A decomposition for plane domains with the quasihyperbolic metric

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It is known that complete Riemannian surfaces can be obtained by pasting three kinds of pieces. In this paper we prove an analogous result in the context of plane domains with their quasihyperbolic metrics. In order to do it, we prove several facts about quasihyperbolic closed geodesics of independent interest; for instance, we characterize the existence of quasihyperbolic minimizers, and we show that images of local quasihyperbolic geodesics are finite graphs.

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1. Introduction

A domain is an open connected subset of Euclidean space. Given a domain $\Omega \subsetneq \mathbb{R}^n$, for each continuous rectifiable curve $\gamma \subset \Omega$ its quasihyperbolic length is the length induced by the density $1/\delta_\Omega(x)$, with $\delta_\Omega(x) = d_{\mathbb{R}^n}(x, \partial\Omega) = d_{\mathbb{R}^n}(x, \Omega^c)$; i.e.,

$$L_\Omega(\gamma) = \int\limits_\gamma \frac{ds}{\delta_\Omega(x)},$$

where ds is the differential of Euclidean arclength. The quasihyperbolic distance in Ω, denoted by k_Ω, is the distance induced by L_Ω, i.e.,

$$k_\Omega(x_1, x_2) = \inf \big\{ \, L_\Omega(\gamma) : \gamma \text{ is a curve joining } x_1 \text{ and } x_2 \text{ in } \Omega \, \big\}.$$

In this paper we identify the complex plane $\mathbb{C}$ with Euclidean plane; thus we have a quasihyperbolic metric defined on every domain $\Omega \subsetneq \mathbb{C}$ as a particular case of the above construction.

The quasihyperbolic metric of a domain in $\mathbb{R}^n$ was introduced by Gehring and Palka [11] in 1976. It has turned out to be a useful tool, for example, in harmonic analysis and many subfields of geometric function theory; for instance: in the study of quasiconformal maps either between Euclidean domains [27], [44] or between domains in Banach spaces [40], analysis of metric spaces [19], hyperbolic type metrics [15], [22], [23], [26], Sobolev spaces [29], [30], and John domains [9]. There is also quite a strong relationship between uniform domains and the quasihyperbolic metric [10], [11], [31].

The quasihyperbolic metric has important invariance properties. Since it depends on the shape of the boundary $\partial\Omega$, it is usually less symmetric than the Poincaré metric [14]. It is nevertheless quasi-invariant under Möbius transformations [11]. Gehring and Osgood [10] established a rough, but very useful, notion of quasi-invariance for quasiconformal distance under quasiconformal maps. See also [27], [44] and the references therein.

The quasihyperbolic distance between two given points may be hard to compute. A useful tool for estimating its value is the j-metric, which is an easier to compute distance function but known to never be a geodesic distance [21, Theorem 2.10]. Gehring and Palka obtained in [11] the comparison result $k \geq j$ between quasihyperbolic distance and j distance. Recent comparison results are proved in [23] and [26].

Definition 1.1. A *quasihyperbolic geodesic* for a domain Ω is a nonconstant path $\alpha : [a, b] \to \Omega$ such that

$$L_\Omega(\alpha) \; = \; k_\Omega\big(\alpha(a), \, \alpha(b) \big). \tag{1.1}$$

If $\beta : [a', b'] \to \Omega$ is a quasihyperbolic geodesic and $\phi : [a, b] \to \mathbb{R}$ is a continuous, monotone function with $\phi([a, b]) = [a', b']$, then the composite path $\alpha(t) \equiv \beta\big(\phi(t)\big)$ is also a quasihyperbolic geodesic and it has the same image as β. We say that the path α is a *reparametrization* of β.

While condition (1.1) is a global one, in the Riemannian setting geodesics are defined by a local property:

Definition 1.2. A path $\gamma : I \to \mathcal{M}$ on a Riemannian manifold $\mathcal{M}$ is a *geodesic path* if there is some fixed constant $c > 0$ and for each $t_0 \in I$ there is an $\varepsilon(t_0) > 0$ such that the Riemannian distance $d_\mathcal{M}\big(\gamma(t_1), \gamma(t_2)\big)$ equals $c|t_1 - t_2|$ for all $t_1, t_2 \in [t_0 - \varepsilon(t_0), \, t_0 + \varepsilon(t_0)] \cap I$. If $c = 1$ then γ is a *unitary geodesic path*.

A *geodesic image* in a Riemannian manifold $\mathcal{M}$ is the set $\gamma(I)$ for some geodesic path $\alpha : I \to \mathcal{M}$.

The word *geodesic* is used for both geodesic paths and geodesic images in $\mathcal{M}$ when there is no danger of confusion.

Definition 1.3. In a Riemannian manifold $\mathcal{M}$, a *closed geodesic* is a geodesic path $\gamma : [a, b] \to \mathcal{M}$ such that $\gamma(a) = \gamma(b)$ and the $(b - a)$-periodic extension $\mathbb{R} \to \mathcal{M}$ of γ is also a geodesic path; which forces $\gamma'(a) = \gamma'(b)$. A *simple closed geodesic* is a closed geodesic $\gamma : [a, b] \to \mathcal{M}$ that is also a simple closed path.

It is well known [10, Lemma 1] that a quasihyperbolic geodesic between given points always exists in Euclidean domains. They also exist in domains of many Banach spaces, and, in the case the domain is convex, the geodesic joining two points is unique [39]. For any planar domain $\Omega \subsetneq \mathbb{R}^2$ Väisälä has the following result:

Theorem A. *([42, Conjecture 1.2 and Theorem 7.7 (2)]) Any two of points in Ω whose quasihyperbolic distance is less than 2 are the endpoints of a unique (up to reparametrization) quasihyperbolic geodesic in Ω.*

G. Martin [31, Corollary 4.8] proved in 1985 that every quasihyperbolic geodesic $\alpha(t)$ in a Euclidean domain is $C^{1,1}$ *smooth,* which exactly means that α is a reparametrization of some path $\beta(u)$ whose velocity $\beta'(u)$ is Lipschitz and nowhere zero; so the path image is an honest $C^{1,1}$ submanifold of Euclidean space. For domains in many Banach spaces, quasihyperbolic geodesics are C^1 smooth [43].

It is shown in [25] that quasihyperbolic balls are C^1 smooth for convex domains in many Banach spaces (including $\mathbb{R}^n$). For arbitrary domains in $\mathbb{R}^n$, it is proved in [41, Theorem 5.10] that the only points where the boundary of a quasihyperbolic ball may not be C^1 smooth are inward pointing cusps.

The study of the convexity properties (in the usual sense of linear algebra) of quasihyperbolic balls has attracted great interest. For instance [20] and [22] study such properties when the domain is punctured space, and [38] deals with convexity properties for quasihyperbolic balls of domains in Banach spaces. In [23] it is proved that quasihyperbolic balls with small radius in Euclidean domains are close to being convex. In the case of arbitrary domains in the Euclidean plane, there is a very strong result of Väisälä:

Theorem B. *([42, Theorem 7.5]) For any domain $\Omega \subsetneq \mathbb{R}^2$, quasihyperbolic balls of radius smaller than 1 are strictly convex.*

In [20] and [42, Theorem 3.6] we find an explicit example of a planar domain whose quasihyperbolic balls of radius greater than 1 are not convex. This makes the radius value 1 optimal for Theorem B.

The proofs of these results required new ideas, since the density $1/\delta_\Omega(x)$ need not be differentiable. An example where this happens is the twice-punctured plane $\Omega = \mathbb{C} \setminus \{0, 2\}$, because $\delta_\Omega(z) = \min(|z|, |z-2|)$ and $1/\delta_\Omega(z) = 1/\min(|z|, |z-2|)$ are not differentiable at any point on the line $\{\mathrm{Re}(z) = 1\}$. The unit disc $\Omega = \{z \in \mathbb{C} : |z| < 1\}$ is a domain with smooth boundary whose quasihyperbolic density $1/\delta_\Omega(z) = 1/(1-|z|)$ is not differentiable at $z = 0$. On the other hand, a once-punctured plane and a halfplane are examples of domains with smooth quasihyperbolic density.

There are good reasons to think that the quasihyperbolic metric in Euclidean domains has negative curvature in a meaningful sense [5], [31], [32]. Also, some papers study the relation of the quasihyperbolic metric with another interesting geometric concept: Gromov hyperbolicity. The main theorem from [3], that extends results from [4], gives a complete characterization of Gromov hyperbolicity of Euclidean domains with the quasihyperbolic metric. In [28], there is a characterization of Gromov hyperbolicity of domains with the quasihyperbolic metric by geometric properties of the Ahlfors regular length metric measure space. See also [16], [17] and [18] for some sufficient Euclidean conditions in order to guarantee the hyperbolicity of some domains with the quasihyperbolic metric.

It is claimed in [6] that an arbitrary plane domain is Gromov hyperbolic in the quasihyperbolic metric if and only if it has the same property in the Poincaré metric.

We also refer to the survey [24] for more properties of the quasihyperbolic metric.

The model space for a *pair of pants* is the result of removing from a sphere the interiors of three pairwise disjoint closed discs (of positive radius). In general, a pair of pants is any compact topological surface with boundary that is homeomorphic to this example.

The celebrated Classification Theorem for compact surfaces says that every connected, orientable, compact topological surface is homeomorphic either to a sphere or to a sphere with handles attached, see e.g. [33]. In particular, if such a surface is neither homeomorphic to a sphere nor to a torus then it contains a finite collection of pairwise disjoint simple closed curves such that by cutting the surface along them we are left with a finite collection of pairs of pants.

If a Riemann metric is specified on a pair of pants and the boundary curves are geodesic images, we call this surface a *Y-piece.* If moreover the metric has constant curvature -1 we speak of a *hyperbolic Y-piece.* In the Riemannian setting one has the following:

Every compact orientable Riemannian surface with constant curvature $K = -1$ can be split into hyperbolic Y-pieces by cutting along a finite collection of pairwise disjoint simple closed geodesics.

A *hyperbolic halfplane* is a Riemannian surface that is isometric with any of the two connected components of $\mathbb{H} \setminus \gamma$, where γ is any complete geodesic in the hyperbolic plane $\mathbb{H}$.

Let S be a surface carrying a complete Riemann metric with constant curvature -1. Then S is called a *hyperbolic funnel* if it is conformally diffeomorphic to a half-closed annulus $\{z \in \mathbb{C} : a \leq |z| < b\}$, for some constants $0 < a < b$, and the circle $\{|z| = a\}$ corresponds under this diffeomorphism to a geodesic image in S. Likewise S is called a *generalized hyperbolic Y-piece* if two conditions are met:

(1) S is diffeomorphic to the result of taking three pairwise disjoint closed discs in a sphere (including single points as closed discs of zero radius), and then removing the discs of zero radius and the interiors of the nontrivial discs.

(2) The diffeomorphism maps the boundaries of the nontrivial discs (if any) to geodesic images in S.

In [2] the following result was obtained

Theorem C. *([2, Theorem 1.2]) A complete orientable Riemannian surface S, with $K \equiv -1$, which is not isometric with the punctured disc, contains a countable collection of geodesics that split it into generalized hyperbolic Y-pieces, hyperbolic funnels, and hyperbolic halfplanes.*

The geodesics in this theorem are either proper embeddings $\mathbb{R} \hookrightarrow S$ or simple closed geodesics, and they have pairwise disjoint open neighborhoods. The generalized Y-pieces and hyperbolic funnels are closed subsets of S, while the halfplanes are open subsets.

Theorem C is generalized in [37] to surfaces with arbitrary curvature.

The present paper deals with the quasihyperbolic metric on plane domains $\Omega \subsetneq \mathbb{C}$. Since this metric is only Lipschitz in general, the behavior of closed geodesics, their existence, and their uniqueness, are not trivial to study.

The main result in this paper, stated with a bit more detail as Theorem 7.3 in Section 7, is a decomposition for plane domains with their quasihyperbolic metric. It is quite analogous to Theorem C:

Main Theorem. *Given a domain $\Omega \subsetneq \mathbb{C}$, endowed with its quasihyperbolic metric, which is neither simply nor doubly connected, there exists a set $H \subseteq \Omega$, union of countably many closed domains each one of which is a Y-piece, a funnel, a puncture or an exterior Y-piece, in such a way that Ω is the disjoint union of the closure $\overline{H}$ and simply connected open sets.*

This decomposition is made possible by other crucial results. In Section 4 we determine when quasihyperbolic length can be minimized within the free homotopy class of a Jordan curve, plus we describe some useful minimizers which we call *quasihyperbolic limit geodesics.* In Section 3 we show that quasihyperbolic geodesics of finite quasihyperbolic length have a finite number of intersections. This is most valuable, as quasihyperbolic geodesics can intersect in the following ways: two distinct quasihyperbolic geodesics may be tangent at a point; further, they can have segments of positive length that coincide for a while and then split apart. These intersection phenomena are well known, and Väisälä has exhibited them explicitly in [42, pages 10-12]. See Section 3 for details.

Just as half-planes are needed in Theorem C, the decomposition in Theorem 7.3 sometimes requires pieces which are simply connected open sets. This is made obvious by Examples 8.1 and 8.2.

2. Background

Let $\Omega \subsetneq \mathbb{C}$ be a domain endowed with its quasihyperbolic metric. The *quasihyperbolic distance from a closed set* $A \subset \Omega$ is the function $k_\Omega(\cdot, A) : \Omega \longrightarrow \mathbb{R}$ given by

$$k_\Omega(z, A) := \inf\{ k_\Omega(z, p) : p \in A \}.$$

For $r > 0$ we define the *quasihyperbolic neighborhoods* of A as follows

$$B_\Omega(A, r) = \{ z \in \Omega : k_\Omega(z, A) < r \},$$
$$\overline{B}_\Omega(A, r) = \{ z \in \Omega : k_\Omega(z, A) \le r \} = \{ z \in \Omega : k_\Omega(z, p) \le r \text{ for all } p \in A \}.$$

The set $B_\Omega(A, r)$ is always open, while $\overline{B}_\Omega(A, r)$ is always closed (and compact when A is compact).

For closed $A, A' \subset \Omega$, the *quasihyperbolic Hausdorff distance* between them is

$$k_{\Omega, H}(A, A') = \inf \left\{ r > 0 : A' \subseteq \overline{B}_\Omega(A, r) \text{ and } A \subseteq \overline{B}_\Omega(A', r) \right\}.$$

The Euclidean analogues $d_\mathbb{C}(\cdot, A)$, $B_\mathbb{C}(A, r)$, $\overline{B}_\mathbb{C}(A, r)$, and $d_{\mathbb{C}, H}(A, B)$ are defined in the obvious way. We next introduce some notions that deal with *ends of surfaces*.

Definition 2.1. Let S be a connected, non-compact topological surface.

An *end* of S is a function E that assigns to each compact subset $K \subset S$ a connected component $E(K)$ of $S \setminus K$, in such a way that whenever $K_1 \subseteq K_2$ we have $E(K_1) \supseteq E(K_2)$.

The intuitive idea behind this definition is that ends are 'points at infinity' of the surface.

Definition 2.2. Let E be an end of a topological surface S.

A *neighborhood* of E is any open set $U \subseteq S$ that contains $E(K)$ for some compact $K \subset S$.

An end E of S is *collared* if some neighborhood of E is homeomorphic to $(0, \infty) \times \mathbb{S}^1$.

Consider the case when the surface is a domain $\Omega \subsetneq \mathbb{C}$. Every isolated point of $\Omega^c = \mathbb{C} \setminus \Omega$ defines a collared end of Ω. If Ω^c is compact, then the point ∞ of $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ defines another collared end of Ω. These are all possible collared ends of a planar domain.

Some people use the term *limit end* to refer to a non-collared end.

We next describe three ways to approach an end of a surface S:

(1) A sequence of points $\{x_n\} \subset S$ *converges to* E if for each neighborhood U of E there is an integer n_U such that the points $x_{n_U}, x_{n_U+1}, x_{n_U+2}, \dots$ all belong to U.
(2) A sequence of non-empty sets $\{X_n\}$ *converges to* E if for each neighborhood U of E there is an integer n_U such that $X_{n_U}, X_{n_U+1}, X_{n_U+2}, \dots$ are all contained in U.
(3) A divergent path $\sigma : [0, \infty) \to S$ *converges* to a unique end E_σ of S; namely, the end that assigns to each compact subset $K \subset S$ the connected component $E_\sigma(K)$ of $S \setminus K$ that contains a terminal segment $\sigma([a, \infty))$, $a \ge 0$.

In the proof of Theorem 7.3 we use the following result, a consequence of the proof of [7, Theorem 4.2].

Theorem D. *Let $\Omega \subsetneq \mathbb{C}$ be a domain, endowed with its quasihyperbolic metric, and let E be an end of Ω. Then E is a collared end if and only if there exists a sequence $\{\alpha_n\}$ of simple closed curves converging to E and representing a single non-trivial free homotopy class in Ω.*

We shall be using two path homotopy notions.

Definition 2.3. Let $\mathcal{M}$ be a connected manifold and consider two paths $\alpha, \beta : [a, b] \to \mathcal{M}$ defined on the same interval.

We say that α and β are *homotopic rel endpoints* in $\mathcal{M}$ if they have the same *initial point* $p = \alpha(a) = \beta(a)$, the same *final point* $q = \alpha(b) = \beta(b)$, and there is a continuous map $F : [a, b] \times [0, 1] \to \mathcal{M}$ such that for each $\lambda \in [0, 1]$ the path $F_\lambda : [a, b] \to \mathcal{M}$ given by $F_\lambda(t) = F(t, \lambda)$ is a path with initial point p, final point q, and, in particular, $F_0 \equiv \alpha$ and $F_1 \equiv \beta$. The map F, if it exists, is a *homotopy rel endpoints* between α and β.

Suppose now that α and β are closed paths, i.e. $\alpha(a) = \alpha(b)$ and $\beta(a) = \beta(b)$. We say that α and β are *freely homotopic* in $\mathcal{M}$ if there is a continuous map $G : [a, b] \times [0, 1] \to \mathcal{M}$ such that for each $\lambda \in [0, 1]$ the path $G_\lambda : [a, b] \to \mathcal{M}$ given by $G_\lambda(t) = G(t, \lambda)$ is a closed path, and, in particular, $G_0 \equiv \alpha$ and $G_1 \equiv \beta$. The map G, if it exists, is a *free homotopy* between α and β.

A homotopy rel endpoints continuously deforms α into β while keeping each endpoint fixed during the deformation. A free homotopy continuously deforms the loop α into the loop β, with all intermediate paths being also loops.

Both properties, being homotopic rel endpoints or being freely homotopic, are equivalence relations and give rise to equivalence classes in $\mathcal{M}$: homotopy classes rel endpoints in $\mathcal{M}$ and free homotopy classes in $\mathcal{M}$, respectively. Given a domain $\Omega \subsetneq \mathbb{C}$, the constant paths $[a, b] \to \Omega$ all belong to the same free homotopy class, which is called *trivial free homotopy class in* Ω. A closed path $\gamma \subset \Omega$ is *null-homotopic,* or *contractible* in Ω, if it belongs to the trivial free homotopy class in Ω.

Fix a point $* \in \mathcal{M}$ and consider the (closed) paths in $\mathcal{M}$ with $*$ as initial and final point. Their homotopy classes rel this endpoint are the elements of a group $\pi_1(\mathcal{M}, *)$, called *fundamental group of* $\mathcal{M}$ *with basepoint* $*$. The multiplication is defined by $[\alpha][\beta] = [\alpha\beta]$, where $\alpha\beta$ is a path that goes first along α and then along β. In this paper we shall not need to keep track of the basepoint and, accordingly, we shall write $\pi_1(\mathcal{M})$ for this group. A domain $\Omega \subseteq \mathbb{C}$ is *simply connected* if $\pi_1(\Omega) = \{0\}$, and it is called *doubly connected* if $\pi_1(\Omega) = \mathbb{Z}$.

Every continuous map $f : \mathcal{M}_1 \to \mathcal{M}_2$ induces a group homomorphism $f_\# : \pi_1(\mathcal{M}_1) \to \pi_1(\mathcal{M}_2)$ defined by $[\alpha] \longmapsto [f \circ \alpha]$. If we have continuous maps $\mathcal{M}_1 \xrightarrow{f} \mathcal{M}_2 \xrightarrow{g} \mathcal{M}_3$, then $(g \circ f)_\# = g_\# \circ f_\#$. An immediate consequence is that if f is a homeomorphism then $f_\#$ is a group isomorphism.

Definition 2.4. Given a connected submanifold (possibly with boundary) $\mathcal{N} \subset \mathcal{M}$ and the inclusion map $i : \mathcal{N} \hookrightarrow \mathcal{M}$, the subgroup $i_\#\big(\pi_1(\mathcal{N})\big)$ of $\pi_1(\mathcal{M})$ will be called the *subgroup induced by* $\mathcal{N}$.

A free loop, as opposed to a based loop used in the definition of the fundamental group, is a map from the circle to the space without the basepoint-preserving restriction. Free homotopy classes of free loops correspond to conjugacy classes in the fundamental group.

We now present some facts about rotation indexes.

Definition 2.5. Consider a point $p \in \mathbb{C}$ and an oriented, closed path $\alpha \subset \mathbb{C} \setminus \{p\}$. The *rotation index,* or *winding number,* of α *around* p is the number

$$i(\alpha, p) \;=\; \frac{1}{2\pi i} \int\limits_{\alpha} \frac{dz}{z - p} \; .$$

Please note that this definition applies to any closed path, not only to simple closed paths.

Lemma 2.6. *The rotation index is always an integer. It remains constant as we deform α within its free homotopy class in $\mathbb{C} \setminus \{p\}$. It also remains constant if we fix α and we move the point p inside one connected component of the open set $\mathbb{C} \setminus \alpha$.*

Proof. In Chapter 9 of [35] we find the definitions of two concepts strongly related to fundamental groups: *covering maps* and *liftings*. Given a covering map $\pi : \mathcal{M} \to \mathcal{M}_0$ and a map $f : \mathcal{N} \to \mathcal{M}_0$, a *lifting of f under π* is a map $\widetilde{f} : \mathcal{N} \to \mathcal{M}$ such that $f \equiv \pi \circ \widetilde{f}$.

The punctured plane $\mathbb{C} \setminus \{p\}$ admits the following covering map:

$$\pi : \mathbb{R}^2 \longrightarrow \mathbb{C} \setminus \{p\} \quad , \quad \pi(l, \theta) = p + e^{l + i\theta} .$$

Each path $\alpha : [a, b] \to \mathbb{C} \setminus \{p\}$ has a continuous lifting $(l, \theta) : [a, b] \to \mathbb{R}^2$ such that $\alpha(t) \equiv p + e^{l(t) + i\theta(t)}$. If moreover $\alpha(t)$ is a closed path, then

$$p + e^{l(a) + i\theta(a)} = \alpha(a) = \alpha(b) = p + e^{l(b) + i\theta(b)} ,$$

thus $l(b) - l(a) = 0$ and the angle change $\theta(a) - \theta(b)$ must be equal to $2\pi k$ for some integer k. It is easily seen that once α is fixed the integer k is the same for all liftings of α.

Let $G : [a, b] \times [0, 1] \to \mathbb{C} \setminus \{p\}$ be a free homotopy between two closed paths $\alpha, \beta : [a, b] \to \mathbb{C} \setminus \{p\}$. As is explained in [35, Lemma 54.2], the map G has a continuous lifting

$$(l, \theta) : [a, b] \times [0, 1] \longrightarrow \mathbb{R}^2 ,$$

such that $G(t, \lambda) \equiv p + e^{l(t, \lambda) + i\theta(t, \lambda)}$. For each $\lambda \in [0, 1]$ we must have $l(b, \lambda) - l(a, \lambda) = 0$, and the angle change $\theta(a, \lambda) - \theta(b, \lambda)$ must be an integer multiple of 2π. We thus have a continuous function

$$[0, 1] \longrightarrow \mathbb{Z} \quad , \quad \lambda \longmapsto \frac{\theta(a, \lambda) - \theta(b, \lambda)}{2\pi} ,$$

from the interval $[0, 1]$ to the discrete space $\mathbb{Z}$. Such a function is necessarily constant and, in particular, has the same value at $\lambda = 0$ and at $\lambda = 1$. This proves that freely homotopic closed paths in $\mathbb{C} \setminus \{p\}$ give rise to the same integer. In other words, every free homotopy class $[\alpha]$ in $\mathbb{C} \setminus \{p\}$ determines a unique integer.

A lifting $(l(t), \theta(t))$ of the closed path $\alpha(t)$ can be used to compute the integral:

$$\int_\alpha \frac{dz}{z - p} = \int_a^b \frac{(l'(t) + i\theta'(t)) e^{l(t) + i\theta(t)}}{e^{l(t) + i\theta(t)}} dt = \int_a^b (l'(t) + i\theta'(t)) dt = 0 + i\theta(b) - i\theta(a) = 2\pi i k,$$

where $k \in \mathbb{Z}$ is the integer determined by the free homotopy class $[\alpha]$ and at the same time equals $i(\alpha, p)$.

The map $\mathbb{C} \setminus \alpha \to \mathbb{Z}$ given by $p \longmapsto i(\alpha, p)$ is continuous, thanks to its integral formula. Therefore it is constant on each connected component of $\mathbb{C} \setminus \alpha$, since $\mathbb{Z}$ is a discrete space. $\quad\square$

In fact, it is shown in [35, Chapter 9] that the map $\pi_1(\mathbb{C} \setminus \{p\}) \to \mathbb{Z}$ that takes each class $[\alpha] \in \pi_1(\mathbb{C} \setminus \{p\})$ to $i(\alpha, p)$ is a group isomorphism. Thus $\pi_1(\mathbb{C} \setminus \{p\})$ is an infinite cyclic group, and a closed path α has rotation index ± 1 around p if and only if its class $[\alpha]$ is a generator for $\pi_1(\mathbb{C} \setminus \{p\})$.

Definition 2.7. Let $\alpha \subset \mathbb{C}$ be a Jordan curve. The *interior* of α is the bounded connected component $\mathrm{Int}(\alpha)$ of $\mathbb{C} \setminus \alpha$. The *exterior of α* is the other connected component $\mathrm{Ext}(\alpha)$ of $\overline{\mathbb{C}} \setminus \alpha$, so that $\infty \in \mathrm{Ext}(\alpha)$.

The meanings of the expressions 'lies interior to α', 'is surrounded by α', and 'lies exterior to α' are the obvious ones.

By a theorem due to Schoenflies [34, Pages 68 and 72], given a Jordan curve $\alpha : [a,b] \to \mathbb{C}$ there is a homeomorphism $\Phi : \mathbb{C} \to \mathbb{C}$ such that the path $\alpha_0 = \Phi \circ \alpha$ goes once around the unit circle. The set $\alpha([a,b]) \cup \text{Int}(\alpha)$ is compact and so is its image under Φ; hence Φ maps $\text{Int}(\alpha)$ to the open disc bounded by α_0. A first consequence is that the interior of any Jordan curve is a simply connected open set.

Lemma 2.8. *Let $\alpha \subset \mathbb{C}$ be a Jordan curve. A point p interior to α gives rotation index $i(\alpha, p) = \pm 1$. An exterior point q gives $i(\alpha, q) = 0$.*

Proof. Let $\Phi : \mathbb{C} \to \mathbb{C}$ be a homeomorphism that maps α to the circular path α_0 and the point p to a point p_0 interior to α_0.

The class $[\alpha_0]$ of α_0 in $\mathbb{C} \setminus \{p_0\}$ clearly is a generator for $\pi_1(\mathbb{C} \setminus \{p_0\})$. The isomorphism $\Phi_\# : \pi_1(\mathbb{C} \setminus \{p\}) \to \pi_1(\mathbb{C} \setminus \{p_0\})$ maps the class $[\alpha]$ to the class $[\alpha_0]$. Therefore the class $[\alpha]$ is a generator for $\pi_1(\mathbb{C} \setminus \{p\})$. As we have explained above, this implies that $i(\alpha, p) = \pm 1$.

Let now U be the unbounded connected component of $\mathbb{C} \setminus \alpha$ and q a point in U. Since connected open sets in the plane are path connected, there is a divergent path $\beta : [0, +\infty) \to U$ starting at $\beta(0) = q$. By Lemma 2.6, the number $i(\alpha, \beta(t))$ remains constant as t goes from 0 to $+\infty$. On the other hand, the integrand in the formula

$$i(\alpha, \beta(t)) \;=\; \frac{1}{2\pi i} \int\limits_\alpha \frac{dz}{z - \beta(t)} \, ,$$

goes uniformly to zero as $t \to +\infty$ because $\beta(t)$ is divergent, therefore $\lim_{t \to +\infty} i(\alpha, \beta(t)) = 0$. We conclude that $i(\alpha, \beta(t))$ is zero for all $t \in [0, +\infty)$; in particular $0 = i(\alpha, \beta(0)) = i(\alpha, q)$. $\square$

Lemma 2.9. *Let $\alpha \subset \Omega$ be a Jordan curve. The free homotopy class of α in Ω is nontrivial if and only if $\text{Int}(\alpha)$ contains at least one point of Ω^c.*

Proof. The set $\alpha \cup \text{Int}(\alpha)$ is homeomorphic to a closed disc. If this set is contained in Ω, it makes α contractible in Ω. Therefore if $[\alpha]$ is nontrivial then at least one point of $\text{Int}(\alpha)$ is not in Ω.

Conversely, if there is a point $p \in \text{Int}(\alpha) \cap \Omega^c$ then $i(\alpha, p) = \pm 1$. Therefore α is not freely homotopic in $\mathbb{C} \setminus \{p\}$ to a constant loop. But $\Omega \subseteq \mathbb{C} \setminus \{p\}$, hence α is not freely homotopic in Ω to a constant loop. $\square$

We finish this Section with the proofs of three facts about uniform path convergence.

Lemma 2.10. *Let $\Omega \subsetneq \mathbb{C}$ be a domain, endowed with its quasi-hyperbolic metric. If $\sigma, \eta : [a,b] \to \Omega$ are closed curves and $d_\Omega(\sigma(t), \eta(t)) < 1$ for every $t \in [a,b]$, then $[\sigma] = [\eta]$.*

Proof. By Theorem B, the quasihyperbolic ball $B_\Omega(\sigma(t), 1)$ is (Euclidean) convex and so, the Euclidean segment joining $\sigma(t)$ and $\eta(t)$ is contained in $B_\Omega(\sigma(t), 1) \subset \Omega$. Hence $\Phi : [a,b] \times [0,1] \to \mathbb{C}$, given by $\Phi(t, s) = (1-s)\,\sigma(t) + s\,\eta(t)$, is a map with values in Ω and it is a free homotopy between σ and η. $\square$

Lemma 2.11. *Let $\Omega \subsetneq \mathbb{C}$ be a domain, endowed with its quasi-hyperbolic metric.*

A point sequence $\{p_n\} \subset \Omega$ converges to a point $p \in \Omega$ with respect to Euclidean distance if and only if it converges to p with respect to quasihyperbolic distance.

A path sequence $\sigma_n : [a,b] \to \Omega$ converges uniformly to a path $\sigma : [a,b] \to \Omega$ with respect to Euclidean distance if and only if it converges uniformly to σ with respect to the quasihyperbolic distance.

Proof. If $z \in B_\mathbb{C}(p, \delta_\Omega(p)/2) \subset \Omega$, then $\frac{1}{2}\delta_\Omega(p) \leq \delta_\Omega(z) \leq \frac{3}{2}\delta_\Omega(p)$.

Given $z_1, z_2 \in B_{\mathbb{C}}(p, \delta_\Omega(p)/2)$ the Euclidean segment γ joining them is contained in $B_{\mathbb{C}}(p, \delta_\Omega(p)/2)$ and we have

$$k_\Omega(z_1, z_2) \leq \int_\gamma \frac{|dz|}{\delta_\Omega(z)} < \frac{2}{\delta_\Omega(p)} \int_\gamma |dz| = \frac{2}{\delta_\Omega(p)} |z_1 - z_2|.$$

A result of G. Martin [31, Theorem 2.2] states that if η is a quasi-hyperbolic geodesic joining z_1, z_2, then $\eta \subset B_{\mathbb{C}}(p, \delta_\Omega(p)/2)$. Thus,

$$k_\Omega(z_1, z_2) = L_\Omega(\eta) = \int_\eta \frac{|dz|}{\delta_\Omega(z)} \geq \frac{2}{3\delta_\Omega(p)} \int_\eta |dz| \geq \frac{2}{3\delta_\Omega(p)} |z_1 - z_2|,$$

and we conclude

$$\frac{2}{3\delta_\Omega(p)} |z_1 - z_2| \leq k_\Omega(z_1, z_2) \leq \frac{2}{\delta_\Omega(p)} |z_1 - z_2|, \tag{2.2}$$

for every $z_1, z_2 \in B_{\mathbb{C}}(p, \delta_\Omega(p)/2)$. This implies the first statement.

Let us prove now the second one. Since the image set of σ is compact and Ω^c is closed, we have

$$m := d_{\mathbb{C}}(\sigma, \Omega^c) = \min_{p \in \sigma} \delta_\Omega(p) > 0, \qquad M := \max_{p \in \sigma} \delta_\Omega(p) < \infty.$$

Hence, (2.2) gives

$$\frac{2}{3M} |z_1 - z_2| \leq d_\Omega(z_1, z_2) \leq \frac{2}{m} |z_1 - z_2|, \tag{2.3}$$

if $z_1, z_2 \in B_{\mathbb{C}}(p, m/2)$ for some $p \in \sigma$. Since $\{z \in \mathbb{C} : d_{\mathbb{C}}(z, \sigma) < m/2\}$ is an open neighborhood of σ, (2.3) implies the second statement. $\square$

We have the following consequence for free homotopy classes.

Corollary 2.12. *Let $\Omega \subsetneq \mathbb{C}$ be a domain, endowed with its quasi-hyperbolic metric, and σ a closed path in Ω. If $\{\sigma_n\}$ is a sequence of closed paths converging uniformly to σ (with respect to the Euclidean or the quasihyperbolic metric), then there is an integer n_0 such that $\sigma_n \subset \Omega$ and $[\sigma_n] = [\sigma]$ for every $n \geq N$.*

3. Intersection behavior of local geodesics

The definition of quasihyperbolic geodesic is quite restrictive. In particular, condition (1.1) forces the path to be an embedding of a Jordan arc. We make a less restrictive definition as follows.

Definition 3.1. Let $\Omega \subsetneq \mathbb{C}$ be a domain, endowed with its quasihyperbolic metric. A path $\alpha : I \to \Omega$ is a *local quasihyperbolic geodesic* if for every $t_0 \in I$ there exists $\varepsilon > 0$ such that $\gamma|_{[t_0-\varepsilon, t_0+\varepsilon] \cap I}$ is a quasihyperbolic geodesic.

In this definition we allow ε to be an arbitrarily small positive number; but actually any value $0 < \varepsilon < 1$ will do, thanks to the following result of Väisälä:

Theorem E. *([42, Theorem 8.10]) If α is a local quasihyperbolic geodesic in Ω and $L_\Omega(\alpha) < 2$, then α is a quasihyperbolic geodesic in Ω.*

A local quasihyperbolic geodesic need not be an injective path. It can have infinite quasihyperbolic length when its domain I is noncompact. It is obvious that Martin's regularity result [31, Corollary 4.8], already mentioned in the Introduction, also holds for local quasihyperbolic geodesics and so they are $C^{1,1}$ smooth.

In this Section, and also in Section 5, we are going to compare the intersection behavior of Riemannian geodesics to that of local quasihyperbolic geodesics.

Riemannian geodesics have a very simple intersection behavior. Let $\mathcal{M}$ be a Riemannian manifold. Let $\alpha : I_1 \to \mathcal{M}$ and $\beta : I_2 \to \mathcal{M}$ be two unitary Riemannian geodesics which have an intersection $\alpha(t_1) = \beta(t_2)$, with $t_2 = t_1 + c$. If moreover $\alpha'(t_1) = \beta'(t_2)$, then

$$\alpha(t) \; \equiv \; \beta\big(t + c\big) \quad \text{for all} \; \; t \in I_1 \cap (-c + I_2) \,.$$

In other words, if two Riemannian geodesics α and β have a tangency then we can fuse $\alpha(t)$ and a time translation $\beta(t + c)$ together into a single geodesic path.

On the contrary, quasihyperbolic geodesics can be tangent at an isolated point of intersection. We recall an example where this happens.

Example 3.2. Consider $\Omega = \mathbb{C} \setminus \{0, 2\}$ with its quasihyperbolic metric. For $0 < r \leq 1$, the following curves are closed local quasihyperbolic geodesics in Ω:

$$\alpha_r = \{\, z \, : \, |z| = r \,\} \quad , \quad \beta_r = \{\, z \, : \, |z - 2| = r \,\} \,,$$

and we see that for $r = 1$ they are tangent at the point $z = 0$. Let D, D' be the closed discs bounded by α_1 and β_1, respectively. One can check that a geodesic joining 1 to any point not in $D \cup D'$ is exterior to both α_1 and β_1, which forces it to be tangent at $z = 1$ to both α_1 and β_1.

What is even more surprising is that two quasihyperbolic geodesic segments may coincide for some nonzero amount of time and then split apart. They can be segments of different quasihyperbolic geodesics or of the same one; more precisely:

There are pairs of local quasihyperbolic geodesics $\alpha_1 : J_1 \to \Omega$ and $\alpha_2 : J_2 \to \Omega$, defined on open intervals J_1, J_2, and compact intervals of positive length $I_1 \subset J_1$, $I_2 \subset J_2$ such that

$$\alpha_1(I_1) \; = \alpha_2(I_2) \quad \text{but} \quad \alpha_1\big(J_1 \setminus I_1\big) \cap \alpha_2\big(J_2 \setminus I_2\big) \; = \; \emptyset \,,$$

see Fig. 2 in Section 5. This even happens for restrictions $\alpha_1 = \alpha|_{J_1}, \alpha_2 = \alpha|_{J_2}$ of a single local quasihyperbolic geodesic α to intervals $J_1 \neq J_2$.

Väisälä describes this previously known phenomenon in [42, pages 10-12], where he exhibits some pairs of quasihyperbolic geodesics in the twice-punctured plane $\mathbb{C} \setminus \{0, 2\}$ that have this behavior.

There are situations which force local quasihyperbolic geodesics to do this. One such condition is the existence of a 'narrow rectangular corridor' in the domain Ω, as in the following Lemma.

Lemma 3.3. *Let $\Omega \subsetneq \mathbb{C}$ be a domain that for some $a > 2$ contains the open rectangle $U_a = (-1, 1) + i(-a, a)$, while the long sides $L_a^{\pm} = \pm 1 + i[-a, a]$ of that rectangle lie entirely in Ω^c.*

If γ is a local quasihyperbolic geodesic in Ω joining a point on the short side $\{x - ia : x \in (-1, 1)\} \cap \Omega$ with a point on the other short side $\{x + ia : x \in (-1, 1)\} \cap \Omega$, and contained in $\overline{U}_a \cap \Omega$, then the segment $L_4 = i[-a + 2, a - 2]$, four units shorter than $[-a, a]$, is part of γ.
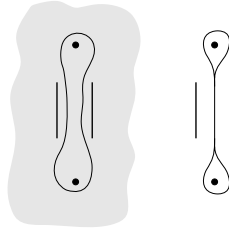
Fig. 1. A Jordan curve and a homotopic closed quasihyperbolic geodesic.

Proof. The segment $L_2 = i[-a+1, a-1]$, two units shorter than $[-a,a]$, separates $Q_a^- := (-1,0] + L_2 \subset U_a$ from $Q_a^+ := [0,1) + L_2 \subset U_a$. In the left rectangle Q_a^- we have $\delta_\Omega(x+iy) = 1+x$, hence the restriction to Q_a^- of the quasihyperbolic metric of Ω coincides with the restriction to Q_a^- of the Poincaré metric in the half-plane $\{x+iy \in \mathbb{C} : x > -1\}$. Thus, the geodesics of Ω in Q_a^- consist of straight segments orthogonal to L_a^-, subarcs of half-circles orthogonal to L_a^-, and segments contained in L_2. A symmetric result holds for the right-side rectangle Q_a^+. All these parts must be put together so that the result is a $C^{1,1}$ curve. One checks, by inspection, that if γ goes all the way from $\{y = 1-a\}$ to $\{y = a-1\}$, then it is the union of a straight segment that contains L_4 and at most two subarcs of circles centered at $L_a^+ \cup L_a^-$ and with radius 1. $\square$

The left part of Fig. 1 shows a Jordan curve inside a domain $\Omega \subsetneq \mathbb{C}$ whose complement Ω^c consists of two points and two vertical straight segments. The right part of the same figure shows a local quasihyperbolic geodesic freely homotopic to the Jordan curve. This local quasihyperbolic geodesic exists, as we shall see in Section 4, and goes twice along a straight segment: first in one direction, then in the opposite direction.

The intersection of local quasihyperbolic geodesics is not arbitrarily bad. For example, they cannot have a convergent infinite sequence of isolated intersection points. In general, two local quasihyperbolic geodesics of finite quasihyperbolic length cannot intersect in a set with infinitely many connected components. Theorem 3.5 below provides a full description for the intersection behavior of local quasihyperbolic geodesics. It is based on the following lemma, obtained from two results of Väisälä: Theorems A and E.

Lemma 3.4. *Let* $\Omega \subsetneq \mathbb{C}$ *be a domain. Let* $\alpha : [a_1, b_1] \to \Omega$ *and* $\beta : [a_2, b_2] \to \Omega$ *be local quasihyperbolic geodesics with quasihyperbolic lengths less than 2. The set* $\alpha([a_1, b_1]) \cap \beta([a_2, b_2])$ *can only be empty, a single point, or a compact Jordan arc.*

Proof. Both α and β are quasihyperbolic geodesics by Theorem E. In particular $\alpha([a_1, b_1])$ and $\beta([a_2, b_2])$ are compact Jordan arcs.

If the lemma did not hold for α, β, then $\alpha([a_1, b_1]) \cap \beta([a_2, b_2])$ would be nonempty with at least two connected components, say C' and C''. Assuming this, choose points $p' \in C'$ and $p'' \in C''$. Then p', p'' would both lie on the Jordan arc $\alpha([a_1, b_1])$, hence they would be the endpoints of a path $\alpha|_{[a_1', b_1']}$ for some interval $[a_1', b_1'] \subseteq [a_1, b_1]$. Likewise they would be the endpoints of $\beta|_{[a_2'', b_2'']}$ for some interval $[a_2'', b_2''] \subseteq [a_2, b_2]$.

It must be $\alpha([a_1', b_1']) \neq \beta([a_2'', b_2''])$, for otherwise we would have an arc $\alpha([a_1', b_1']) = \beta([a_2'', b_2''])$ contained in $\alpha([a_1, b_1]) \cap \beta([a_2, b_2])$ and joining C' to C'', which is impossible since C', C'' are distinct connected components of $\alpha([a_1, b_1]) \cap \beta([a_2, b_2])$.

Thus $\alpha|_{[a_1', b_1']}$ and $\beta|_{[a_2'', b_2'']}$ would be quasihyperbolic geodesics, different even up to reparametrization, with quasihyperbolic length less than 2 but with the same endpoints. As this contradicts Theorem A, the lemma follows. $\square$

The theorem we now state says that the joint image of one or several local quasihyperbolic geodesics, each with a finite quasihyperbolic length, can be described as a finite graph embedded in the plane.

Theorem 3.5. *Let $\alpha_i : [a_i, b_i] \to \Omega$, $i = 1, \ldots, k$, be a finite collection of local quasihyperbolic geodesics whose quasihyperbolic lengths are finite. There exists a graph Γ, with finitely many vertices and edges, and an embedding $\kappa : \Gamma \hookrightarrow \Omega$, such that $\kappa(\Gamma) = \bigcup_{i=1}^{k} \alpha_i([a_i, b_i])$.*

Proof. By splitting each path into segments of quasihyperbolic length less than 2, we may assume without loss of generality that $L_\Omega(\alpha_i) < 2$ for $i = 1, \ldots, k$. We assume this in the rest of the proof and we proceed by induction on k.

Case $k = 1$. We have only one local quasihyperbolic geodesic α_1 with $L_\Omega(\alpha_1) < 2$, hence a quasihyperbolic geodesic. The image set $\alpha([a_1, b_1])$ is a compact Jordan arc; i.e., the result of embedding into the plane a graph that consists of two vertices joined by one edge.

General case: $k > 1$ and the theorem is true for $k - 1$. The set $K \subset \Omega$, joint image of the local quasihyperbolic geodesics $\alpha_1, \ldots, \alpha_{k-1}$, is the result of embedding a finite graph Γ_{k-1} into the plane. If $\alpha_k([a_k, b_k]) \subseteq K$, we are done. If K and $\alpha_k([a_k, b_k])$ are disjoint, then the joint image of $\alpha_1, \ldots, \alpha_k$ is the result of embedding into the plane the disjoint union of Γ_{k-1} and the graph with two vertices joined by one edge. Assume now that the Jordan arc $\alpha_k([a_k, b_k])$ is not contained in K but it intersects K.

Since each intersection $\alpha_i([a_i, b_i]) \cap \alpha_k([a_k, b_k])$, $i = 1, \ldots, k - 1$, is either empty or connected, it follows that $K \cap \alpha_k([a_k, b_k])$ is a compact set with at least one and at most $k - 1$ connected components. Therefore $\alpha_k([a_k, b_k]) \setminus \big(K \cap \alpha_k([a_k, b_k]) \big)$ is the union of some pairwise disjoint Jordan arcs $\gamma_1, \ldots, \gamma_r \subset \alpha_k([a_k, b_k])$, with $0 < r \leq (k - 1) + 1 = k$. None of the arcs γ_j is compact, although there may be one including the point $\alpha_k(a_k)$ and there may be another one including $\alpha_k(b_k)$. We have a set equality:

$$K \cup \alpha_k([a_k, b_k]) \; = \; K \cup \gamma_1 \cup \cdots \cup \gamma_r \, ,$$

where the right-hand side is a disjoint union.

Each arc γ_j has two endpoints in $\alpha_k([a_k, b_k])$. Of these endpoints, the ones different from $\alpha_k(a_k)$ and $\alpha_k(b_k)$ correspond to points $v_h \in \Gamma_{k-1}$, thus leading to a finite list $v_1, \ldots, v_s \in \Gamma_{k-1}$ with $s \leq 2r \leq 2k$.

We now enlarge Γ_{k-1} by the following finite process. Do nothing for those points v_h that are vertices of Γ_{k-1}. If v_h is interior to an edge of Γ_{k-1}, add it as a new vertex and split that edge accordingly. If some γ_j contains $\alpha_k(a_k)$ or $\alpha_k(b_k)$, add a new isolated vertex. Once this has been done for all v_h and all γ_j, we have a new finite graph Γ'_{k-1} that has an embedding into the plane whose image is one of the following:

$$K \quad , \quad K \cup \{\alpha_k(a_k)\} \quad , \quad K \cup \{\alpha_k(b_k)\} \quad , \quad K \cup \{\alpha_k(a_k), \alpha_k(b_k)\} \, .$$

By construction, the joint image set

$$\alpha_1([\alpha_1, b_1]) \cup \cdots \cup \alpha_k([a_k, b_k]) \; = \; K \cup \gamma_1 \cup \cdots \cup \gamma_r \, ,$$

is the result of embedding into the plane a graph Γ_k obtained from Γ'_{k-1} by adding r new edges, so that for $j = 1, \ldots, r$ the new edge corresponding to γ_j joins the vertices of Γ'_{k-1} that came from the two endpoints of γ_j in $\alpha_k([a_k, b_k])$. $\square$

4. Closed minimizers

Our second way to replace the definition of quasihyperbolic geodesic by a less restrictive one is to consider minimization of a functional (e.g. quasihyperbolic length) within a given path homotopy class.

Definition 4.1. Let $\mathfrak{a}$ be a path homotopy class (free or rel endpoints) in some manifold $\mathcal{M}$. Let $\mathcal{F}$ be a functional that maps paths on $\mathcal{M}$ to numbers. A *minimizer for $\mathcal{F}$ in the class $\mathfrak{a}$* is a path α on $\mathcal{M}$ such that

$$\alpha \in \mathfrak{a} \quad \text{and} \quad \mathcal{F}[\alpha] \; = \; \min \big\{ \mathcal{F}[\sigma] \, : \, \sigma \in \mathfrak{a} \big\} \, .$$

We shall apply this definition to only two functionals: Riemannian length and quasihyperbolic length.

Proposition 4.2. *Let $\Omega \subsetneq \mathbb{C}$ be a domain, endowed with its quasihyperbolic metric. Let α be a closed path in Ω that is a minimizer for quasihyperbolic length in its free homotopy class $[\alpha]$. If α is non-constant, then it is a local quasihyperbolic geodesic, hence $C^{1,1}$.*

Proof. Let σ be one such minimizer. Fix $z_0 \in \sigma$ and consider an open arc $\sigma_0 \subset \sigma$ with $z_0 \in \sigma_0$ and $L_\Omega(\sigma_0) < 1/2$ (this arc exists because σ is non-constant). Let η_0 be a quasihyperbolic geodesic joining the endpoints of σ_0.

By Theorem B, the quasihyperbolic ball $B_\Omega(z_0, 1)$ is convex in the usual sense of linear algebra. Then, since $\sigma_0 \cup \eta_0 \subset B_\Omega(z_0, 1)$, we have that σ_0 and η_0 are homotopic rel their endpoints. If $L_\Omega(\eta_0) < L_\Omega(\sigma_0)$, then $(\sigma \setminus \sigma_0) \cup \eta_0$ is in the free homotopy class $[\sigma]$ and has strictly less quasihyperbolic length than σ, a contradiction. Therefore, $L_\Omega(\eta_0) = L_\Omega(\sigma_0)$ and the arc σ_0 is a quasihyperbolic geodesic. Since this is valid for all points $z_0 \in \sigma$, we see that σ is a local quasihyperbolic geodesic. $\square$

The above proof shows that arcs in α with quasihyperbolic length lesser that $1/2$ are quasihyperbolic geodesics. But, once we know that α is a local quasihyperbolic geodesic, we can apply Theorem E and conclude that arcs in α with quasihyperbolic length lesser that 2 are quasihyperbolic geodesics.

4.1. Existence of closed minimizers

In this subsection we give existence and nonexistence results for minimizers of quasihyperbolic length in the free homotopy class represented by a Jordan curve $\alpha_0 \subset \Omega$. Recall that the *interior* of α_0 is the bounded connected component $\text{Int}(\alpha_0)$ of $\mathbb{C} \setminus \alpha_0$, while the *exterior of α_0* is the other connected component $\text{Ext}(\alpha_0)$ of $\overline{\mathbb{C}} \setminus \alpha_0$, so that $\infty \in \text{Ext}(\alpha_0)$.

It is natural to consider three possible cases. In the third case we construct these minimizers as limits of minimizing Riemannian geodesics, a feature that will be very important later in the paper.

Case 1 (the easy case). $\Omega^c \cap \text{Int}(\alpha_0)$ consists of exactly one point p. In this case it does not matter how many points of Ω^c lie exterior to α_0. The quasihyperbolic metric near an isolated point p of Ω^c is isometric to a product cylinder $(0, \infty) \times \mathbb{S}^1$. If we define polar coordinates (r, θ) by $z = p + re^{i\theta}$, then near p we have $\delta_\Omega \equiv r$ and the quasihyperbolic metric in $\{r_0 > r > 0\}$ can be seen as the following Riemann metric:

$$\frac{|dz|^2}{r^2} = \frac{(dr)^2 + r^2\,(d\theta)^2}{r^2} = (d\ell)^2 + (d\theta)^2 \quad, \quad \text{where } \ell = \log r_0 - \log r \in (0, \infty).$$

This is very well known; it appeared in [32, page 38] and has been used, for example, in [42] and [20]. Notice that the circles $\{r = \text{constant}\}$ all have quasihyperbolic length 2π and are minimizers for quasihyperbolic length within the class $[\alpha_0]$.

Case 2 (nonexistence case). Ω^c has two or more points interior to α_0 and no point exterior to α_0. This can only happen when Ω^c is compact. If for a moment we consider the Gauss sphere $\overline{\mathbb{C}}$, then ∞ is the only point exterior to α_0; we say that α_0 *surrounds the puncture at infinity.*

Let us see that there is no minimizer in this case.

Choose two points $z_0, z_1 \in \Omega^c$, and let $\gamma(t) = z_0 + r(t)e^{i\theta(t)} \subset \Omega$ be any counterclockwise Jordan curve that surrounds Ω^c, then:

$$L_\Omega(\gamma) = \int_\gamma \frac{|dz|}{\delta_\Omega(z)} \geq \int_\gamma \frac{|dz|}{|z - z_0|} = \int_0^1 \left| \frac{r'(t)}{r(t)} + i\theta'(t) \right| dt \geq \int_0^1 \theta'(t)\,dt = 2\pi. \tag{4.4}$$

The value 2π is a lower bound for the quasihyperbolic length of all curves in the class $[\alpha_0]$. Of the two inequalities in formula (4.4), the second one is strict unless $r'(t) \equiv 0$. The curve γ thus has $L_\Omega(\gamma) > 2\pi$ if it is not a circle centered at z_0. Likewise it has $L_\Omega(\gamma) > 2\pi$ if it is not a circle centered at z_1. But a circle cannot be centered at both points simultaneously; thus the equality $L_\Omega(\gamma) = 2\pi$ is never achieved.

On the other hand 2π is indeed the infimum. In this case Ω^c is a compact set, hence contained in a ball $\Omega^c \subset B_{\mathbb{C}}(p, r_0)$. For each $R > r_0$, the path $\alpha_R(t) = p + Re^{it}$, $0 \le t \le 2\pi$, is a circle freely homotopic to α_0. From the estimate:

$$\delta_\Omega(\alpha_R(t)) \; = \; d_{\mathbb{C}}(p + re^{it}, \Omega^c) \; \in \; [R - r_0, \, R + r_0],$$

we get:

$$\frac{1}{R + r_0} \int\limits_{\alpha_R} |dz| \;\le\; \int\limits_{\alpha_R} \frac{|dz|}{\delta_\Omega(z)} \;\le\; \frac{1}{R - r_0} \int\limits_{\alpha_R} |dz|,$$

which together with $\displaystyle\int\limits_{\alpha_R} |dz| = 2\pi R$ leads to

$$2\pi \frac{R}{R + r_0} \;\le\; L_\Omega(\alpha_R) \;\le\; 2\pi \frac{R}{R - r_0},$$

and so $L_\Omega(\alpha_R) \to 2\pi$ as $R \to \infty$. This proves that 2π is the (non attainable) infimum of quasihyperbolic length within the class $[\alpha_0]$.

We state the third case as a theorem. It contains a statement dealing with uniform limits that will be made precise during the proof.

Theorem 4.3. (Case 3). *Let $\Omega \subsetneq \mathbb{C}$ be a domain, endowed with its quasihyperbolic metric. Let $\mathfrak{a} = [\alpha_0]$ be the free homotopy class in Ω of a Jordan curve $\alpha_0 \subset \Omega$, such that Ω^c contains at least two points interior to α_0 and at least one point exterior to α_0. Then $\mathfrak{a}$ contains a minimizer for quasihyperbolic length.*

In fact, there is one such minimizer that is a uniform limit of minimizers for Riemannian metrics.

Proof. Since $\Omega^c \subset \mathbb{C} \setminus \alpha_0$, the map $\Omega^c \to \mathbb{Z}$ given by $x \longmapsto i(x, \alpha_0)$ is continuous. Therefore the sets

$$\mathrm{Int}(\alpha_0) \cap \Omega^c \;=\; \{\, x \in \Omega^c : i(x, \alpha_0) = \pm 1 \,\} \quad, \quad \mathrm{Ext}(\alpha_0) \cap \Omega^c \;=\; \{\, x \in \Omega^c : i(x, \alpha_0) = 0 \,\},$$

are both closed.

Consider any path $\beta : [a_0, b_0] \to \mathbb{C}$ with $\beta(a_0) \in \mathrm{Int}(\alpha_0) \cap \Omega^c$ and $\beta(b_0) \in \mathrm{Ext}(\alpha_0) \cap \Omega^c$. We first trim the initial part of β out. For that purpose, we define the number

$$a \;=\; \sup \{\, t \in [a_0, b_0] \;:\; \beta(t) \in \mathrm{Int}(\alpha_0) \cap \Omega^c \,\}.$$

We have $a = \lim_{k \to \infty} t_k$ for a sequence $\{t_k\}_{k=1}^\infty \subset [a_0, b_0]$ such that $\beta(t_k) \in \mathrm{Int}(\alpha_0) \cap \Omega^c$ for all k. We deduce that $\beta(a) \in \mathrm{Int}(\alpha_0) \cap \Omega^c$, because $\mathrm{Int}(\alpha_0) \cap \Omega^c$ is closed. Actually, we have:

$$a_0 \le a < b_0 \quad, \quad \beta(a) \in \mathrm{Int}(\alpha_0) \cap \Omega^c \quad, \quad \beta((a, b_0]) \subset \Omega \cup \mathrm{Ext}(\alpha_0).$$

Now we trim the final part of β out with the help of the following number:

$$b \;=\; \inf \{\, t \in [a, b_0] \;:\; \beta(t) \in \mathrm{Ext}(\alpha_0) \cap \Omega^c \,\}.$$

By an argument similar to the previous one

$$a < b \leq b_0 \quad , \quad \beta(b) \in \mathrm{Ext}(\alpha_0) \cap \Omega^c \quad , \quad \beta\big((a,b)\big) \subset \Omega \, .$$

The points $p_1 = \beta(a)$ and $q = \beta(b)$ are the only ones outside Ω on the trimmed path $\beta|_{[a,b]}$. Since Ω^c contains at least two points interior to α_0, there is a point $p_2 \in \mathrm{Int}(\alpha_0) \cap \Omega^c$ with $p_2 \neq p_1$.

Since $\Omega \subseteq \mathbb{C} \setminus \{p_1, p_2, q\}$, any loop $\alpha \in \mathfrak{a}$ is also free homotopic to α_0 in $\mathbb{C} \setminus \{p_1, p_2, q\}$, hence

$$i(\alpha, p_1) = i(\alpha_0, p_1) = \pm 1 \quad , \quad i(\alpha, p_2) = i(\alpha_0, p_2) = \pm 1 \quad , \quad i(\alpha, q) = i(\alpha_0, q) = 0 \, . \qquad (4.5)$$

Fix a radius $r > 0$ such that the following sets are pairwise disjoint

$$\overline{D}_1 = \overline{B}_{\mathbb{C}}(p_1, r) \quad , \quad \{p_2\} \quad , \quad \overline{D} = \overline{B}_{\mathbb{C}}(q, r) \, .$$

Any loop $\alpha \in \mathfrak{a}$ contains a point $q_\alpha \in \alpha$ outside $\overline{D} \cup \overline{D}_1$. If such a point did not exist, we would have $\alpha \subset \overline{D}$ or $\alpha \subset \overline{D}_1$ because $\overline{D}$ and $\overline{D}_1$ are different path components of $\overline{D} \cup \overline{D}_1$. In both of these cases α would be null-homotopic in $\mathbb{C} \setminus \{p_2\}$, thereby leading to $i(\alpha, p_2) = 0$ which contradicts (4.5).

For $L \geq L_\Omega(\alpha_0)$ we define $\mathfrak{a}_L = \big\{ \alpha \in \mathfrak{a} : L_\Omega(\alpha) \leq L \big\}$, which is a nonempty class of loops. Minimizing quasihyperbolic length in $\mathfrak{a}$ is equivalent to minimizing it in $\mathfrak{a}_L$.

Define also $\overline{D}_1(L) = \overline{B}_{\mathbb{C}}\big(p_1, re^{-L}\big) \subset \overline{D}_1$ and $\overline{D}(L) = \overline{B}_{\mathbb{C}}\big(q, re^{-L}\big) \subset \overline{D}$. Suppose that some loop $\alpha \in \mathfrak{a}$ reaches $\overline{D}(L)$; then it contains a subpath $\alpha_1 \subsetneq \alpha$ going from the point q_α to some point in $\overline{D}(L)$. This subpath has to intersect $\partial \overline{D}$ and $\partial \overline{D}(L)$, because its initial point q_α lies outside $\overline{D}$ and its final point is on $\overline{D}(L)$. Thus we would have:

$$L_\Omega(\alpha) > L_\Omega(\alpha_1) \geq \inf\big\{ k_\Omega(x,y) : x \in \Omega \cap \partial\overline{D}, \, y \in \Omega \cap \partial\overline{D}(L) \big\} \geq$$
$$\geq \inf\big\{ k_{\mathbb{C}\setminus\{q\}}(x,y) : x \in \Omega \cap \partial\overline{D}, \, y \in \Omega \cap \partial\overline{D}(L) \big\} \geq$$
$$\geq \inf\big\{ k_{\mathbb{C}\setminus\{q\}}(x,y) : x \in \partial\overline{D}, \, y \in \partial\overline{D}(L) \big\} =$$
$$= \log r - \log\big(re^{-L}\big) = L \, .$$

One proves in the same way that if $\alpha \in \mathfrak{a}$ reaches $\overline{D}_1(L)$ then $L_\Omega(\alpha) > L$. We deduce that all loops $\alpha \in \mathfrak{a}_L$ are disjoint from $\overline{D}(L) \cup \overline{D}_1(L)$. For such loops $\overline{D}(L)$ is contained in the same path component of $\mathbb{C} \setminus \alpha$ as q, hence $i(\alpha, z) = 0$ for all $z \in \overline{D}(L)$. Likewise $i(\alpha, z_1) = \pm 1$ for $\alpha \in \mathfrak{a}_L$ and all $z_1 \in \overline{D}_1(L)$.

Since the path $\beta : [a,b] \to \Omega$ starts at the center p_1 of $\overline{D}_1(L)$ and ends at the center q of $\overline{D}(L)$, there is an interval $[a_L, b_L] \subset [a,b]$ such that the subpath $\beta|_{[a_L,b_L]}$ starts at a point of $\partial\overline{D}_1(L)$ and ends at a point of $\partial\overline{D}(L)$. For each $\alpha \in \mathfrak{a}_L$, we have:

$$i\big(\alpha, \beta(a_L)\big) = \pm 1 \quad , \quad i\big(\alpha, \beta(b_L)\big) = 0 \, . \qquad (4.6)$$

We claim that the compact arc $\Gamma_{\mathfrak{a},L} = \beta\big([a_L, b_L]\big) \subset \Omega$ intersects every loop $\alpha \in \mathfrak{a}_L$. Otherwise $\Gamma_{\mathfrak{a},L}$ would be contained in a path component of $\mathbb{C} \setminus \alpha$ and $i(\alpha, \beta(t))$ would be constant for $t \in [a_L, b_L]$, thereby contradicting (4.6).

We have constructed a compact arc $\Gamma_{\mathfrak{a},L} \subset \Omega$ that intersects every loop $\alpha \in \mathfrak{a}_L$. It follows that all these loops are contained in the compact connected set $\overline{B}_\Omega\big(\Gamma_{\mathfrak{a},L}, L\big) \subset \Omega$. It is connected because $\Gamma_{\mathfrak{a},L}$ is an arc and quasihyperbolic distance is realized by quasihyperbolic geodesics.

We shall now approximate the quasihyperbolic metric by smooth Riemann metrics. Let $\varphi_1 : \mathbb{C} \to \mathbb{R}$ be a smooth function with the following properties:

$$\varphi_1 \geq 0 \quad , \quad \text{support}(\varphi_1) \subseteq \overline{B}_{\mathbb{C}}(0, 1) \quad , \quad \int_{\mathbb{C}} \varphi_1(x) \, dx \ = \ 1 \, ,$$

and for each positive integer ν define $\varphi_\nu(x) \equiv \nu^3 \varphi(\nu^3 x)$. We have:

$$\varphi_\nu \geq 0 \quad , \quad \text{support}(\varphi_\nu) \subseteq \overline{B}_{\mathbb{C}}\left(0, 1/\nu^3\right) \quad , \quad \int_{\mathbb{C}} \varphi_\nu(x) \, dx \ = \ 1 \, .$$

The function $\delta_\Omega(x) = d_{\mathbb{C}}(x, \Omega^c)$ is defined and Lipschitz on all of $\mathbb{C}$, with Lipschitz constant 1. It is easy to check that for every positive integer ν the function $\max\left(1/\nu, \delta_\Omega(x)\right) > 0$ is also Lipschitz with Lipschitz constant 1, and that the function

$$f_\nu(x) \ = \ \frac{1}{\max\left(1/\nu, \delta_\Omega(x)\right)} \ = \ \min\left\{\nu, \frac{1}{\delta_\Omega(x)}\right\} \, ,$$

is Lipschitz with Lipschitz constant ν^2.

For the convolution $f_\nu * \varphi_\nu(x) = \int f_\nu(x - y) \varphi_\nu(y) \, dy$ we have $f_\nu * \varphi_\nu(x) > 0$ and the following estimate, valid on all of $\mathbb{C}$:

$$\left| f_\nu * \varphi_\nu(x) - f_\nu(x) \right| = \left| \int \left(f_\nu(x - y) - f(x) \right) \varphi_\nu(y) \, dy \right| \leq$$

$$\leq \text{Lip}(f_\nu) \int \left| (x - y) - x \right| \varphi_\nu(y) \, dy \ =$$

$$= \nu^2 \int |y| \, \varphi_\nu(y) \, dy \ \leq$$

$$\leq \nu^2 \frac{1}{\nu^3} \int \varphi_\nu(y) \, dy \ = \ \frac{1}{\nu} \, .$$

Let $\{K_j\}_{j=1}^\infty$ be a sequence of compact sets such that $K_j \subset \text{Int}(K_{j+1})$ and $\bigcup_{j=1}^\infty K_j = \Omega$. We need these sets to be smooth compact domains (in particular, connected), because we shall later apply Theorem F of Section 5 to them. A possible construction, among many, for the sequence $\{K_j\}$ is the following. Every domain $\Omega \subsetneq \mathbb{C}$, with two or more points in the complement Ω^c, has a complete, analytic Riemann metric ρ with constant curvature -1 called Poincaré metric. Analyticity implies that given a point $p_0 \in \Omega$ there is a countable set $X \subset (0, +\infty)$ such that for $r \in (0, +\infty) \setminus X$ the boundary of the Poincaré ball $B_\rho(r)$ centered at p_0 is a finite union of pairwise disjoint Jordan curves, each of them analytic except for a finite number of corner points (see [12, Theorem 1.2]). Take a divergent sequence of radii $\{r_j\}_{j=1}^\infty \subset (0, +\infty) \setminus X$ and for each j transform $\overline{B}_\rho(r_j)$ into K_j by smoothing out the corner points.

For each j define numbers $0 < d_j < \Delta_j$ as follows

$$d_j \ = \ \min_{x \in K_j} \delta_\Omega(x) \quad , \quad \Delta_j \ = \ \max_{x \in K_j} \delta_\Omega(x) \, .$$

Construct a divergent sequence of integers $\{\nu_j\}_{j=1}^\infty$ with

$$\nu_j \ > \ \max\left\{ j, \frac{1}{d_j}, j\Delta_j \right\} \quad , \quad \text{for all } j \, .$$

On K_j we have $f_{\nu_j} = 1/\delta_\Omega$ and

$$\frac{1}{\delta_\Omega} - \frac{1}{\nu_j} \ \leq \ f_{\nu_j} * \varphi_{\nu_j} \ \leq \ \frac{1}{\delta_\Omega} + \frac{1}{\nu_j} \, , \tag{4.7}$$

which implies

$$\left(1 - \frac{1}{j}\right)\frac{1}{\delta_\Omega} \;\leq\; f_{\nu_j} * \varphi_{\nu_j} \;\leq\; \left(1 + \frac{1}{j}\right)\frac{1}{\delta_\Omega}\,.$$

For each j we have a smooth Riemann metric $\left(f_{\nu_j} * \varphi_{\nu_j}(z)\right)^2 |dz|^2$ defined on all of $\mathbb{C}$. These are our Riemannian approximations to the quasihyperbolic metric. Denote by L_j the Riemannian length functional that corresponds to the metric $\left(f_{\nu_j} * \varphi_{\nu_j}(z)\right)^2 |dz|^2$. Any path $\gamma \subset K_j$ satisfies:

$$\left(1 - \frac{1}{j}\right) L_\Omega(\gamma) \;\leq\; L_j(\gamma) \;\leq\; \left(1 + \frac{1}{j}\right) L_\Omega(\gamma)\,.$$

We now fix j and minimize the functional L_j among the loops $\alpha \in \mathfrak{a}$ that are contained in K_j and have $L_j(\alpha) \leq L_j(\alpha_0)$. These loops satisfy:

$$L_\Omega(\alpha) \;\leq\; \frac{1}{1 - \frac{1}{j}} L_j(\alpha) \;\leq\; \frac{1}{1 - \frac{1}{j}} L_j(\alpha_0) \;\leq\; \frac{1 + \frac{1}{j}}{1 - \frac{1}{j}} L_\Omega(\alpha_0) \;=\; \frac{j+1}{j-1} L_\Omega(\alpha_0)\,,$$

which for $j \geq 2$ implies $L_\Omega(\alpha) \leq 2L_\Omega(\alpha_0)$.

For the rest of the proof make $L = 2L_\Omega(\alpha_0)$. We have just shown that the loops $\alpha \in \mathfrak{a}$ with $L_j(\alpha) \leq L_j(\alpha_0)$ satisfy $L_\Omega(\alpha) \leq L$, hence they are all contained in $\overline{B}(\Gamma_{\mathfrak{a},L}, L)$. In addition, for j large enough the interior of K_j contains $\overline{B}(\Gamma_{\mathfrak{a},L}, L)$.

The loops $\alpha \in \mathfrak{a}$ with $L_j(\alpha) \leq L_j(\alpha_0)$ also satisfy:

$$L_j(\alpha) \;\leq\; L_j(\alpha_0) \;\leq\; \left(1 + \frac{1}{j}\right) L_\Omega(\alpha_0) \;\leq\; L\,,$$

which implies that they are reparametrizations of closed paths $\gamma : [0, L] \to K_j$ with

$$\|\gamma'(t)\| \;\leq\; \frac{1}{(f_{\nu_j} * \varphi_{\nu_j})(\gamma(t))} \quad, \quad \text{for all } t \in [0, L]\,.$$

Since reparametrization does not change Riemannian length, in order to minimize L_j among loops in $\mathfrak{a}$ contained in K_j we need only consider these closed paths $\gamma(t)$. This defines a set P_j of closed paths with a common domain $[0, L]$, contained in a compact set that lies interior to K_j, and sharing the Lipschitz constant

$$\max\left\{\frac{1}{(f_{\nu_j} * \varphi_{\nu_j})(x)} \;:\; x \in K_j\right\}\,.$$

Choose a sequence $\{\gamma_{j,m}\}_{m=1}^\infty \subset P_j$ with $\lim_{m\to\infty} L_j\left(\gamma_{j,m}\right) = \ell_j := \inf\left\{L_\nu(\gamma) \;:\; \gamma \in P_j\right\}$. We can use the Arzelá-Ascoli theorem to obtain a subsequence $\{\gamma_{j,m_k}\}_{k=1}^\infty$ that converges uniformly to a Lipschitz path $\gamma_j : [0, L] \to K_j$. The uniform convergence is in Euclidean distance, but thanks to Lemma 2.11 it is also uniform convergence in quasihyperbolic distance. Also, Corollary 2.12 ensures that $\gamma_j \in \mathfrak{a}$.

From $\gamma_j \in \mathfrak{a}$ we get $\ell_j \leq L_j(\gamma_j)$. As is well known, Riemannian length is lower semicontinuous under uniform convergence; therefore:

$$\ell_j \;\leq\; L_j(\gamma_j) \;\leq\; \lim_{k\to\infty} L_j\left(\gamma_{j,m_k}\right) \;=\; \ell_j\,,$$

and so $L_j(\gamma_j) = \ell_j$. The closed path γ_j minimizes L_j among all loops in $\mathfrak{a}$ contained in K_j. Moreover γ_j lies inside the compact set $\overline{B}(\Gamma_{\mathfrak{a},L}, L)$ that is contained in the interior of K_j. This implies that γ_j is an honest Riemannian geodesic for the Riemann metric $\left(f_{\nu_j} * \varphi_{\nu_j}(z)\right)^2 |dz|^2$.

We arrive at a sequence $\{\gamma_j\}_{j=2}^{\infty}$ made of closed paths $\gamma_j : [0, L] \to K_j$ with common domain $[0, L]$ and lying inside the compact set $\overline{B}(\Gamma_{\mathfrak{a},L}, L)$ that is independent of j. They share the following finite Lipschitz constant:

$$\sup\left\{ \frac{1}{(f_{\nu_j} * \varphi_{\nu_j})(x)} \ : \ x \in \overline{B}(\Gamma_{\mathfrak{a},L}, L) \ , \ j = 2, 3, 4 \ldots \right\},$$

and thus there is a subsequence $\{\gamma_{j_k}\}_{k=1}^{\infty}$ that converges uniformly as $k \to \infty$ to a closed path $\gamma_{\mathfrak{a}} \in \mathfrak{a}$. We claim that $\gamma_{\mathfrak{a}}$ is a minimizer for quasihyperbolic length in the class $\mathfrak{a}$.

Let $\ell_{\mathfrak{a}} = \inf\left\{ L_\Omega(\alpha) \ : \ \alpha \in \mathfrak{a} \right\} \leq L_\Omega(\alpha_0)$. Given $0 < \varepsilon < L_\Omega(\alpha_0)$, there is $\alpha \in \mathfrak{a}$ with

$$L_\Omega(\alpha) \ < \ \ell_{\mathfrak{a}} + \varepsilon \ < \ 2L_\Omega(\alpha_0) \ = \ L \, ,$$

and in particular $\alpha \subset \overline{B}(\Gamma_{\mathfrak{a},L}, L) \subset K_{j_k}$, hence

$$L_{j_k}(\gamma_{j_k}) \ \leq \ L_{j_k}(\alpha) \ \leq \ \left(1 + \frac{1}{j_k}\right) (\ell_{\mathfrak{a}} + \varepsilon) \, ,$$

and

$$L_\Omega(\gamma_{j_k}) \ \leq \ \frac{1}{1 - \frac{1}{j_k}} L_{j_k}(\gamma_{j_k}) \ \leq \ \frac{1 + \frac{1}{j_k}}{1 - \frac{1}{j_k}} (\ell_{\mathfrak{a}} + \varepsilon) \ = \ \frac{j_k + 1}{j_k - 1} (\ell_{\mathfrak{a}} + \varepsilon) \, .$$

But $\dfrac{j_k + 1}{j_k - 1} \to 1$ as $k \to \infty$, thus $L_\Omega(\gamma_{j_k}) \leq \ell_{\mathfrak{a}} + 2\varepsilon$ for k large enough (depending on $\ell_{\mathfrak{a}}$ and ε). Since this is valid for all ε, we have $\limsup_{k\to\infty} L_\Omega(\gamma_{j_k}) \leq \ell_{\mathfrak{a}}$. Using now the lower semicontinuity of quasihyperbolic length under uniform convergence, we get $\ell_{\mathfrak{a}} \leq L_\Omega(\gamma_{\mathfrak{a}}) \leq \limsup_{k\to\infty} L_\Omega(\gamma_{j_k}) \leq \ell_{\mathfrak{a}}$ and so $L_\Omega(\gamma_{\mathfrak{a}}) = \ell_{\mathfrak{a}}$.   □

5. Absence of crossings

We start by giving a name to the special quasihyperbolic minimizers we have constructed.

Definition 5.1. Given a domain $\Omega \subsetneq \mathbb{C}$, a *quasihyperbolic limit geodesic* in Ω is any quasihyperbolic minimizer in Ω constructed as in the proof of Theorem 4.3.

We now recall a very strong result in the Riemannian setting.

Theorem F. *([8]) Let $\mathcal{R} \subset \mathbb{C}$ be a smooth, compact domain with boundary. Fix an arbitrary smooth Riemann metric on $\mathcal{R}$.*

Let $\mathfrak{a}$ be a nontrivial free homotopy class in $\mathcal{R}$ represented by a Jordan curve. If $\mathfrak{a}$ has a minimizer α for Riemannian length, and α stays in the interior of $\mathcal{R}$, then α is a Jordan curve.

Let $\mathfrak{a} \neq \mathfrak{b}$ be two distinct nontrivial free homotopy classes in $\mathcal{R}$, represented by disjoint Jordan curves. If these classes have minimizers α, β for Riemannian length, both lying in the interior of $\mathcal{R}$, then α and β are disjoint Jordan curves.

It is remarkable that Theorem F is valid for all smooth Riemann metrics, no matter what the Gaussian curvature is. It is a direct consequence of [8, Theorem 2.1], [8, Corollary 3.4], [8, first paragraph of §4], and the following lemma.

Fig. 2. Touching curves.

Lemma 5.2. *If α, β are two Jordan curves with nontrivial free homotopy classes in a planar region $\mathcal{R}$, then α is not freely homotopic in $\mathcal{R}$ to a loop that goes several times along β.*

Proof. By Lemma 2.9 we know that $\mathrm{Int}(\alpha)$ contains at least one point $p \in \mathcal{R}^c$. Notice that $p \in \mathbb{C} \setminus \beta$ and $\mathcal{R} \subseteq \mathbb{C} \setminus \{p\}$.

By Lemma 2.8 we have $i(\alpha, p) = \pm 1$ and $i(\beta, p) \in \{-1, 0, 1\}$. Let β^k be a path that goes k times along β, with $|k| > 1$. If α were freely homotopic in $\mathcal{R}$ to β^k, then α would also be freely homotopic to β^k in $\mathbb{C} \setminus \{p\}$ and we would have $\pm 1 = i(\alpha, p) = k \cdot i(\beta, p) \in \{-k, 0, k\}$, which is impossible. $\square$

Quasihyperbolic metrics satisfy a weakened version of Theorem F, were 'simple closed curve' is replaced by 'closed curve with no selfcrossings' and 'disjoint curves' is replaced by 'curves that do not cross'. We now make these notions precise.

Definition 5.3. Let $\Omega \subsetneq \mathbb{C}$ be a domain.

We say that a closed path $\alpha : [a, b] \to \Omega$ *does not cross itself,* or that it is *without selfcrossings,* if it is a uniform limit of simple closed paths.

Given two closed paths $\alpha : [a, b] \to \Omega$, $\beta : [a', b'] \to \Omega$, each without selfcrossings, we say that they *do not cross* if the configuration of the two is a uniform limit of pairs of disjoint simple closed paths.

Thanks to Lemma 2.11, uniform path convergence is the same in Euclidean or quasihyperbolic distance.

Fig. 2 exhibits two pairs of curves. In each pair one curve is the thin solid line while the other is the thick dashed line. The curves on the left touch each other but do not cross. The curves on the right do cross; i.e., they have a 'robust' intersection: it is impossible to make them disjoint by a small perturbation.

Theorem 5.4. *Let $\Omega \subsetneq \mathbb{C}$ de a domain, endowed with its quasihyperbolic metric.*

Let $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_m\}$ be a finite family of pairwise distinct free homotopy classes in Ω, each one of them in the hypotheses of Theorem 4.3. If $m \geq 2$, suppose further that they are represented by pairwise disjoint Jordan curves. We can choose minimizers $\gamma_{\mathfrak{a}_i} \in \mathfrak{a}_i$, $i = 1, \ldots, m$, in such a way that no $\gamma_{\mathfrak{a}_i}$ crosses itself and whenever $i \neq i'$ the curves $\gamma_{\mathfrak{a}_i}$ and $\gamma_{\mathfrak{a}_{i'}}$ do not cross.

Proof. We start with a family $\{\mathfrak{a}\}$ of one single free homotopy class. In the proof of Theorem 4.3 the quasihyperbolic minimizer $\gamma_{\mathfrak{a}}$ for $\mathfrak{a}$ is a uniform limit of curves γ_{j_k} with three properties:

– Each γ_{j_k} belongs to the class $\mathfrak{a}$.
– Each γ_{j_k} is a minimizer within $\mathfrak{a}$ for a Riemannian metric on a smooth compact domain $K_{j_k} \subset \Omega$.
– There is a compact set $\overline{B}_\Omega(\Gamma_{\mathfrak{a}, L}, L)$ that contains all curves γ_{j_k} and lies in the interior of all K_{j_k}.

Then we can apply Theorem F, to conclude that the γ_{j_k} are all Jordan curves. Thus the minimizer constructed in the proof of Theorem 4.3 is a uniform limit of Jordan curves. This means, by definition, that $\gamma_{\mathfrak{a}}$ does not cross itself.

Let now $\mathfrak{a} \neq \mathfrak{b}$ be two distinct free homotopy classes, both in the hypotheses of Theorem 4.3, and represented by disjoint Jordan curves. Let $\gamma_{\mathfrak{a}}$ and $\gamma_{\mathfrak{b}}$ be the quasihyperbolic minimizers constructed in the proof of Theorem 4.3. We have sequences of Riemannian minimizers $\{\gamma_j\}$ and $\{\eta_j\}$, two numbers L_1, L_2, and two compact Jordan arcs $\Gamma_{\mathfrak{a}, L_1}, \Gamma_{\mathfrak{b}, L_2} \subset \Omega$ such that the γ_j remain inside $A_1 = \overline{B}_\Omega(\Gamma_{\mathfrak{a}, L_1}, L_1)$, while the

η_j remain inside $A_2 = \overline{B}_\Omega\left(\Gamma_{\mathfrak{b}, L_2}, L_2\right)$. For j large the interior of the compact domain K_j contains both A_1 and A_2, therefore γ_j and η_j are Riemannian minimizers for the same Riemann metric on the same domain K_j. Theorem F says that γ_j and η_j are disjoint Jordan curves for each large j.

There is an infinite set $J_1 = \{j_k : k = 1, 2, 3, \dots\}$ such that the subsequence $\left\{\gamma_j\right\}_{j \in J_1} = \left\{\gamma_{j_k}\right\}_{k=1}^{\infty}$ converges uniformly to the minimizer $\gamma_{\mathfrak{a}}$. Consider the subsequence with the same indexes $\left\{\eta_{j_k}\right\}_{k=1}^{\infty}$, which we can write as $\left\{\eta_j\right\}_{j \in J_1}$. There is an infinite subset $J_2 = \{j_{k_s} : s = 1, 2, 3, \dots\} \subseteq J_1$ such that the sub-subsequence $\left\{\eta_{j_{k_s}}\right\}_{s=1}^{\infty} = \left\{\eta_j\right\}_{j \in J_2}$ converges uniformly to the minimizer $\gamma_{\mathfrak{b}}$. Going back to the γ_j, the corresponding sub-subsequence $\left\{\gamma_{j_{k_s}}\right\}_{s=1}^{\infty} = \left\{\gamma_j\right\}_{j \in J_2}$ still converges uniformly to $\gamma_{\mathfrak{a}}$. Theorem F says that for each s the curves $\gamma_{j_{k_s}}$ and $\eta_{j_{k_s}}$ are disjoint Jordan curves. This implies that the limit curves $\gamma_{\mathfrak{a}}$ and $\gamma_{\mathfrak{b}}$ do not have self-crossings and do not cross.

For a general finite family $\{\mathfrak{a}_1, \dots, \mathfrak{a}_m\}$ we generate a nested list $J_1 \supseteq J_2 \supseteq \cdots \supseteq J_m$ of infinite sets of indices; then get, for each $\mathfrak{a}_i$, a sequence of Riemannian minimizers with J_m as index set. These m sequences are put together into a sequence of configurations $\left\{C_j\right\}_{j \in J_m}$, where each C_j consists of m pairwise disjoint Jordan curves, whose limit $\lim C_j = \left\{\gamma_{\mathfrak{a}_1}, \dots, \gamma_{\mathfrak{a}_m}\right\}$ is the configuration of the quasihyperbolic minimizers. It follows that the curves $\gamma_{\mathfrak{a}_i}$ do not cross themselves and do not cross with one another. $\square$

6. Discs with holes

We have explained in section 4 how to obtain quasihyperbolic minimizers by passing to the limit of Riemannian minimizers. In Section 5 we have described the special nature of these limit minimizers. In the present section we want to pass to the limit of a sequence of domains bounded by Riemannian geodesics.

Definition 6.1. We shall call *disc with k holes* the result of removing from a compact, simply connected, smooth domain $\mathcal{R}$ in $\mathbb{C}$ the interiors of k pairwise disjoint, compact, simply connected, smooth domains contained in $\mathcal{R}$.

Let $\Omega \subsetneq \mathbb{C}$ be a domain endowed with some Riemann metric. A *Riemannian geodesic domain* is a compact subset of Ω bounded by a finite collection of pairwise disjoint Jordan curves that are geodesics for the given Riemann metric.

Notice that, although the word 'domain' is part of the name, Riemannian geodesic domains are not open sets. Instead, they are compact subsets of the plane.

Of course, every Riemannian geodesic domain is a disc with holes, but in addition to its topological nature it also has geometric features.

We first consider the case of zero holes. This has two possible characterizations:

(1) The set $\mathcal{R}$ is simply connected.
(2) $\mathcal{R}$ is the compact set $\overline{\text{Int}}(\alpha) := \alpha \cup \text{Int}(\alpha)$ bounded by a smooth Jordan curve α.

Lemma 6.2. *Let $\Omega \subsetneq \mathbb{C}$ be a domain. Let $\alpha_n : [a, b] \to \mathbb{C}$, $n = 1, 2, 3, \dots$ be a sequence of simple closed paths that converges uniformly to a closed path $\alpha : [a, b] \to \Omega$. Then the sequence $\left\{\overline{\text{Int}}(\alpha_n)\right\}$ is a Cauchy sequence with respect to Euclidean Hausdorff distance.*

Proof. For any Jordan curve γ and $c > 0$, we define the following compact subset of $\text{Int}(\gamma)$:

$$\text{Int}_c(\gamma) = \left\{p \in \mathbb{C} : d_{\mathbb{C}}(p, \gamma) \geq c, \ i(\gamma, p) = \pm 1\right\}.$$

Given any $\varepsilon > 0$ and $t \in [a, b]$, there is n_ε such that if $n, m \geq n_\varepsilon$, then

$$\|\alpha_n(t) - \alpha_m(t)\| \leq \varepsilon.$$

We claim that the following also holds: If $n, m \geq n_\varepsilon$, then

$$\mathrm{Int}_{2\varepsilon}(\alpha_n) \subseteq \mathrm{Int}_\varepsilon(\alpha_m) . \tag{6.8}$$

If $p \in \mathrm{Int}_{2\varepsilon}(\alpha_n)$, then the map

$$F : [a, b] \times [0, 1] \longrightarrow \mathbb{C} \quad , \quad F(t, \lambda) = (1 - \lambda)\, \alpha_n(t) + \lambda\, \alpha_m(t) ,$$

is a free homotopy from α_n to α_m that never touches p, because

$$d_\mathbb{C}\big(p, F(t, \lambda)\big) \geq d_\mathbb{C}\big(p, \alpha_n(t)\big) - d_\mathbb{C}\big(\alpha_n(t), F(t, \lambda)\big) \geq 2\varepsilon - \varepsilon > 0 ,$$

and so F is a free homotopy in $\mathbb{C} \setminus \{p\}$ from α_n to α_m. By Lemma 2.6, we have

$$i(\alpha_m, p) = i(\alpha_n, p) = \pm 1 ,$$

and also $d_\mathbb{C}(p, \alpha_m) \geq d(p, \alpha_n) - d_{\mathbb{C}, H}(\alpha_n, \alpha_m) \geq 2\varepsilon - \varepsilon = \varepsilon$. We conclude $p \in \mathrm{Int}_\varepsilon(\alpha_m)$. This proves (6.8).

We finish the proof of the lemma by showing:

$$n, m \geq n_\varepsilon \implies d_{\mathbb{C}, H}\big(\overline{\mathrm{Int}}(\alpha_n), \overline{\mathrm{Int}}(\alpha_m)\big) \leq 3\varepsilon .$$

This is a statement symmetric in n and m; therefore it will be proved if we can show:

$$p \in \overline{\mathrm{Int}}(\alpha_n) \implies d_\mathbb{C}\big(p, \overline{\mathrm{Int}}(\alpha_m)\big) \leq 3\varepsilon .$$

Consider any $p \in \alpha_n \cup \mathrm{Int}(\alpha_n)$. If $p \in \alpha_n$, then $d_\mathbb{C}(p, \alpha_m) \leq \varepsilon$ and so $d_\mathbb{C}\big(p, \overline{\mathrm{Int}}(\alpha_m)\big) \leq \varepsilon$. If $d_\mathbb{C}\big(p, \alpha_n\big) \geq 2\varepsilon$, then $p \in \mathrm{Int}_{2\varepsilon}(\alpha_n)$ and formula (6.8) says that $p \in \mathrm{Int}(\alpha_m)$, hence $d_\mathbb{C}\big(p, \overline{\mathrm{Int}}(\alpha_m)\big) = 0$. The only remaining possibility is $0 < d(p, \alpha_n) < 2\varepsilon$, but then $d_\mathbb{C}\big(p, \alpha_m\big) \leq 2\varepsilon + d_{\mathbb{C}, H}(\alpha_n, \alpha_m) \leq 3\varepsilon$. □

This lemma says that the sets $\overline{\mathrm{Int}}(\alpha_n)$ have a compact Hausdorff limit, which we call *limit interior of α* and denote $\mathrm{limint}(\alpha)$. The above proof, in turn, implies that $\mathrm{limint}(\alpha)$ includes the limit path α and the sets $\mathrm{Int}_\varepsilon(\alpha_{n_\varepsilon})$ for all $\varepsilon > 0$.

A finite collection of pairwise disjoint Jordan curves bounds a disc with holes if and only if two conditions are met:

(1) The collection has an element α_0, called the *outer boundary,* that surrounds all other curves $\alpha_1, \ldots, \alpha_k$ in that collection.
(2) The interiors of $\alpha_1, \ldots, \alpha_k$ are pairwise disjoint.

Consider now $k + 1$ uniformly convergent sequences $\{\alpha_{0,n}\} \ldots, \{\alpha_{k,n}\}$ of Jordan curves. Suppose that for each n the curves $\alpha_{0,n}, \ldots, \alpha_{k,n}$ are pairwise disjoint and bound a compact set A_n. Then $\{A_n\}$ is a Cauchy sequence with respect to Hausdorff distance, where the Hausdorff limit set is described in terms of the limit interiors of the limit curves $\alpha_i = \lim_n \alpha_{i,n}$, for $i = 0, 1, \ldots k$.

Definition 6.3. Let $\Omega \subsetneq \mathbb{C}$ be a domain, endowed with its quasihyperbolic metric. A *quasihyperbolic geodesic domain* is a compact set $G \subset \Omega$ which is bounded by $k + 1$ limit quasihyperbolic geodesics $\gamma_0, \gamma_1, \ldots, \gamma_k$, where the whole configuration (domain and boundary curves) is the limit of a sequence $\{G_n\}$ of Riemannian geodesic domains whose boundary geodesics are distributed into $k + 1$ uniformly convergent sequences.

Proposition 6.4. *A set G constructed as in the above definition is always path connected.*

Fig. 3. A simple Y-piece.

Proof. Suffices to prove that any two points in ∂G are joined by a path in G.

Fix two quasihyperbolic geodesics $\alpha_1, \beta_1 \subset \partial G_1$. For each $n > 1$ let $\alpha_n, \beta_n \subset \partial G_n$ be the Riemannian geodesics homotopic in Ω to α_1 and β_1, respectively. Let M be an a priori bound for the quasihyperbolic distance between α_n and β_n, and choose paths $\xi_n \subset \Omega$ that join α_n to β_n and have $L_\Omega(\xi_n) \le M$. Every time ξ_n exits G_n through a geodesic $\gamma \subset \partial G_n$, it must re-enter through the same γ and we can replace the part of ξ_n between those two events with an arc of γ. Doing this as many times as necessary, we get a new path $\overline{\xi}_n \subset G_n$ joining α_n to β_n.

If $\overline{\xi}_n$ visits a geodesic $\gamma \subset \partial G_n$ more than once, we can replace the part between the first and last visit by a single arc of γ. This procedure leads to yet another path $\widetilde{\xi}_n \subset G_n$, still joining α_n to β_n and having $L_\Omega(\widetilde{\xi}_n) \le M + L$, where L is an a priori bound for $L_\Omega(\partial G_n)$. This bound exists because the boundary curves of the G_n minimize quasihyperbolic length in the limit. The domains G_n are all contained inside a compact set K, where an estimate like formula (4.7) of Section 4 holds. Therefore there is a constant N such that $L_{\mathbb{C}}(\widetilde{\xi}_n) \le N$ for all n.

We can reparametrize $\widetilde{\xi}_n : [0, N] \to \Omega$ so that the sequence becomes uniformly Lipschitz. By Arzelá-Ascoli there is a subsequence converging uniformly to a Lipschitz path $\xi_\infty \subset G_n$ that joins the two boundary curves $\alpha_\infty = \lim_{n \to \infty} \alpha_n$ and $\beta_\infty = \lim_{n \to \infty} \beta_n$ homotopic in Ω to α_1 and β_1, respectively. □

7. Structure theorem

Definition 7.1. Given any domain $\Omega \subsetneq \mathbb{C}$, endowed with its quasihyperbolic metric, we define:

A *funnel* is the closed subset of Ω placed between a quasihyperbolic geodesic and a connected component of $\overline{\mathbb{C}} \setminus \Omega = \Omega^c \cup \{\infty\}$ with more than one point.

A *puncture* in Ω is the closed subset of Ω lying between a simple closed quasihyperbolic geodesic with length 2π and an isolated point in Ω^c.

The *puncture at infinity of* Ω exists only when Ω^c is bounded, in which case it is the collared end in Ω defined by the inclusion $\mathbb{C} \setminus D \subset \Omega$, where D is any closed disc that contains Ω^c.

Generalizing the classical definition with negative curvature to our context, we call *Y-piece* a compact geodesic domain bounded by three limit geodesics. Likewise we call *exterior Y-piece* a non-compact geodesic domain that contains the puncture at infinity and is bounded by two limit geodesics.

Fig. 3 shows a Y-piece bounded by three quasihyperbolic geodesics: the one on the right of Fig. 1 and two circles, each surrounding a puncture of the domain (Case 1 in Section 4).

Near a puncture $p \in \Omega^c$, the quasihyperbolic metric is cylindrical; in particular, this neighborhood is filled with quasihyperbolic minimizers that surround p and have quasihyperbolic length equal to 2π. By contrast, the Poincaré metric near an isolated point $p \in \Omega^c$ has a cusp-like shape and there is no Poincaré geodesic surrounding p.

According to the results in Section 4, the puncture at infinity is the only collared end with no quasihyperbolic minimizer in its homotopy class.

An exterior Y-piece can be cut by a Jordan curve into a pair of pants and a puncture at infinity, but the cutting curve can never be a quasihyperbolic minimizer.

In the arguments of Section 6, the outer boundary can be replaced by the point ∞ of $\overline{\mathbb{C}}$ and everything works, giving rise to noncompact sets that are still closed subsets of $\mathbb{C}$. That is how we construct funnels or generalized Y-pieces as limits of the Riemannian analogues.

Proposition 7.2. *Let $\Omega \subsetneq \mathbb{C}$ be a domain, endowed with its quasihyperbolic metric. Every quasihyperbolic geodesic domain $G \subset \Omega$ is a finite union (with pairwise disjoint interiors) of Y-pieces and, at most, an exterior Y-piece. Furthermore, the exterior Y-piece appears in this union if and only if Ω^c is a compact set and the quasihyperbolic geodesic domain contains a neighborhood of infinity in Ω.*

Proof. We denote by $\gamma_1, \gamma_2, \ldots, \gamma_k$ the quasihyperbolic limit geodesics in ∂G. We can choose pairwise disjoint Jordan curves $g_1, g_2, \ldots, g_k$ in Ω such that $g_j \in [\gamma_j]$ for each j. Let G' be the closed region with boundary $g_1 \cup \cdots \cup g_k$. Topologically G' is a disc with $k-1$ holes (or the complex plane with k holes) and we can cut it into finitely many pairs of pants $Y_1, \ldots, Y_s$ (and maybe also an exterior Y-piece Y_0). We consider the set $\{g_1, \ldots, g_k, \eta_1, \ldots, \eta_h\}$ of pairwise disjoint Jordan curves in $\cup_n \partial Y_n$ and modify it in the following way. For $i = 1, \ldots, k$ replace g_i with γ_i. For $j = 1, \ldots, h$ choose a quasihyperbolic limit geodesic $\gamma_{k+j} \in [\eta_j]$, that exists because η_j separates two pieces none of which is a puncture at infinity. By Theorem 5.4, the quasihyperbolic limit geodesics $\gamma_1, \ldots, \gamma_{k+h}$ do not cross; therefore $\gamma_{k+1}, \ldots, \gamma_{k+h}$ lie inside G and in fact split it into the required finite union of Y-pieces and, perhaps, one exterior Y-piece in addition. $\square$

Theorem 7.3. *Let $\Omega \subsetneq \mathbb{C}$ be a domain, endowed with its quasihyperbolic metric, which is neither simply nor doubly connected. There exists a set $H \subseteq \Omega$, union of countably many closed domains each one of which is a Y-piece, a funnel, a puncture or an exterior Y-piece, in such a way that Ω is the disjoint union of the closure $\overline{H}$ and simply connected open sets. Any two of the closed domains that make up H either are disjoint or share exactly one boundary curve.*

Furthermore, the exterior Y-piece appears in this decomposition if and only if Ω^c is a compact set.

Assume first that the fundamental group of Ω is finitely generated. Then we prove that $\overline{\mathbb{C}} \setminus \Omega$ has a finite number of connected components. We do it by contradiction, the references here are [36] and [13]. Suppose that $\overline{\mathbb{C}} \setminus \Omega$ has infinitely many components; then the Čech cohomology group $\check{H}^1(\overline{\mathbb{C}} \setminus \Omega)$ would have infinite rank. The rank of the reduced Čech cohomology group $\tilde{\check{H}}^1(\overline{\mathbb{C}} \setminus \Omega)$, which equals that of $\check{H}^1(\overline{\mathbb{C}} \setminus \Omega)$ minus one, would also be infinite. By Alexander-Pontryagin duality [36, page 445] the reduced homology group $\tilde{H}_1(\Omega)$ and the homology group $H_1(\Omega)$ would both have infinite rank. But $H_1(\Omega)$ is the abelianization of $\pi_1(\Omega)$ [13, page 63] and is thus finitely generated. This contradiction shows that $\mathbb{C} \setminus \Omega$ has a finite number $k+1$ of connected components $C_0, C_1, \ldots, C_k$, with $\infty \in C_0$. Since $\pi_1(\Omega)$ is a free group in at least two generators, its abelianization $H_1(\Omega)$ is a free abelian group of rank at least two. This implies $k+1 \geq 2$, that is $k \geq 1$.

For each $1 \leq j \leq k$, let F_j be a funnel or puncture in Ω such that C_j is contained in the interior of the quasihyperbolic limit geodesic ∂F_j. If Ω^c is not compact, i.e. $C_0 \neq \{\infty\}$, let F_0 be the funnel in Ω between C_0 and a quasihyperbolic limit geodesic ∂F_0 that separates $C_1, \ldots, C_k$ from C_0. Then the closure of $\Omega \setminus \cup_{j=1}^n F_j$ (if Ω^c is compact) or $\Omega \setminus \cup_{j=0}^n F_j$ (if Ω^c is not compact) is a quasihyperbolic geodesic domain, and Proposition 7.2 gives the result in this case.

Assume now that Ω has infinitely generated fundamental group. The proof in this case will take up the rest of this section, including proofs of some lemmas and a proposition. Fix a point $p_0 \in \Omega$. Since Ω^c has more than one point, we can consider the Poincaré metric ρ in Ω, a complete Riemannian metric with constant curvature -1. As ρ is real analytic, the boundary of the Poincaré ball $B_\rho(r)$ centered at p_0 is a finite union of pairwise disjoint Jordan curves (see [12, Theorem 1.2]) except for $r \in X$ where X is some countable set of numbers. Start with $r_1 \notin X$ such that the fundamental group of the ball $B_\rho(r_1)$ induces a subgroup of $\pi_1(\Omega)$ with at least two generators. Inductively, once r_{n-1} has been chosen we take $r_n \notin X$ with

$r_n > \max\{r_{n-1}, n\}$. Each $B_n = B_\rho(r_n)$ induces a non-cyclic subgroup of $\pi_1(\Omega)$ and has boundary made of finitely many pairwise disjoint Jordan curves.

Call a boundary component of ∂B_n *inessential* if it is contractible in Ω, and *essential* if it is not contractible in Ω. Let $\widehat{B}_n \subset \Omega$ be the union of B_n with the closures of the interiors of its inessential boundary components.

Now $\partial \widehat{B}_n$ is made of the essential boundary components $\{\eta_i^n\}_{i \in I_n^0}$ of ∂B_n. In particular, write η_0^n for the outer component; which surrounds all other boundary components. Replace the family of curves $\{\eta_i^n\}_{i \in I_n^0}$ by a family of quasihyperbolic limit geodesics $\{\gamma_i^n\}_{i \in I_n}$, using the following inductive rules. If the class $[\eta_i^n]$ is not in Case of Section 4, and it does not appear among the classes $[\eta_j^{n-1}]$ (in particular, if $n = 1$), then let γ_i^n be a quasihyperbolic limit geodesic in this class and include the index i in I_n. If $n > 1$ and there is $\gamma_j^{n-1} \in [\eta_i^n]$, then choose $\gamma_i^n = \gamma_j^{n-1}$ and include the index i in I_n. If the outer component η_0^n is in Case 1 of Theorem 4.3, forget this curve and exclude the index 0 from I_n. When done, either $I_n = I_n^0$ or $I_n = I_n^0 \setminus \{0\}$.

Fix a ball B_n. Since $\{\eta_i^n\}_{i \in I_n}$ are pairwise disjoint Jordan curves, Theorem 4.3 says that the quasihyperbolic limit geodesics $\{\gamma_i^n\}_{i \in I_n}$ do not cross, and there is a quasihyperbolic geodesic domain G_n bounded by them.

Lemma 7.4. *Let γ be a quasihyperbolic limit geodesic for which there is a natural number N with $\gamma \subset \partial G_n$ for every $n \geq N$. Then γ is the border of a funnel or a puncture in Ω.*

Proof. It is well-known that $\mathrm{dist}_\rho \leq 2 d_\Omega$ (see, e.g., [1, Theorem 1-11]). For $n \geq N$, let us consider the Jordan curve $\eta_n \subset \partial B_\rho(r_n)$ which is freely homotopic to γ. Since $2 \liminf_{n \to \infty} d_\Omega(p_0, \eta_n) \geq \lim_{n \to \infty} \mathrm{dist}_\rho(p_0, \eta_n) = \lim_{n \to \infty} r_n = \infty$, and η_n belongs to a single non-trivial free homotopy class for every $n \geq N$, Theorem D gives that $\{\eta_n\}$ converges to a collared end F. Since γ is a quasihyperbolic limit geodesic and $\eta_n \in [\gamma]$ for every $n \geq N$, the collared end F must be a funnel or a puncture. $\square$

Let us continue now with the proof of Theorem 7.3. By construction we have $G_n \subseteq G_{n+1}$. We can take a subsequence of radii $\{r_h\}$ such that $G_h \subsetneq G_{h+1}$, and besides, if $\partial G_h \cap \partial G_{h+1}$ contains some quasihyperbolic limit geodesic γ, then γ is also in ∂G_N for all $N > h$ (such γ is, by Lemma 7.4, the border of a funnel or a puncture). This subsequence can be constructed because, once we have arrived at the quasihyperbolic geodesic domain G_h, we only need to examine the long-term behavior of a finite number of boundary components, namely, those of G_h.

By Proposition 7.2, each connected component of the closure of $G_{h+1} \setminus G_h$ is a finite union (with pairwise disjoint interiors) of Y-pieces and, at most, an exterior Y-piece.

For each h, let us define H_h as the closed subset of Ω obtained as the union of G_h and the funnels and punctures whose boundaries are contained in ∂G_h. Define also H as the union $H := \cup_h H_h$.

By construction, any two quasihyperbolic limit geodesics $\gamma_h \subset \partial H_h$ and $\gamma_{h+1} \subset \partial H_{h+1}$ are non-homotopic in Ω.

If $\Omega = H$ there is nothing else to prove, but $\Omega \setminus \overline{H}$ can be a non-empty set, see Examples 8.1 and 8.2. In any case H "captures all the homotopy of Ω"; let us see that it captures even more.

Lemma 7.5. *Every Jordan curve $\alpha_0 \subset \Omega$ with non-trivial homotopy class intersects the set H.*

Proof. Choose a radius r_h, so that the ball $B = B_\rho(r_h)$ contains α_0. Let G_h be the quasihyperbolic geodesic domain that corresponds to B (each closed geodesic in ∂G_h is freely homotopic to a closed curve in ∂B).

Part 1. Let us see that α_0 intersects G_h or is homotopic to an essential boundary component of B.

For each connected component η_i of ∂B we have either $\eta_i \subset \operatorname{Int}(\alpha_0)$ (possible only for the inner components of ∂B) or $\eta_i \cap \operatorname{Int}(\alpha_0) = \emptyset$. If no essential component η_i is contained in the interior of α_0, then we would have $\operatorname{Int}(\alpha_0) \subset \widehat{B} \subset \Omega$, and $[\alpha_0]$ would be trivial, contrary to our hypotheses. If only one essential component η_{i_0} lies interior to α_0, then $\alpha_0 \cup \eta_{i_0}$ is the boundary of an annulus contained in $\widehat{B}$ and α_0 is homotopic to η_{i_0}, as claimed. If all essential inner components of ∂B lie interior to α_0, then α_0 is homotopic to the outer component of ∂B, again proving our claim.

The remaining possibility is that there are three essential inner components of ∂B, say η_1, η_2, η_3, the first two lying interior to α_0 and the third one exterior to α_0. It is possible to choose, for $j = 1, 2, 3$, a point $z_j \in \Omega^c \cap \operatorname{Int}(\eta_j)$.

For $j = 2, 3$, let $\gamma_j \subset \partial G_h$ be the quasihyperbolic limit geodesic chosen in $[\eta_j]$. If α_0 intersects γ_2 or γ_3, the claim is true. Assume that α_0 is disjoint from $\gamma_2 \cup \gamma_3$. If γ_3 is contained in the interior of α_0, then we would have $i(\gamma_3, z_3) = 0 \neq i(\eta_3, z_3)$, impossible, hence γ_3 lies exterior to α_0. If γ_2 lied exterior to α_0, then a close enough Jordan curve $\widetilde{\gamma}_2$ would also lie exterior to α_0 and either $\operatorname{Int}(\alpha_0) \subset \operatorname{Int}(\widetilde{\gamma}_2)$ or $\operatorname{Int}(\alpha_0) \cap \operatorname{Int}(\widetilde{\gamma}_2) = \emptyset$; in the first case we would have $i(\widetilde{\gamma}_2, z_1) = \pm 1 \neq i(\gamma_2, z_1)$, impossible; in the second case we would have $i(\widetilde{\gamma}_2, z_2) = 0 \neq i(\gamma_2, z_2)$, again impossible; therefore γ_2 lies interior to α_0.

Now γ_2 lies interior to α_0 while γ_3 lies exterior to α_0. The set G_h thus visits the interior and the exterior of α_0 and, since by Proposition 6.4 it is path connected, it must intersect α_0.

Part 2. Suppose α_0 is homotopic to an essential boundary component η_i of B, hence homotopic to a quasihyperbolic limit geodesic γ_i in ∂G_h or perhaps to the puncture at infinity.

Assume first that α_0 is homotopic to the puncture at infinity, which in turn is contained in an exterior Y-piece P. It is impossible to have $\alpha_0 \cap P = \emptyset$, because then α_0 could not be homotopic to η_i. Hence, α_0 intersects H in this case.

Assume now that α_0 is homotopic to a quasihyperbolic limit geodesic γ_i in ∂G_h. If γ_i is not in ∂G_{h+1}, then α_0 is not homotopic to any essential boundary component of $B_\rho(r_{h+1})$ and, by Part 1, it intersects G_{h+1}. By Lemma 7.4, the only alternative option for γ_i is to be the boundary of a funnel or puncture F. If some non-empty part $\xi \subset \alpha_0$ lies on γ_i or is on the side of γ_i where F is, then α intersects F.

Finally, α_0 could be a Jordan curve homotopic to the boundary γ_i of F but disjoint from F. But on the side of γ_i opposite to F we must have another piece P of the decomposition, and if α_0 intersects P, then it intersects H and we are done. Now P cannot be a funnel or a puncture, because then we would have $\Omega = F \cup P$, a domain with cyclic fundamental group. Thus P is either a Y-piece or an exterior Y-piece. It is impossible that α_0 be disjoint from $F \cup P$, because then α_0 could not be homotopic to γ_i. Hence, α_0 intersects H and the proof is finished. □

The following result completes the proof of Theorem 7.3.

Proposition 7.6. *Each connected component V of $\Omega \setminus \overline{H}$ is simply connected.*

Proof. Let $\gamma_0 \subset V$ be any loop. Slightly perturb γ_0 into a closed path $\gamma \subset V$ in general position. This does not change the homotopy class, but now γ has a finite number of transverse self-intersections and $\mathbb{C} \setminus \gamma$ has finitely many bounded components, each contractible. Let U be any of those components.

The Jordan curve $\partial U \subset \gamma \subset V$ is disjoint from $\overline{H}$ and, by Lemma 7.5, it is contractible in Ω. Thus $U \subset \Omega$.

Since the connected set $\overline{H}$ is disjoint from ∂U, either $\overline{H} \subset U$ or $\overline{H} \cap U = \emptyset$. But if we had $\overline{H} \subset U$ then H would induce the trivial subgroup in $\pi_1(\Omega)$, which is false, hence $\overline{H}$ and U are disjoint. Then U is a connected open subset of $\Omega \setminus \overline{H}$ and it intersects V, therefore $U \subset V$.

Since all bounded components of $\mathbb{C} \setminus \gamma$ are contained in V, the path γ is contractible in V and so is γ_0. □
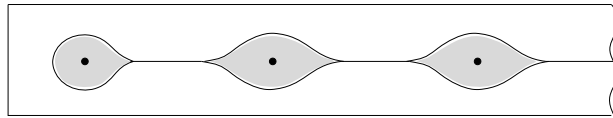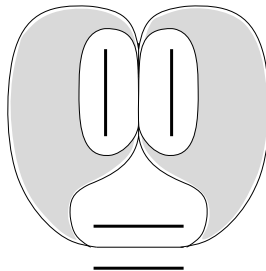
Fig. 4. A decomposition with nonempty V.

Fig. 5. A Y-piece.

8. Examples

The following example shows that $\Omega \setminus \overline{H}$ may be non-empty.

Example 8.1. Let us define

$$\mathbb{D} = \big\{ z \in \mathbb{C} : |z| < 1 \big\}, \quad \Omega = \mathbb{D} \setminus \cup_{n=1}^{\infty} \big\{ -1 + 1/n \big\}, \quad W = \big\{ x + iy \in \mathbb{D} : x > 1/2 \big\}.$$

The set H corresponding to this domain contains no outer funnel and no exterior Y-piece. It consists only of quasihyperbolic geodesic domains G_h and the punctures around the points $-1 + 1/n$. We claim that every quasihyperbolic limit geodesic in Ω is disjoint from W, hence the G_h are disjoint from W. Since the punctures are also disjoint from W, at least one simply connected piece is needed in the decomposition of Ω.

Let γ be a quasihyperbolic limit geodesic in Ω and suppose that there exists $z_0 = x_0 + iy_0 \in \gamma$ with $x_0 > 1/2$. The connected component γ_0 of $\gamma \cap \overline{W}$ that contains z_0 is an arc joining two points $1/2 + iy_1, 1/2 + iy_2 \in \mathbb{D}$. For each $x + iy \in \gamma_0$ we have $d_{\mathbb{C}}(x + iy, \partial\Omega) \le d_{\mathbb{C}}(1/2 + iy, \partial\Omega)$, with strict inequality for $x_0 + iy_0$. Therefore, if g is the Euclidean segment joining $1/2 + iy_1$ and $1/2 + iy_2$, we have $L_{\Omega}(g) < L_{\Omega}(\gamma_0)$ and, since g and γ_0 are homotopic in Ω rel endpoints, we would deduce that γ is not minimizing for quasihyperbolic length in its homotopy class.

Example 8.2. Let Ω be the plane domain

$$\Omega = \big\{ x + iy : x > 0, \, |y| < 1 \big\} \setminus \cup_{n=1}^{\infty}\{5n\}.$$

Lemma 3.3 implies that the decomposition is $\Omega = H \cup V$, where the closed set H is the union of the shaded regions and straight segments shown in Fig. 4 while the simply-connected open set V is the (non-shaded) rest of the domain. The boundary ∂V is a local quasihyperbolic geodesic with infinite quasihyperbolic length, and it touches itself along infinitely many straight arcs.

Example 8.3. An application of Lemma 3.3 is the existence of a Y-piece with the shape shown in Fig. 5. The domain Ω is the complex plane minus two pairs of parallel straight segments, indicated by thick lines in the figure, which delimit narrow rectangular corridors in Ω. The Y-piece consists of the shaded area plus the thin segments (one vertical, one horizontal). One of the boundary quasihyperbolic geodesics goes twice

along each corridor, touching itself in a whole arc. The other two boundary quasihyperbolic geodesics are Jordan curves. The three boundary quasihyperbolic geodesics have a common touch along the thin vertical segment.

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