}\big)}\,, \tag{E.25}$$

with $\nu^{E_0+1}_{\text{LGRAV}_2}$ corresponding to the LGRAV_2's with dimension E_0+1, etc. Expanding these functions at fixed ℓ_1 and ℓ_2 in powers of u and x, the multiplicity m, R-charge y_0 and flavour charge f of a multiplet can be read off from the term $m\, u^{y_0} x^f$ in the expansion of its associated function ν.

	E_0	$[0]^{(0,0)}$
Q	$E_0 + \frac{1}{2}$	$[\frac{1}{2}]^{(\frac{1}{2},\frac{1}{2})}$
Q^2	$E_0 + 1$	$[1]^{(1,0)} + [1]^{(0,1)}$
		$[0]^{(1,1)} + [0]^{(0,0)}$
Q^3	$E_0 + \frac{3}{2}$	$[\frac{3}{2}]^{(\frac{1}{2},\frac{1}{2})}$
		$[\frac{1}{2}]^{(\frac{3}{2},\frac{1}{2})} + [\frac{1}{2}]^{(\frac{1}{2},\frac{3}{2})} + [\frac{1}{2}]^{(\frac{1}{2},\frac{1}{2})}$
Q^4	$E_0 + 2$	$[2]^{(0,0)}$
		$[1]^{(1,1)} + [1]^{(1,0)} + [1]^{(0,1)}$
		$[0]^{(2,0)} + [0]^{(0,2)} + [0]^{(1,1)} + [0]^{(0,0)}$
Q^5	$E_0 + \frac{5}{2}$	$[\frac{3}{2}]^{(\frac{1}{2},\frac{1}{2})}$
		$[\frac{1}{2}]^{(\frac{3}{2},\frac{1}{2})} + [\frac{1}{2}]^{(\frac{1}{2},\frac{3}{2})} + [\frac{1}{2}]^{(\frac{1}{2},\frac{1}{2})}$
Q^6	$E_0 + 3$	$[1]^{(1,0)} + [1]^{(0,1)}$
		$[0]^{(1,1)} + [0]^{(0,0)}$
Q^7	$E_0 + \frac{7}{2}$	$[\frac{1}{2}]^{(\frac{1}{2},\frac{1}{2})}$
Q^8	$E_0 + 4$	$[0]^{(0,0)}$

Table E.4: States in the long graviton supermultiplet $\mathrm{LGRAV}_4[E_0, 0, 0]$.

	E_0	$[0]^{\left(\frac{1}{2},0\right)}$
Q	$E_0+\frac{1}{2}$	$[\frac{1}{2}]^{\left(1,\frac{1}{2}\right)}+[\frac{1}{2}]^{\left(0,\frac{1}{2}\right)}$
Q^2	E_0+1	$[1]^{\left(\frac{3}{2},0\right)}+[1]^{\left(\frac{1}{2},0\right)}+[1]^{\left(\frac{1}{2},1\right)}$
		$[0]^{\left(\frac{3}{2},1\right)}+[0]^{\left(\frac{1}{2},1\right)}+[0]^{\left(\frac{1}{2},0\right)}$
Q^3	$E_0+\frac{3}{2}$	$[\frac{3}{2}]^{\left(1,\frac{1}{2}\right)}+[\frac{3}{2}]^{\left(0,\frac{1}{2}\right)}$
		$[\frac{1}{2}]^{\left(2,\frac{1}{2}\right)}+2\,[\frac{1}{2}]^{\left(1,\frac{1}{2}\right)}+[\frac{1}{2}]^{\left(1,\frac{3}{2}\right)}+[\frac{1}{2}]^{\left(0,\frac{3}{2}\right)}+[\frac{1}{2}]^{\left(0,\frac{1}{2}\right)}$
Q^4	E_0+2	$[2]^{\left(\frac{1}{2},0\right)}$
		$[1]^{\left(\frac{3}{2},1\right)}+2\,[1]^{\left(\frac{1}{2},1\right)}+[1]^{\left(\frac{3}{2},0\right)}+[1]^{\left(\frac{1}{2},0\right)}$
		$[0]^{\left(\frac{5}{2},0\right)}+[0]^{\left(\frac{3}{2},0\right)}+[0]^{\left(\frac{1}{2},2\right)}+[0]^{\left(\frac{3}{2},1\right)}+[0]^{\left(\frac{1}{2},1\right)}+[0]^{\left(\frac{1}{2},0\right)}$
Q^5	$E_0+\frac{5}{2}$	$[\frac{3}{2}]^{\left(1,\frac{1}{2}\right)}+[\frac{3}{2}]^{\left(0,\frac{1}{2}\right)}$
		$[\frac{1}{2}]^{\left(2,\frac{1}{2}\right)}+2\,[\frac{1}{2}]^{\left(1,\frac{1}{2}\right)}+[\frac{1}{2}]^{\left(1,\frac{3}{2}\right)}+[\frac{1}{2}]^{\left(0,\frac{3}{2}\right)}+[\frac{1}{2}]^{\left(0,\frac{1}{2}\right)}$
Q^6	E_0+3	$[1]^{\left(\frac{3}{2},0\right)}+[1]^{\left(\frac{1}{2},0\right)}+[1]^{\left(\frac{1}{2},1\right)}$
		$[0]^{\left(\frac{3}{2},1\right)}+[0]^{\left(\frac{1}{2},1\right)}+[0]^{\left(\frac{1}{2},0\right)}$
Q^7	$E_0+\frac{7}{2}$	$[\frac{1}{2}]^{\left(1,\frac{1}{2}\right)}+[\frac{1}{2}]^{\left(0,\frac{1}{2}\right)}$
Q^8	E_0+4	$[0]^{\left(\frac{1}{2},0\right)}$

Table E.5: States in the long graviton supermultiplet $\mathrm{LGRAV}_4[E_0,\frac{1}{2},0]$.

	E_0	$[0]^{\left(\frac{1}{2},\frac{1}{2}\right)}$
Q	$E_0+\frac{1}{2}$	$[\frac{1}{2}]^{(1,1)}+[\frac{1}{2}]^{(1,0)}+[\frac{1}{2}]^{(0,1)}+[\frac{1}{2}]^{(0,0)}$
Q^2	E_0+1	$[1]^{\left(\frac{3}{2},\frac{1}{2}\right)}+[1]^{\left(\frac{1}{2},\frac{3}{2}\right)}+2\,[1]^{\left(\frac{1}{2},\frac{1}{2}\right)}$
		$[0]^{\left(\frac{3}{2},\frac{3}{2}\right)}+[0]^{\left(\frac{3}{2},\frac{1}{2}\right)}+[0]^{\left(\frac{1}{2},\frac{3}{2}\right)}+2\,[0]^{\left(\frac{1}{2},\frac{1}{2}\right)}$
Q^3	$E_0+\frac{3}{2}$	$[\frac{3}{2}]^{(1,1)}+[\frac{3}{2}]^{(1,0)}+[\frac{3}{2}]^{(0,1)}+[\frac{3}{2}]^{(0,0)}$
		$[\frac{1}{2}]^{(2,1)}+[\frac{1}{2}]^{(1,2)}+[\frac{1}{2}]^{(2,0)}+[\frac{1}{2}]^{(0,2)}+3[\frac{1}{2}]^{(1,1)}+2[\frac{1}{2}]^{(1,0)}+2[\frac{1}{2}]^{(0,1)}+[\frac{1}{2}]^{(0,0)}$
Q^4	E_0+2	$[2]^{\left(\frac{1}{2},\frac{1}{2}\right)}$
		$[1]^{\left(\frac{3}{2},\frac{3}{2}\right)}+2\,[1]^{\left(\frac{3}{2},\frac{1}{2}\right)}+2\,[1]^{\left(\frac{1}{2},\frac{3}{2}\right)}+3\,[1]^{\left(\frac{1}{2},\frac{1}{2}\right)}$
		$[0]^{\left(\frac{5}{2},\frac{1}{2}\right)}+[0]^{\left(\frac{1}{2},\frac{5}{2}\right)}+[0]^{\left(\frac{3}{2},\frac{3}{2}\right)}+2\,[0]^{\left(\frac{3}{2},\frac{1}{2}\right)}+2\,[0]^{\left(\frac{1}{2},\frac{3}{2}\right)}+2\,[0]^{\left(\frac{1}{2},\frac{1}{2}\right)}$
Q^5	$E_0+\frac{5}{2}$	$[\frac{3}{2}]^{(1,1)}+[\frac{3}{2}]^{(1,0)}+[\frac{3}{2}]^{(0,1)}+[\frac{3}{2}]^{(0,0)}$
		$[\frac{1}{2}]^{(2,1)}+[\frac{1}{2}]^{(1,2)}+[\frac{1}{2}]^{(2,0)}+[\frac{1}{2}]^{(0,2)}+3[\frac{1}{2}]^{(1,1)}+2[\frac{1}{2}]^{(1,0)}+2[\frac{1}{2}]^{(0,1)}+[\frac{1}{2}]^{(0,0)}$
Q^6	E_0+3	$[1]^{\left(\frac{3}{2},\frac{1}{2}\right)}+[1]^{\left(\frac{1}{2},\frac{3}{2}\right)}+2\,[1]^{\left(\frac{1}{2},\frac{1}{2}\right)}$
		$[0]^{\left(\frac{3}{2},\frac{3}{2}\right)}+[0]^{\left(\frac{3}{2},\frac{1}{2}\right)}+[0]^{\left(\frac{1}{2},\frac{3}{2}\right)}+2\,[0]^{\left(\frac{1}{2},\frac{1}{2}\right)}$
Q^7	$E_0+\frac{7}{2}$	$[\frac{1}{2}]^{(1,1)}+[\frac{1}{2}]^{(1,0)}+[\frac{1}{2}]^{(0,1)}+[\frac{1}{2}]^{(0,0)}$
Q^8	E_0+4	$[0]^{\left(\frac{1}{2},\frac{1}{2}\right)}$

Table E.6: States in the long graviton supermultiplet $\mathrm{LGRAV}_4[E_0,\frac{1}{2},\ \frac{1}{2}]$.

	E_0	$[0]^{(1,0)}$
Q	$E_0 + \frac{1}{2}$	$[\frac{1}{2}]^{\left(\frac{3}{2},\frac{1}{2}\right)} + [\frac{1}{2}]^{\left(\frac{1}{2},\frac{1}{2}\right)}$
Q^2	$E_0 + 1$	$[1]^{(2,0)} + [1]^{(1,1)} + [1]^{(1,0)} + [1]^{(0,0)}$
		$[0]^{(2,1)} + [1]^{(1,1)} + [1]^{(1,0)} + [1]^{(0,1)}$
Q^3	$E_0 + \frac{3}{2}$	$[\frac{3}{2}]^{\left(\frac{3}{2},\frac{1}{2}\right)} + [\frac{3}{2}]^{\left(\frac{1}{2},\frac{1}{2}\right)}$
		$[\frac{1}{2}]^{\left(\frac{5}{2},\frac{1}{2}\right)} + [\frac{1}{2}]^{\left(\frac{3}{2},\frac{3}{2}\right)} + 2\,[\frac{1}{2}]^{\left(\frac{3}{2},\frac{1}{2}\right)} + [\frac{1}{2}]^{\left(\frac{1}{2},\frac{3}{2}\right)} + 2\,[\frac{1}{2}]^{\left(\frac{1}{2},\frac{1}{2}\right)}$
Q^4	$E_0 + 2$	$[2]^{(1,0)}$
		$[1]^{(2,1)} + [1]^{(2,0)} + 2\,[1]^{(1,1)} + [1]^{(1,0)} + [1]^{(0,1)} + [1]^{(0,0)}$
		$[0]^{(3,0)} + [0]^{(2,1)} + [0]^{(2,0)} + [0]^{(1,2)} + [0]^{(1,1)} + 2\,[0]^{(1,0)} + [0]^{(0,1)}$
Q^5	$E_0 + \frac{5}{2}$	$[\frac{3}{2}]^{\left(\frac{3}{2},\frac{1}{2}\right)} + [\frac{3}{2}]^{\left(\frac{1}{2},\frac{1}{2}\right)}$
		$[\frac{1}{2}]^{\left(\frac{5}{2},\frac{1}{2}\right)} + [\frac{1}{2}]^{\left(\frac{3}{2},\frac{3}{2}\right)} + 2\,[\frac{1}{2}]^{\left(\frac{3}{2},\frac{1}{2}\right)} + [\frac{1}{2}]^{\left(\frac{1}{2},\frac{3}{2}\right)} + 2\,[\frac{1}{2}]^{\left(\frac{1}{2},\frac{1}{2}\right)}$
Q^6	$E_0 + 3$	$[1]^{(2,0)} + [1]^{(1,1)} + [1]^{(1,0)} + [1]^{(0,0)}$
		$[0]^{(2,1)} + [1]^{(1,1)} + [1]^{(1,0)} + [1]^{(0,1)}$
Q^7	$E_0 + \frac{7}{2}$	$[\frac{1}{2}]^{\left(\frac{3}{2},\frac{1}{2}\right)} + [\frac{1}{2}]^{\left(\frac{1}{2},\frac{1}{2}\right)}$
Q^8	$E_0 + 4$	$[0]^{(1,0)}$

Table E.7: States in the long graviton supermultiplet $\mathrm{LGRAV}_4[E_0, 1, 0]$.

	E_0	$[0]^{(1,\frac{1}{2})}$
Q	$E_0+\frac{1}{2}$	$[\frac{1}{2}]^{(\frac{3}{2},1)} + [\frac{1}{2}]^{(\frac{3}{2},0)} + [\frac{1}{2}]^{(\frac{1}{2},1)} + [\frac{1}{2}]^{(\frac{1}{2},0)}$
Q^2	E_0+1	$[1]^{(2,\frac{1}{2})} + [1]^{(1,\frac{3}{2})} + 2\,[1]^{(1,\frac{1}{2})} + [1]^{(0,\frac{1}{2})}$
		$[0]^{(2,\frac{3}{2})} + [0]^{(2,\frac{1}{2})} + [0]^{(1,\frac{3}{2})} + 2\,[0]^{(1,\frac{1}{2})} + [0]^{(0,\frac{3}{2})} + [0]^{(0,\frac{1}{2})}$
Q^3	$E_0+\frac{3}{2}$	$[\frac{3}{2}]^{(\frac{3}{2},1)} + [\frac{3}{2}]^{(\frac{3}{2},0)} + [\frac{3}{2}]^{(\frac{1}{2},1)} + [\frac{3}{2}]^{(\frac{1}{2},0)}$
		$[\frac{1}{2}]^{(\frac{5}{2},1)} + [\frac{1}{2}]^{(\frac{5}{2},0)} + [\frac{1}{2}]^{(\frac{3}{2},2)} + 3\,[\frac{1}{2}]^{(\frac{3}{2},1)} + 2\,[\frac{1}{2}]^{(\frac{3}{2},0)} + [\frac{1}{2}]^{(\frac{1}{2},2)}$
		$+\,3\,[\frac{1}{2}]^{(\frac{1}{2},1)} + 2\,[\frac{1}{2}]^{(\frac{1}{2},0)}$
Q^4	E_0+2	$[2]^{(1,\frac{1}{2})}$
		$[1]^{(2,\frac{3}{2})} + 2\,[1]^{(2,\frac{1}{2})} + 2\,[1]^{(1,\frac{3}{2})} + 3\,[1]^{(1,\frac{1}{2})} + [1]^{(0,\frac{3}{2})} + 2\,[1]^{(0,\frac{1}{2})}$
		$[0]^{(3,\frac{3}{2})} + [0]^{(2,\frac{3}{2})} + 2\,[0]^{(2,\frac{1}{2})} + [0]^{(1,\frac{5}{2})} + 2\,[0]^{(1,\frac{3}{2})} + 3\,[0]^{(1,\frac{1}{2})}$
		$+\,[0]^{(0,\frac{3}{2})} + [0]^{(0,\frac{1}{2})}$
Q^5	$E_0+\frac{5}{2}$	$[\frac{1}{2}]^{(\frac{5}{2},1)} + [\frac{1}{2}]^{(\frac{5}{2},0)} + [\frac{1}{2}]^{(\frac{3}{2},2)} + 3\,[\frac{1}{2}]^{(\frac{3}{2},1)} + 2\,[\frac{1}{2}]^{(\frac{3}{2},0)} + [\frac{1}{2}]^{(\frac{1}{2},2)}$
		$+\,3\,[\frac{1}{2}]^{(\frac{1}{2},1)} + 2\,[\frac{1}{2}]^{(\frac{1}{2},0)}$
Q^6	E_0+3	$[1]^{(2,\frac{1}{2})} + [1]^{(1,\frac{3}{2})} + 2\,[1]^{(1,\frac{1}{2})} + [1]^{(0,\frac{1}{2})}$
		$[0]^{(2,\frac{3}{2})} + [0]^{(2,\frac{1}{2})} + [0]^{(1,\frac{3}{2})} + 2\,[0]^{(1,\frac{1}{2})} + [0]^{(0,\frac{3}{2})} + [0]^{(0,\frac{1}{2})}$
Q^7	$E_0+\frac{7}{2}$	$[\frac{1}{2}]^{(\frac{3}{2},1)} + [\frac{1}{2}]^{(\frac{3}{2},0)} + [\frac{1}{2}]^{(\frac{1}{2},1)} + [\frac{1}{2}]^{(\frac{1}{2},0)}$
Q^8	E_0+4	$[0]^{(1,\frac{1}{2})}$

Table E.8: States in the long graviton supermultiplet $\mathrm{LGRAV}_4[E_0,1,\frac{1}{2}]$.

	E_0	$[0]^{(1,1)}$
Q	$E_0+\frac{1}{2}$	$[\frac{1}{2}]^{\left(\frac{3}{2},\frac{3}{2}\right)} + [\frac{1}{2}]^{\left(\frac{3}{2},\frac{1}{2}\right)} + [\frac{1}{2}]^{\left(\frac{1}{2},\frac{3}{2}\right)} + [\frac{1}{2}]^{\left(\frac{1}{2},\frac{1}{2}\right)}$
Q^2	E_0+1	$[1]^{(2,1)} + [1]^{(1,2)} + 2\,[1]^{(1,1)} + [1]^{(1,0)} + [1]^{(0,1)}$
		$[0]^{(2,2)} + [0]^{(2,1)} + [0]^{(2,0)} + [0]^{(1,2)} + 2\,[0]^{(1,1)} + [0]^{(1,0)}$
		$+\,[0]^{(0,2)} + [0]^{(0,1)} + [0]^{(0,0)}$
Q^3	$E_0+\frac{3}{2}$	$[\frac{3}{2}]^{\left(\frac{3}{2},\frac{3}{2}\right)} + [\frac{3}{2}]^{\left(\frac{3}{2},\frac{1}{2}\right)} + [\frac{3}{2}]^{\left(\frac{1}{2},\frac{3}{2}\right)} + [\frac{3}{2}]^{\left(\frac{1}{2},\frac{1}{2}\right)}$
		$[\frac{1}{2}]^{\left(\frac{5}{2},\frac{3}{2}\right)} + [\frac{1}{2}]^{\left(\frac{5}{2},\frac{1}{2}\right)} + [\frac{1}{2}]^{\left(\frac{3}{2},\frac{5}{2}\right)} + 3\,[\frac{1}{2}]^{\left(\frac{3}{2},\frac{3}{2}\right)} + 3\,[\frac{1}{2}]^{\left(\frac{3}{2},\frac{1}{2}\right)}$
		$+\,[\frac{1}{2}]^{\left(\frac{1}{2},\frac{5}{2}\right)} + 3\,[\frac{1}{2}]^{\left(\frac{3}{2},\frac{1}{2}\right)} + [\frac{1}{2}]^{\left(\frac{1}{2},\frac{1}{2}\right)}$
Q^4	E_0+2	$[2]^{(1,1)}$
		$[1]^{(2,2)} + 2\,[1]^{(2,1)} + [1]^{(2,0)} + 2\,[1]^{(1,2)} + 3\,[1]^{(1,1)} + 2\,[1]^{(1,0)}$
		$+\,[1]^{(0,2)} + 2\,[1]^{(0,1)} + [1]^{(0,0)}$
		$[0]^{(3,1)} + [0]^{(2,2)} + 2\,[0]^{(2,1)} + [0]^{(2,0)} + [0]^{(1,3)} + 2\,[0]^{(1,2)} + 4\,[0]^{(1,1)}$
		$+\,[0]^{(1,0)} + [0]^{(0,2)} + [0]^{(0,1)} + [0]^{(0,0)}$
Q^5	$E_0+\frac{5}{2}$	$[\frac{3}{2}]^{\left(\frac{3}{2},\frac{3}{2}\right)} + [\frac{3}{2}]^{\left(\frac{3}{2},\frac{1}{2}\right)} + [\frac{3}{2}]^{\left(\frac{1}{2},\frac{3}{2}\right)} + [\frac{3}{2}]^{\left(\frac{1}{2},\frac{1}{2}\right)}$
		$[\frac{1}{2}]^{\left(\frac{5}{2},\frac{3}{2}\right)} + [\frac{1}{2}]^{\left(\frac{5}{2},\frac{1}{2}\right)} + [\frac{1}{2}]^{\left(\frac{3}{2},\frac{5}{2}\right)} + 3\,[\frac{1}{2}]^{\left(\frac{3}{2},\frac{3}{2}\right)} + 3\,[\frac{1}{2}]^{\left(\frac{3}{2},\frac{1}{2}\right)}$
		$+\,[\frac{1}{2}]^{\left(\frac{1}{2},\frac{5}{2}\right)} + 3\,[\frac{1}{2}]^{\left(\frac{3}{2},\frac{1}{2}\right)} + [\frac{1}{2}]^{\left(\frac{1}{2},\frac{1}{2}\right)}$
Q^6	E_0+3	$[1]^{(2,1)} + [1]^{(1,2)} + 2\,[1]^{(1,1)} + [1]^{(1,0)} + [1]^{(0,1)}$
		$[0]^{(2,2)} + [0]^{(2,1)} + [0]^{(2,0)} + [0]^{(1,2)} + 2\,[0]^{(1,1)} + [0]^{(1,0)}$
		$+\,[0]^{(0,2)} + [0]^{(0,1)} + [0]^{(0,0)}$
Q^7	$E_0+\frac{7}{2}$	$[\frac{1}{2}]^{\left(\frac{3}{2},\frac{3}{2}\right)} + [\frac{1}{2}]^{\left(\frac{3}{2},\frac{1}{2}\right)} + [\frac{1}{2}]^{\left(\frac{1}{2},\frac{3}{2}\right)} + [\frac{1}{2}]^{\left(\frac{1}{2},\frac{1}{2}\right)}$
Q^8	E_0+4	$[0]^{(1,1)}$

Table E.9: States in the long graviton supermultiplet $\mathrm{LGRAV}_4[E_0,1,1]$.

	E_0	$[0]^{(\ell_1,0)}$
Q	$E_0 + \frac{1}{2}$	$[\frac{1}{2}]^{\left(\ell_1 \pm \frac{1}{2}, \frac{1}{2}\right)}$
Q^2	$E_0 + 1$	$[1]^{(\ell_1 \pm 1,0)} + [1]^{(\ell_1,1)} + [0]^{(\ell_1,0)}$
		$[0]^{(\ell_1 \pm 1,1)} + [0]^{(\ell_1,1)} + [0]^{(\ell_1,0)}$
Q^3	$E_0 + \frac{3}{2}$	$[\frac{3}{2}]^{\left(\ell_1 \pm \frac{1}{2}, \frac{1}{2}\right)}$
		$[\frac{1}{2}]^{\left(\ell_1 \pm \frac{3}{2}, \frac{1}{2}\right)} + [\frac{1}{2}]^{\left(\ell_1 \pm \frac{1}{2}, \frac{3}{2}\right)} + 2\,[\frac{1}{2}]^{\left(\ell_1 \pm \frac{1}{2}, \frac{1}{2}\right)}$
Q^4	$E_0 + 2$	$[2]^{(\ell_1,0)}$
		$[1]^{(\ell_1 \pm 1,1)} + [1]^{(\ell_1 \pm 1,0)} + 2\,[1]^{(\ell_1,1)} + [1]^{(\ell_1,0)}$
		$[0]^{(\ell_1 \pm 2,0)} + [0]^{(\ell_1 \pm 1,1)} + [0]^{(\ell_1 \pm 1,0)} + [0]^{(\ell_1,2)} + [0]^{(\ell_1,1)} + 2\,[0]^{(\ell_1,0)}$
Q^5	$E_0 + \frac{5}{2}$	$[\frac{3}{2}]^{\left(\ell_1 \pm \frac{1}{2}, \frac{1}{2}\right)}$
		$[\frac{1}{2}]^{\left(\ell_1 \pm \frac{3}{2}, \frac{1}{2}\right)} + [\frac{1}{2}]^{\left(\ell_1 \pm \frac{1}{2}, \frac{3}{2}\right)} + 2\,[\frac{1}{2}]^{\left(\ell_1 \pm \frac{1}{2}, \frac{1}{2}\right)}$
Q^6	$E_0 + 3$	$[1]^{(\ell_1 \pm 1,0)} + [1]^{(\ell_1,1)} + [0]^{(\ell_1,0)}$
		$[0]^{(\ell_1 \pm 1,1)} + [0]^{(\ell_1,1)} + [0]^{(\ell_1,0)}$
Q^7	$E_0 + \frac{7}{2}$	$[\frac{1}{2}]^{\left(\ell_1 \pm \frac{1}{2}, \frac{1}{2}\right)}$
Q^8	$E_0 + 4$	$[0]^{(\ell_1,0)}$

Table E.10: States in the long graviton supermultiplet $\mathrm{LGRAV}_4[E_0, \ell_1, 0]$, with $\ell_1 \geq \frac{3}{2}$. For $\ell_1 = \frac{3}{2}$, the negative Dynkin label at the Q^4 level is absent.

	E_0	$[0]^{\left(\ell_1,\,\frac{1}{2}\right)}$
Q	$E_0+\frac{1}{2}$	$[\frac{1}{2}]^{\left(\ell_1\pm\frac{1}{2},1\right)}+[\frac{1}{2}]^{\left(\ell_1\pm\frac{1}{2},0\right)}$
Q^2	E_0+1	$[1]^{\left(\ell_1\pm1,\,\frac{1}{2}\right)}+[1]^{\left(\ell_1,\,\frac{3}{2}\right)}+2\,[1]^{\left(\ell_1,\,\frac{1}{2}\right)}$
		$[0]^{\left(\ell_1\pm1,\,\frac{3}{2}\right)}+[0]^{\left(\ell_1\pm1,\,\frac{1}{2}\right)}+[0]^{\left(\ell_1,\,\frac{3}{2}\right)}+2\,[0]^{\left(\ell_1,\,\frac{1}{2}\right)}$
Q^3	$E_0+\frac{3}{2}$	$[\frac{3}{2}]^{\left(\ell_1\pm\frac{1}{2},1\right)}+[\frac{3}{2}]^{\left(\ell_1\pm\frac{1}{2},0\right)}$
		$[\frac{1}{2}]^{\left(\ell_1\pm\frac{3}{2},1\right)}+[\frac{1}{2}]^{\left(\ell_1\pm\frac{3}{2},0\right)}+[\frac{1}{2}]^{\left(\ell_1\pm\frac{1}{2},2\right)}+3\,[\frac{1}{2}]^{\left(\ell_1\pm\frac{1}{2},1\right)}+[\frac{1}{2}]^{\left(\ell_1\pm\frac{1}{2},0\right)}$
Q^4	E_0+2	$[2]^{\left(\ell_1,\,\frac{1}{2}\right)}$
		$[1]^{\left(\ell_1\pm1,\,\frac{3}{2}\right)}+[1]^{\left(\ell_1\pm1,\,\frac{1}{2}\right)}+2\,[1]^{\left(\ell_1,\,\frac{3}{2}\right)}+3\,[1]^{\left(\ell_1,\,\frac{1}{2}\right)}$
		$[0]^{\left(\ell_1\pm2,\,\frac{1}{2}\right)}+[0]^{\left(\ell_1\pm1,\,\frac{3}{2}\right)}+2\,[0]^{\left(\ell_1\pm1,\,\frac{1}{2}\right)}+[0]^{\left(\ell_1,\,\frac{5}{2}\right)}+2\,[0]^{\left(\ell_1,\,\frac{3}{2}\right)}+3\,[0]^{\left(\ell_1,\,\frac{1}{2}\right)}$
Q^5	$E_0+\frac{5}{2}$	$[\frac{3}{2}]^{\left(\ell_1\pm\frac{1}{2},1\right)}+[\frac{3}{2}]^{\left(\ell_1\pm\frac{1}{2},0\right)}$
		$[\frac{1}{2}]^{\left(\ell_1\pm\frac{3}{2},1\right)}+[\frac{1}{2}]^{\left(\ell_1\pm\frac{3}{2},0\right)}+[\frac{1}{2}]^{\left(\ell_1\pm\frac{1}{2},2\right)}+3\,[\frac{1}{2}]^{\left(\ell_1\pm\frac{1}{2},1\right)}+[\frac{1}{2}]^{\left(\ell_1\pm\frac{1}{2},0\right)}$
Q^6	E_0+3	$[1]^{\left(\ell_1\pm1,\,\frac{1}{2}\right)}+[1]^{\left(\ell_1,\,\frac{3}{2}\right)}+2\,[1]^{\left(\ell_1,\,\frac{1}{2}\right)}$
		$[0]^{\left(\ell_1\pm1,\,\frac{3}{2}\right)}+[0]^{\left(\ell_1\pm1,\,\frac{1}{2}\right)}+[0]^{\left(\ell_1,\,\frac{3}{2}\right)}+2\,[0]^{\left(\ell_1,\,\frac{1}{2}\right)}$
Q^7	$E_0+\frac{7}{2}$	$[\frac{1}{2}]^{\left(\ell_1\pm\frac{1}{2},1\right)}+[\frac{1}{2}]^{\left(\ell_1\pm\frac{1}{2},0\right)}$
Q^8	E_0+4	$[0]^{\left(\ell_1,\,\frac{1}{2}\right)}$

Table E.11: States in the long graviton supermultiplet $\mathrm{LGRAV}_4[E_0,\ell_1,\frac{1}{2}]$, with $\ell_1\geq\frac{3}{2}$. For $\ell_1=\frac{3}{2}$, the negative Dynkin label at the Q^4 level is absent.

	E_0	$[0]^{(\ell_1,1)}$
Q	$E_0 + \frac{1}{2}$	$[\frac{1}{2}]^{\left(\ell_1 \pm \frac{1}{2}, \frac{3}{2}\right)} + [\frac{1}{2}]^{\left(\ell_1 \pm \frac{1}{2}, \frac{1}{2}\right)}$
Q^2	$E_0 + 1$	$[1]^{(\ell_1 \pm 1,1)} + [1]^{(\ell_1,2)} + 2\,[1]^{(\ell_1,1)} + [1]^{(\ell_1,0)}$
		$[0]^{(\ell_1 \pm 1,2)} + [0]^{(\ell_1 \pm 1,1)} + [0]^{(\ell_1 \pm 1,0)} + [0]^{(\ell_1,2)} + 2\,[0]^{(\ell_1,1)} + [0]^{(\ell_1,0)}$
Q^3	$E_0 + \frac{3}{2}$	$[\frac{3}{2}]^{\left(\ell_1 \pm \frac{1}{2}, \frac{3}{2}\right)} + [\frac{3}{2}]^{\left(\ell_1 \pm \frac{1}{2}, \frac{1}{2}\right)}$
		$[\frac{1}{2}]^{\left(\ell_1 \pm \frac{3}{2}, \frac{3}{2}\right)} + [\frac{1}{2}]^{\left(\ell_1 \pm \frac{3}{2}, \frac{1}{2}\right)} + [\frac{1}{2}]^{\left(\ell_1 \pm \frac{1}{2}, \frac{5}{2}\right)} + 3\,[\frac{1}{2}]^{\left(\ell_1 \pm \frac{1}{2}, \frac{3}{2}\right)} + 2\,[\frac{1}{2}]^{\left(\ell_1 \pm \frac{1}{2}, \frac{1}{2}\right)}$
Q^4	$E_0 + 2$	$[2]^{(\ell_1,1)}$
		$[1]^{(\ell_1 \pm 2,1)} + 2\,[1]^{(\ell_1 \pm 1,1)} + [1]^{(\ell_1 \pm 1,0)} + 2\,[1]^{(\ell_1,2)} + 3\,[1]^{(\ell_1,1)} + 2\,[1]^{(\ell_1,0)}$
		$[0]^{(\ell_1 \pm 2,1)} + [0]^{(\ell_1 \pm 1,2)} + 2\,[0]^{(\ell_1 \pm 1,1)} + [0]^{(\ell_1 \pm 1,0)} + [0]^{(\ell_1,3)} + 2\,[0]^{(\ell_1,2)}$
		$+\,4\,[0]^{(\ell_1,1)} + [0]^{(\ell_1,0)}$
Q^5	$E_0 + \frac{5}{2}$	$[\frac{3}{2}]^{\left(\ell_1 \pm \frac{1}{2}, \frac{3}{2}\right)} + [\frac{3}{2}]^{\left(\ell_1 \pm \frac{1}{2}, \frac{1}{2}\right)}$
		$[\frac{1}{2}]^{\left(\ell_1 \pm \frac{3}{2}, \frac{3}{2}\right)} + [\frac{1}{2}]^{\left(\ell_1 \pm \frac{3}{2}, \frac{1}{2}\right)} + [\frac{1}{2}]^{\left(\ell_1 \pm \frac{1}{2}, \frac{5}{2}\right)} + 3\,[\frac{1}{2}]^{\left(\ell_1 \pm \frac{1}{2}, \frac{3}{2}\right)} + 2\,[\frac{1}{2}]^{\left(\ell_1 \pm \frac{1}{2}, \frac{1}{2}\right)}$
Q^6	$E_0 + 3$	$[1]^{(\ell_1 \pm 1,1)} + [1]^{(\ell_1,2)} + 2\,[1]^{(\ell_1,1)} + [1]^{(\ell_1,0)}$
		$[0]^{(\ell_1 \pm 1,2)} + [0]^{(\ell_1 \pm 1,1)} + [0]^{(\ell_1 \pm 1,0)} + [0]^{(\ell_1,2)} + 2\,[0]^{(\ell_1,1)} + [0]^{(\ell_1,0)}$
Q^7	$E_0 + \frac{7}{2}$	$[\frac{1}{2}]^{\left(\ell_1 \pm \frac{1}{2}, \frac{3}{2}\right)} + [\frac{1}{2}]^{\left(\ell_1 \pm \frac{1}{2}, \frac{1}{2}\right)}$
Q^8	$E_0 + 4$	$[0]^{(\ell_1,1)}$

Table E.12: States in the long graviton supermultiplet $\mathrm{LGRAV}_4[E_0, \ell_1, 1]$, with $\ell_1 \geq \frac{3}{2}$. For $\ell_1 = \frac{3}{2}$, the negative Dynkin label at the Q^4 level is absent.

	E_0	$[0]^{(\ell_1,\,\ell_2)}$
Q	$E_0+\frac{1}{2}$	$[\frac{1}{2}]^{\left(\ell_1\pm\frac{1}{2},\,\ell_2\pm\frac{1}{2}\right)}$
Q^2	E_0+1	$[1]^{(\ell_1\pm1,\,\ell_2)}+[1]^{(\ell_1,\,\ell_2\pm1)}+2\,[1]^{(\ell_1,\,\ell_2)}$
		$[0]^{(\ell_1\pm1,\,\ell_2\pm1)}+[0]^{(\ell_1\pm1,\,\ell_2)}+[0]^{(\ell_1,\,\ell_2\pm1)}+2\,[0]^{(\ell_1,\,\ell_2)}$
Q^3	$E_0+\frac{3}{2}$	$[\frac{3}{2}]^{\left(\ell_1\pm\frac{1}{2},\,\ell_2\pm\frac{1}{2}\right)}$
		$[\frac{1}{2}]^{\left(\ell_1\pm\frac{3}{2},\,\ell_2\pm\frac{1}{2}\right)}+[\frac{1}{2}]^{\left(\ell_1\pm\frac{1}{2},\,\ell_2\pm\frac{3}{2}\right)}+3\,[\frac{1}{2}]^{\left(\ell_1\pm\frac{1}{2},\,\ell_2\pm\frac{1}{2}\right)}$
Q^4	E_0+2	$[2]^{(\ell_1,\,\ell_2)}$
		$[1]^{(\ell_1\pm1,\,\ell_2\pm1)}+2\,[1]^{(\ell_1\pm1,\,\ell_2)}+2\,[1]^{(\ell_1,\,\ell_2\pm1)}+3\,[1]^{(\ell_1,\,\ell_2)}$
		$[0]^{(\ell_1\pm2,\,\ell_2)}+[0]^{(\ell_1,\,\ell_2\pm2)}+[0]^{(\ell_1\pm1,\,\ell_2\pm1)}+2\,[0]^{(\ell_1\pm1,\,\ell_2)}+2\,[0]^{(\ell_1,\,\ell_2\pm1)}+4\,[0]^{(\ell_1,\,\ell_2)}$
Q^5	$E_0+\frac{5}{2}$	$[\frac{3}{2}]^{\left(\ell_1\pm\frac{1}{2},\,\ell_2\pm\frac{1}{2}\right)}$
		$[\frac{1}{2}]^{\left(\ell_1\pm\frac{3}{2},\,\ell_2\pm\frac{1}{2}\right)}+[\frac{1}{2}]^{\left(\ell_1\pm\frac{1}{2},\,\ell_2\pm\frac{3}{2}\right)}+3\,[\frac{1}{2}]^{\left(\ell_1\pm\frac{1}{2},\,\ell_2\pm\frac{1}{2}\right)}$
Q^6	E_0+3	$[1]^{(\ell_1\pm1,\,\ell_2)}+[1]^{(\ell_1,\,\ell_2\pm1)}+2\,[1]^{(\ell_1,\,\ell_2)}$
		$[0]^{(\ell_1\pm1,\,\ell_2\pm1)}+[0]^{(\ell_1\pm1,\,\ell_2)}+[0]^{(\ell_1,\,\ell_2\pm1)}+2\,[0]^{(\ell_1,\,\ell_2)}$
Q^7	$E_0+\frac{7}{2}$	$[\frac{1}{2}]^{\left(\ell_1\pm\frac{1}{2},\,\ell_2\pm\frac{1}{2}\right)}$
Q^8	E_0+4	$[0]^{(\ell_1,\,\ell_2)}$

Table E.13: States in the long graviton supermultiplet $\mathrm{LGRAV}_4[E_0,\ell_1,\,\ell_2]$, with $\ell_1,\,\ell_2\geq\frac{3}{2}$. For $\ell_i=\frac{3}{2}$, negative Dynkin labels at the Q^4 level are absent.

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