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Stability, Global Dynamics and Markov Equilibrium in Models of Economic Growth

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Stability, Global Dynamics and Markov Equilibrium in Models of Economic Growth

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Abstract

This paper studies the local and a global dynamics of two-sector models of endogenous growth with economy-wide external effects and taxes on capital and labor. The local analysis classifies the parameter space depending on the number of stationary solutions and local stability of equilibria. Taxes on labor and subsidies to education may determine the existence of poverty traps and indeterminacy. The global analysis shows that if externalities and taxes are not too big then the equilibrium path is monotone and therefore a continuous Markov equilibrium can be defined.

Journal of Economic Literature Classification Numbers: C62, E62, O41.

Key Words: competitive equilibrium; stability; Markov equilibrium; externalities; taxes.

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1 Introduction

This paper is concerned with the local and global dynamics of a wide class of two-sector endogenous growth models with economy-wide external effects and distortionary taxation. Our analysis comprises the Lucas [22] framework to which we have added new elements. First, we include economy-wide external effects in the production function of the physical good and in the law of motion of physical capital. A number of authors have studied the influence of external effects in models of economic growth; however, most of them have limited the scope of these external effects to the sector-specific level (see Ben-Gad [2], Benhabib, Meng and Nishimura [4] and Benhabib and Nishimura [6]). The inclusion of sector-specific instead of economy-wide external effects simplifies the analysis, but implies that the external effects at the aggregate level are not taken into account. In contrast, our model includes economy-wide external effects on the average level of physical and human capital. The presence of economy-wide external effects has been used to explain some of the observed empirical regularities. Lucas [22] explains the wage differences for workers in different countries through the external effect on the average level of human capital. García-Belenguer and Santos [15] use this sort of external effects to explain the different growth rates of physical capital, human capital and output. Another reason to include economy-wide instead of sector-specific external effects is that the decisions in the educational sector seem to have a global effect that affects the productivity of the whole economy and that, given the small relative size of the educational sector, surpass the sector-specific level.

Besides external effects, our analysis also considers distortionary taxes on physical capital and labor returns and subsidies to education. Several papers have studied the effect of taxes and externalities on local dynamics, but most of them have considered these effects separately. For instance, Bond, Wang and Yip [8], Mino [26] and Raurich [31] study the influence of taxes on local dynamics, while Benhabib and Gali [3], Benhabib and Perli [7] and Chamley [10] include external effects in their analysis. Thus, the influence of economy-wide external effects coupled with distortionary taxes on the dynamics of growth models remains to be explored (see Ben-Gad [2] for a local analysis with taxes and sector-specific external effects). Our analysis differs from previous ones in that we study both the local and the global dynamics of distortionary taxation together with economy-wide external effects. Our results show that the influence of taxes on labor and subsidies to education on the growth rate of the economy depends on the parameter structure of the economy. More precisely, we find that a rise in taxes and/or subsidies to education increases the growth rate of the economy in most cases. But if the production function of human capital exhibits constant returns to scale, sufficiently high values of the elasticity of intertemporal substitution may entail that the rise in taxes and/or subsidies lowers the growth rate of the economy, provided that external effects are high enough.

A main concern of the literature on local dynamics has been the necessary conditions for the existence of indeterminacy. An equilibrium is indeterminate if there is a continuum of equilibrium paths that converge to the same stationary solution, implying that the agents may coordinate in different equilibria with different growth rates. Bond, Wang and Yip [8] prove the existence of indeterminate equilibria under asymmetric taxation of production factors that implies factor intensity reversals. However, the necessary conditions for the existence of indeterminacy that they find require that the size of the asymmetries is big. Raurich [31] introduces non-productive government spending and obtains indeterminate equilibria for more realistic parameter values. Among the studies that include sector-specific external effects Ben-Gad [2] considers distortionary taxes; both Ben-Gad [2] and Raurich [31] obtain indeterminacy for values of the elasticity of intertemporal substitution less than one. The functional specification of our model differs from these three works. First, we consider economy-wide instead of sector-specific external effects. Secondly, Bond, Wang and Yip [8] and Raurich [31] include physical capital in the production of human capital. This assumption implies that in order to have a balanced growth path (BGP henceforth) no external effect can be present. Therefore, in order to allow for the existence of a BGP in which capital stocks grow at different rates, we do not consider physical capital in the production of human capital.

Our local analysis considers the possibility of more than one stationary solution through the existence of decreasing returns to scale in the effort devoted to schooling. This sort of production function for the human capital technology has already been used by Alonso-Carrera [1] and generates multiple stationary solutions. Also, Ladrón de Guevara, Ortigueira and Santos [20] and [21] obtain multiple stationary solutions in two-sector models with leisure. Our results show that there could be two stationary solutions if the elasticity of intertemporal substitution is sufficiently high (greater than one). Regarding the stability of the stationary solutions, indeterminacy is only possible if the elasticity of intertemporal substitution is greater than one. The more realistic case occurs when there are two steady states. If there is only one steady state we need higher values of the external effects in order to have indeterminacy. We obtain indeterminate equilibria for a broader parameter range than in Benhabib and Perli [7] but we need a higher elasticity of substitution than in Ben-Gad [2] and Raurich [31]. Finally, our analysis shows that there is a wide region of the parameter space in which the economy displays no stationary solution. Moreover, we find that subsidies to education and taxes on labor may determine whether the economy belongs to this region or not.

Besides the local properties of the stationary equilibrium, we also offer a qualitative global study of the equilibrium dynamics for points arbitrarily far from the stationary solution. Global dynamics is more complex to analyze than local dynamics since linearizations are not effective and closed form solutions are usually not available. Research on global dynamics has usually been concerned with studying the existence of Markov -or recursive- equilibrium. Since the work of Lucas and Prescott [23] several authors have studied this existence problem in one sector models (e.g., Lucas and Stokey [24], Coleman [11], Greenwood and Huffman [17] and Datta, Mirman and Reffett [13]). In these papers the most common method of proof relies on the monotonicity of the equilibrium path. Conditions that insure the monotonicity of the equilibrium dynamics in two-sector models are less well known. For instance, Ortigueira and Santos [30] and Santos [32] have shown the lack of existence of Markov equilibrium for non-optimal economies with one and two sectors. Finally, since the work of Benhabib and Nishimura [5], several other works have obtained conditions for the existence of cycles in overlapping generations models (e.g., Michel and Venditti [25]) or in the presence of external effects (e.g., Venditti [34] and Nishimura and Venditti [29]).

Most of the research on global analysis have focussed on one-sector models. Results for two-sector models are in Caballé and Santos [9] and Xie [35]. Caballé and Santos [9] show the monotonicity of the solution in an optimal two-sector growth model. Xie [35] considers an specific parametrization of the model in Lucas [22] with the human capital externality and obtains the closed form solution of the model. In our paper we obtain results on the global dynamics of two-sector endogenous growth models with economy-wide external effects and distortionary taxation that include the frameworks in [9], [22] and [35]. More precisely, we give a set of sufficient conditions that guarantee the monotonicity of the equilibrium solution and that allow to define a Markov equilibrium. These sufficient conditions require that the size of the external effects and taxes on capital is not too big. We also show that if there are no external effects there exits a Markov equilibrium for any tax scheme.

The rest of the paper is organized as follows: Section 2 describes the model, solves the competitive equilibrium and defines the BGP. Section 3 is concerned with the global dynamics and the existence of Markov equilibrium. Section 4 is devoted to the local analysis and contains the main results concerning the uniqueness and stability of the BGP. Section 5 concludes. The proofs of the main results are relegated to the appendix.

2 The Model

The model economy is assumed to consist of a dynasty of representative agents. The number of agents at period t is represented by L_t and its law of motion is given by the exogenous process $L_t = L_o e^{nt}$, where $n \geq 0$ is the population growth rate. Each agent obtains income from the working activities, the rents of capital and the subsidies to education. This income is devoted to consumption, investment and tax payments. The stream of consumption yields utility represented by the function,

$$\int_0^\infty \frac{c_t^{1-\sigma}}{1-\sigma} e^{-\rho t} L_t. \tag{1}$$

Note that $\sigma > 0$ represents the inverse of the elasticity of intertemporal substitution, c_t is per capita consumption in period t and ρ is the discount factor.

The production of the aggregate good is described by a standard Cobb-Douglas neoclassical production function which includes two external effects, corresponding to the possible influence of the average levels of physical and human capital on the level of productivity. More precisely,

$$Y_t = A(k_t^e)^\varphi (h_t^e)^\phi K_t^\alpha \left(L_t u_t h_t\right)^{1-\alpha} \tag{2}$$

where Y_t is real aggregate output, K_t is the total stock of physical capital in the economy and h_t is the average level of human capital. All agents are endowed with one unit of time which can be allocated between working and schooling activities. Then, u_t is the average fraction of time devoted to work whereas $(1-u_t)$ is the average fraction of time devoted to schooling. Physical capital per unit of labor is represented by $k_t = K_t/(L_t u_t)$. Superscript e refers to the external or spillover effects, which are not directly taken into account for the decisions made by the individual. The size of the external effects is represented by the coefficients φ and ϕ, which are supposed to be non-negative. Note that there are constant returns to scale at the individual level, and $0 < \alpha < 1$. The term A represents the level of productivity available to the economy.

At each date t, output in the economy may be either consumed or invested:

$$Y_t = L_t c_t + I_t. \tag{3}$$

The stock of physical capital in the economy evolves according to the following law of motion,

$$\dot{K}_t = q_t^{1-\varepsilon} \left(h_t^e\right)^{\varepsilon} I_t - \pi_K K_t. \tag{4}$$

Here $\pi_K > 0$ denotes physical capital depreciation and q_t is the level of exogenous technological progress for physical capital accumulation. This exogenous process represents the current state of the technology for producing capital goods and expresses the idea that new vintages of capital may be more productive than existing ones; e.g., see Hulten [18] and Greenwood, Hercowitz and Krusell [16]. We assume that variable q_t evolves according to the law $q_t = q_0 e^{gt}$ for $g \geq 0$ and for all t. Also, the process q_t is embodied more efficiently the greater the average level of human capital. This effect is represented by the externality h_t^e and the parameter $\varepsilon \in [0, 1)$ reflects its size. This type of external effect has been employed in the empirical investigation carried out by García-Belenguer and Santos [15]. The influence of human capital on the adoption of new technologies conveys the idea that education enhances one's ability to receive and understand information, and that this information processing is necessary to use new technologies. There are several reasons to believe that this adoption process is not carried out at the individual level and that some degree of spillover effects are present (see Nelson and Phelps [28]).

The technology for the production of human capital is represented by a linear function in the stock of human capital. Thus, the stock of human capital evolves according to the following law of motion,

$$\dot{h}_t = B\,(1-u_t)^{\gamma}\,h_t - \pi_h h_t \tag{5}$$

where $B > 0$ represents the level of the productivity available and $(1-u_t)$ is the average fraction of time devoted to schooling. Since $\gamma \in (0,1]$ there may be decreasing returns to scale. Human capital depreciates at the rate $\pi_h > 0$.

Assumption 1 : $1-\varphi-\alpha > 0$

Assumption 2 : $B-\pi_h > 0$

Assumption 1 implies that there are decreasing returns to scale in physical capital. Assumption 2 is a necessary condition to have positive growth in the human capital sector.

At every moment in time each dynasty must satisfy the following budget balance so that total expenditures on consumption and investment cannot be greater than total income.

$$L_t c_t + I_t = (1-\tau_K) r_t K_t + (1-\tau_l) w_t u_t h_t L_t + \xi w_t (1-u_t) h_t L_t + T_t. \tag{6}$$

In (6) total income is the sum of after tax returns to physical capital $(1-\tau_K)r_t K_t$, after tax returns to labor $(1-\tau_l)w_t u_t h_t L_t$, subsidies to education $\xi w_t(1-u_t)h_t L_t$ and transfers from the government T_t. $\tau_K \in [0,1)$ represents the tax rate on the returns to physical capital $r_t K_t$, while $\tau_l \in [0,1)$ is the tax rate on returns to labor. The competitive salary in efficiency units is w_t and $\xi \geq 0$ represents subsidies to education. The next assumption restricts the set of feasible values for taxes and subsidies. In the absence of this condition the agent has no incentive to work and devotes all the time to schooling.

Assumption 3 : $1-\tau_l-\xi > 0$.

2.1 Competitive Equilibria

Definition 1 : *A competitive equilibrium for this economy is a path for the quantities* $\{c_t, I_t, u_t, K_t, h_t, T_t\}_{t=0}^{\infty}$*, prices* $\{r_t, w_t\}_{t=0}^{\infty}$ *and constant tax rates and subsidies* τ_K*,* τ_l *and* ξ *such that taking prices as given:*

(i) *The representative agent chooses the path for consumption c_t, investment I_t and effort devoted to working activities u_t for each period t that maximize (1) subject to (2)-(6).*

(ii) *The firm chooses the amount of physical capital $\mathcal{K}_t$ and labor N_t that maximize profits at every period t,*

$$\max_{\mathcal{K}_t, N_t} \{A(k_t^e)^{\varphi}(h_t^e)^{\phi}\mathcal{K}_t^{\alpha}N_t^{1-\alpha} - r_t\mathcal{K}_t - w_tN_t\}. \tag{7}$$

(iii) *Markets clear and agent's beliefs must be consistent at every period t,*

$k_t^e = k_t, \quad h_t^e = h_t, \quad \mathcal{K}_t = K_t, \quad N_t = u_th_tL_t, \quad T_t = \tau_K r_t K_t + \tau_l w_t u_t h_t L_t - \xi w_t(1 - u_t)h_tL_t.$

Definition 2 : *A balanced growth path for this economy is a competitive equilibrium such that all the variables $\{c_t, I_t, K_t, h_t, T_t\}$ always grow at a constant rate and u_t remains constant.*

Let η_Y, η_c, η_I, η_K and η_h be the growth rates of output, consumption, investment, physical capital and human capital in the BGP. Then, using (2)-(4) we obtain the following conditions,

$$\eta_Y = \eta_I = \eta_c + n \tag{8}$$

$$\eta_K = [g(1-\varepsilon) + (1-\alpha+\phi+\varepsilon)\eta_h]\frac{1}{1-\varphi-\alpha} + n \tag{9}$$

$$\eta_c = \frac{g(1-\varepsilon)(\varphi+\alpha)}{1-\varphi-\alpha} + \frac{1-\alpha+\phi+\varepsilon(\varphi+\alpha)}{1-\varphi-\alpha}\eta_h. \tag{10}$$

The growth rate of human capital in the BGP is $\eta_h = B(1-u^*)^{\gamma-1} - \pi_h$, where superscript $*$ represents a steady state value.

Assumption 4 : $(1-\sigma)\eta_c(0) < \rho - n < \iota(\gamma) + (1-\sigma)\eta_c(1)$.

Here $\eta_c(0)$ and $\eta_c(1)$ are the values of the growth rate of consumption when $u = 0$ and $u = 1$. $\iota(\gamma)$ is the derivative of the production function of human capital evaluated at $u = 1$; when $\gamma = 1$ this derivative is Bh_t while if $\gamma < 1$ its value is ∞. The first inequality in assumption 4 plays the role of a transversality condition, and it is equivalent to condition (12) in Uzawa [33]. If this condition is not satisfied $(1-\sigma)\eta_c(0) > \rho - n$ and (1) may take an unbounded value. Conversely, the second inequality gives a lower bound for the productivity of the human capital sector. If investment in human capital is not profitable then an interior BGP is not achieved.[1]

The problem of the agent can be solved by maximizing the current value Hamiltonian,

$$
\begin{aligned}
\mathcal{H} &= \frac{c_t^{1-\sigma}}{1-\sigma} + \lambda_t \left\{ q_t^{1-\varepsilon} \left(h_t^e\right)^\varepsilon \left[\left(1-\tau_K\right) r_t K_t + \left(1-\tau_l\right) w_t u_t h_t L_t + \xi w_t \left(1-u_t\right) h_t L_t \right.\right. \\
&\quad \left.\left. + T_t - L_t c_t\right] - \pi_K K_t \right\} + \mu_t \left\{ B \left(1-u_t\right)^\gamma h_t - \pi_h h_t \right\}
\end{aligned}
$$

where λ_t and μ_t represent the costate variables. Given the utility and production functions, the maximization problem is concave and the maximum principle provides necessary and sufficient conditions that define the set of solutions to the problem.

The first-order conditions provide a system in variables c_t, K_t, h_t and u_t. For convenience this system is presented in the appendix. To facilitate the analysis we follow Mulligan and Sala-i-Martin [27] and define two new stationary variables. These variables are $z_t = [K_t^{\varphi+\alpha-1} L_t^{1-\varphi-\alpha} h_t^{1-\alpha+\phi+\varepsilon} q_t^{1-\varepsilon}]^{\frac{1}{\varphi+\alpha-1}}$ and $a_t = \frac{L_t c_t q_t^{1-\varepsilon} (h_t)^\varepsilon}{K_t}$. Using (8)-(10) one can check that both ratios remain constant along the BGP. Moreover, taking logarithms and differentiating the expressions above we arrive to the following reduced system in variables z, a and u.

$$
\begin{aligned}
\frac{\dot{z}_t}{z_t} &= \frac{\varepsilon-1}{1-\varphi-\alpha} g - \left(n+\pi_K\right) - \frac{1-\alpha+\phi+\varepsilon}{1-\varphi-\alpha} \left[B\left(1-u_t\right)^\gamma - \pi_h\right] \\
&\quad + A z_t^{\varphi+\alpha-1} u_t^{1-\varphi-\alpha} - a_t \\
\frac{\dot{a}_t}{a_t} &= n + (1-\frac{1}{\sigma})\{\left(1-\varepsilon\right) g + \pi_K\} - \frac{\rho}{\sigma} + \left[\frac{\left(1-\tau_K\right)\alpha}{\sigma} - 1\right] A z_t^{\varphi+\alpha-1} u_t^{1-\alpha-\varphi} + \\
&\quad (1-\frac{1}{\sigma})\varepsilon \left[B\left(1-u_t\right)^\gamma - \pi_h\right] + a_t \qquad (11) \\
\frac{\dot{u}_t}{u_t} &= \frac{1-u_t}{\left(1-\gamma\right) u_t + \left(\varphi+\alpha\right)\left(1-u_t\right)} \left\{ B\left(1-u_t\right)^{\gamma-1} \left[\left(1-\alpha+\phi+\varepsilon\right)\left(1-u_t\right) \right.\right. \\
&\quad \left. + \gamma u_t + \frac{\xi\gamma}{1-\tau_l-\xi}\right] + (\varphi+\tau_K\alpha) A z_t^{\varphi+\alpha-1} u_t^{1-\varphi-\alpha} - (\varphi+\alpha) a_t + \left(1-\varepsilon\right) g \\
&\quad \left. - (1-\alpha+\phi+\varepsilon)\pi_h + \left(1-\varphi-\alpha\right)\left(\pi_K+n\right) \right\}.
\end{aligned}
$$

[1]In section 4 and in the appendix we further investigate the influence of these two conditions on the existence of a BGP.

2.2 Calibration

We finally provide some baseline parameters which will be useful in future discussions. We let the inverse of the elasticity of intertemporal substitution $\sigma = 1$, the discount factor $\rho = 0.08$ and the population growth rate $n = 0.015$. For the share of physical capital in the production function of the physical good we take the value $\alpha = 0.35$. In the law of motion of physical capital we set the growth rate of the exogenous technological change to $g = 0.02$ and the depreciation rate of physical capital to $\pi_K = 0.06$. For the law of motion of human capital we have chosen parameter values that imply a growth rate of human capital around 1 per cent in the BGP, thus, we have $B = 0.081$, $\pi_h = 0.035$ and $\gamma = 0.4$. All these parameter values are quite standard [e.g., Lucas (1988)]. More controversial are the values of the external effects φ, ϕ and ε. Regarding the external effects in the production function of the physical good, φ and ϕ, there are many studies that have obtained estimates for these parameters, we follow the work of García-Belenguer and Santos [15] and set $\varphi = \phi = 0.2$. For the external effect of the average level of human capital in the adoption of the exogenous technological progress ε, there is not so much empirical evidence and we take the estimate of Costa Carpena and Santos [12] for a sample of Latin American and OECD countries $\varepsilon = 0.45$. Finally we set the values of the tax rates and subsidies to $\tau_K = \tau_l = 0.2$ and $\xi = 0.05$. This parametrization implies that there is a unique BGP with $u^* = 0.76$ and a growth rate for human capital of 1 per cent.

3 Global Dynamics

Before analyzing the local dynamics of system (11) in section 4, in this section we perform a novel analysis of the global dynamics of growth models. This analysis exploits the homogeneity of the Cobb-Douglas production function and allows us to derive some monotonicity properties of the solution path. More precisely, we give a set of sufficient conditions that guarantee the monotonicity of the solution trajectory at any point for $t \geq 0$. We also show that these conditions insure the existence of a Markov equilibrium. In the second part of this section we give some examples of economies that do not satisfy our sufficient conditions and that may display non-monotone dynamics.

For two sector models, global analysis has been carried out by Caballé and Santos [9] in an optimal version of Lucas [22] model. They prove, using the properties of the value function, the monotonicity of the solution path. Also, Xie [35] obtains the closed form solution for the Lucas [22] model. But his results are only valid for a specific parametrization since he assumes that the share of physical capital is equal to the inverse of the elasticity of intertemporal substitution, in our model $\alpha = \sigma$. The results presented in this section apply to the three models mentioned above and include a wide class of growth models with economy-wide external effects and distortionary taxes and subsidies.

Assumption 5 : $\gamma = 1$.

To simplify the analysis, we make assumption 5 and only consider the case of constant returns to scale in the effort devoted to schooling.

To facilitate the analysis it is convenient to define the stationary variable $x_t = z_t^{\varphi+\alpha-1} u_t^{1-\varphi-\alpha}$. This variable and assumption 5 allow us to define the growth rates of the variables as linear functions in the reduced model. Taking logarithms and differentiating the definition of x_t, we obtain from (11) the following system,

$$\begin{aligned} \frac{\dot{x}_t}{x_t} &= \Delta_x + \Omega_x x_t + \Sigma_x u_t \\ \frac{\dot{a}_t}{a_t} &= \Delta_a + \Omega_a x_t + a_t + \Sigma_a u_t \\ \frac{\dot{u}_t}{u_t} &= \Delta_u + \Omega_u x_t - a_t + \Sigma_u u_t \end{aligned} \tag{12}$$

where the parameters are defined in the appendix. System (12) is useful to summarize some of the results in the literature on global dynamics. As pointed out above, Caballé and Santos [9] study the model in Lucas [22] without external effects.[2] Their parameter selection and the absence of external effects implies that $\Sigma_x = \Sigma_a = 0$, it follows that system (12) can be solved since now it is a triangular system. Regarding the work of Xie [35], if there are no taxes on physical capital, his assumption on α implies that $\Omega_a = \Sigma_a = \Omega_u = 0$, and again system (12) is triangular. Nonetheless, as taxes and spillover effects are introduced in the analysis, the dynamics of the model becomes more complex and the methods used in these works are not valid to study global dynamics.

[2] Caballé and Santos choose $\phi = 0$, Lucas assumes $\phi > 0$. Both have $\varphi = \varepsilon = \tau_K = \tau_l = \xi = 0$.

We turn now to the study of system (12). The main result on monotonicity is stated in theorem 1 and it enables us to prove the existence of a continuous Markov equilibrium in theorem 2. The following set of assumptions on the parameter space will be used in the next theorems.

Assumption 6 : $\sigma > \alpha(1-\tau_K)$

Assumption 7 : $\alpha > \varepsilon + \phi$

Assumption 8 : $\tau_K < \frac{-\varphi(\varphi+\varepsilon+\phi)+(\alpha-\varepsilon-\phi)\alpha(1-\varphi-\alpha)}{\alpha(\varphi+\varepsilon+\phi)+(\alpha-\varepsilon-\phi)\alpha(1-\varphi-\alpha)}$.

Assumption 6 sets an upper bound for the elasticity of intertemporal substitution. Assumption 7 requires that the sum of the external effects on the average level of human capital is less than the share of physical capital in the production function. Assumption 8 sets an upper bound for the tax rate on physical capital.

Our construct of the Markov equilibrium relies on the following result.

Theorem 1 : *Under assumptions 1-8 if the Jacobian matrix of system (12) evaluated at the steady state has at least one eigenvalue with negative real part, the equilibrium path is strictly monotone in variables x, a and u. Moreover, at every initial condition $\dot{x}_t$, $\dot{a}_t$ and $\dot{u}_t$ have the same sign for all $t \geq 0$.*

Proof: See the appendix.

The condition of one eigenvalue with negative real part implies that the manifold that converges to the BGP has at least dimension 1. If there is more than one eigenvalue with negative real part the equilibrium is indeterminate and there is a continuum of equilibrium paths converging to the BGP. In any case, theorem 1 states the monotonicity of the equilibrium trajectories at all points. In section 4 we provide the characterization of the stability of the BGP and describe the set of parameter values that imply at least one eigenvalue with negative real part. The bound imposed on τ_K by assumption 8 includes a wide range of values and in the absence of external effects is 1. This means that taxes on physical capital alone are not enough to have non-monotone dynamics or cycles and that some degree of external effects is needed.

A Markov equlibrium is a list of decision rules $\{a, u\}$ that depend on the state variables (K, h, L, q) such that every trajectory generated by these decision rules is a competitive equilibrium. Given the utility function and the production functions we can define the state variable as the ratio $m = K^{\varphi+\alpha-1}h^{1-\alpha+\phi+\varepsilon}L^{1-\varphi-\alpha}q^{1-\varepsilon}$. Hence, if there is a continuous Markov equilibrium the law of motion of variable m can be expressed by a continuous function $\dot{m} = f(m)$.

THEOREM 2 : *Under assumptions 1-8 there exists a continuous Markov equilibrium for our model economy.*

PROOF: See the appendix.

It is known from Coleman [11] and Greenwood and Huffman [17] that for one-sector economies with flat tax rates on capital returns there always exists a continuous Markov equilibrium. Theorems 1 and 2 imply that these results can be extended to the two-sector economies considered in this section. We state this result in the next corollary.

COROLLARY 3 : *Under the assumptions of theorem 2, if there are no external effects, that is* $\phi = \varphi = \varepsilon = 0$*, a continuous Markov equilibrium does exist for any tax scheme* (τ_K, τ_l, ξ)*.*

3.1 Counterexamples

Theorem 2 provides a set of sufficient conditions that guarantee the existence of a continuous Markov equilibrium. In this subsection we construct three economies that do not satisfy some of these conditions. Our aim is to evaluate the behavior of our model economy when assumptions 5 to 8 are no satisfied. In examples 1 and 2 we consider two parametrizations that do not satisfy assumptions 6 and 7. In example 1 we present an economy that displays non-monotone dynamics while example 2 shows that if those assumptions are not satisfied, it is possible to have closed orbits. In the last example we perform a computational experiment of an economy in which assumption 5 is not satisfied and has decreasing returns to scale in the effort devoted to education.

Example 1

In this first example we consider the case $\gamma = 1$, $\sigma = (1-\tau_K)\alpha$ and $\varepsilon = 0$. This parametrization includes the case of Xie [35][3] and implies that assumption 6 is not satisfied. The main difference between our economy and that of Xie is that, under the parametrization chosen by Xie, it can be obtained a single differential equation for variable u which allows to characterize the complete equilibrium path of the economy. In our case, this simple differential equation in one variable cannot be obtained since there are other external effects ($\varphi > 0$) and taxes ($\tau_K > 0$).[4]

Lemma 4 : *When $\gamma = 1$, $\sigma = (1 - \tau_K)\alpha$ and $\varepsilon = 0$, along the equilibrium path, the quantity $L_t c_t q_t$ is always proportional to the capital stock K_t. Moreover, we have that $a_t = -\Delta_a = \frac{\rho}{\sigma} + (\frac{1}{\sigma} - 1)(g + \pi_K) - n$ for all $t \geq 0$.*

Proof: Substituting $\sigma = (1 - \tau_K)\alpha$ and $\varepsilon = 0$ in the second equation of system (12), one obtains that the only value of a that satisfies the transversality conditions is $a_t = -\Delta_a$ for all $t \geq 0$.

From the previous lemma it follows that (12) reduces to the following planar system,

$$\begin{aligned} \frac{\dot{u}_t}{u_t} &= \Delta_{u+a} + \Omega_u x_t + \Sigma_u u_t \\ \frac{\dot{x}_t}{x_t} &= \Delta_x + \Omega_x x_t + \Sigma_x u_t \end{aligned}$$

where $\Delta_{u+a} = \Delta_u + \Delta_a$. In figure 1 we show the phase diagram of this system.

Linearizing the new system around the steady state we can obtain the expression of the discriminant Λ of the characteristic equation,

$$\Lambda = (x^*\Omega_x + u^*\Sigma_u)^2 - 4x^*u^*(\Omega_x\Sigma_u - \Sigma_x\Omega_u).$$

Therefore, there will be complex eigenvalues only if the second term is negative. From the signs of Ω_x, Σ_x, Ω_u and Σ_u it follows that there will be complex eigenvalues if ϕ, φ or τ_K

[3]Xie assumes $\sigma = \alpha < \phi$ and $\varepsilon = \varphi = \tau_K = \tau_l = \xi = 0$.

[4]Using lemma 4 below and the parametrization of Xie, the differential equation on u is given by the last equation of (12). It can be easily seen that if φ or τ_K are positive, x_t enters the equation.

are high enough. Also, when $\phi > \alpha$ the coefficients of the characteristic polynomial are all positive, implying that both eigenvalues have negative real part.

Now we consider the parametrization of the benchmark economy of section 2 with $\alpha = 0.3$, $B = 0.09$, $\varphi = 0.09$, $\sigma = 0.24$, $\gamma = 1$, $\varepsilon = 0$, $\phi = 0.4$ and $\xi = 0.1$. It can be easily checked that this economy satisfies that $\sigma = (1 - \tau_K)\alpha$ and $\phi > \alpha$. These parameter values imply that assumptions 7 and 8 are not satisfied either. Moreover, there is only one interior BGP at $u^* = 0.6027$ (see section 4 for more details) at which the eigenvalues of the Jacobian matrix are complex and have negative real part, implying that the trajectories spiral in toward the steady state (u^*, x^*). The equilibrium paths are represented in figure 1.

Let us consider now the state variable $m = K^{\varphi+\alpha-1}h^{1-\alpha+\phi}L^{1-\varphi-\alpha}q$. It follows from the previous analysis and the definition of x_t that, along any equilibrium path, the value of m oscillates around the stationary value. Thus, the next proposition shows that there does not exist a Markov equilibrium for this economy.

Proposition 5 : *For the economy described in this example, there does not exist a Markov equilibrium.*

Proof: Let us define our candidate equilibrium function as $\dot{m} = f(m)$. Since the stationary solution is a spiral sink, function $\dot{m} = f(m)$ will circle around the steady state generating an additional countable number of steady state equilibria. But this is not possible since point (u^*, x^*) is assumed to be the only non-degenerate steady state equilibrium. The proof is complete.

Example 2

In this example we present an economy whose equilibrium trajectories are closed-orbits. We choose again $\gamma = 1$, $\varepsilon = 0$ and $\sigma = (1 - \tau_K)\alpha$, but now we also assume $\alpha + \varphi = 1$ and $\alpha = \phi$. These two new assumptions imply that the production function of the physical good exhibits constant returns to scale in physical capital and that the share of physical capital is equal to the external effects on the average level of human capital. All these assumptions imply that now x_t is the state variable since $x_t = m_t = h_t q_t$. Besides, $\Omega_x = \Sigma_u = 0$ entail that the planar system is,

$$\begin{aligned}\dot{u}_t &= u_t(\Delta_{u+a} + \Omega_u x_t)\\ \dot{x}_t &= x_t(\Delta_x + \Sigma_x u_t).\end{aligned}$$

We assume $\Delta_{u+a} < 0$ in order to have an interior steady state. This unique interior steady state is located at $(\frac{-\Delta_x}{\Sigma_x}, \frac{-\Delta_{u+a}}{\Omega_u})$. Also, it follows from the expression of the discriminant in example 1 that the eigenvalues of the Jacobian matrix at the steady state are purely imaginary, which gives no information about stability. The phase diagram of this system is represented in figure 2. The lines that define the stationary solution divide the region $u > 0$, $x > 0$ into four quadrants.

The equations of this system are equivalent to the *predator-prey equations* of Volterra and Lotka (see Hirsch and Smale [19, chapter 12]). From figure 2 it follows that each solution curve (u_t, x_t) moves clockwise around the stationary solution.

Let us consider now a trajectory (u_t, x_t) starting at point $u_0 > \frac{-\Delta_x}{\Sigma_x} > 0$ and $x_0 > \frac{-\Delta_{u+a}}{\Omega_u} > 0$ in quadrant I. Following Hirsch and Smale [19, pp. 261], it can be proved that the trajectory enters quadrant II, and similarly for other quadrants.

We propose the following function,

$$H(u,x) = \Delta_{u+a} log x + \Omega_u x - (\Delta_x log u + \Sigma_x u)$$

defined for $u > 0$ and $x > 0$. By considering the signs of $\partial H/\partial u$ and $\partial H/\partial x$ it is easy to see that the stationary solution (u^*, x^*) is an absolute minimum for H. Also, it can be checked that $H(u,x)$ is constant on the solution curves of our system, that is, $\dot{H}(u,x) = 0$. It then follows that $H - H(u^*, x^*)$ is a Liapunov function and therefore (u^*, x^*) is a stable equilibrium. We can then state the following theorem,

THEOREM 6 : *Every interior trajectory of the system considered in this example is a closed orbit.*

PROOF: See Hirsch and Smale [19, theorem 1, pp. 262].

The previous theorem entails that for any given initial conditions (u_0, x_0) different from (0,0) and (u^*, x^*), the economy will oscillate cyclically. This theorem enables us to prove in the next proposition that no Markov equilibrium exists for this economy. As pointed out above x is now the state variable and it follows that, along any equilibrium path, the value of x oscillates around the stationary value.

Proposition 7 : *For the economy described in this example, there does not exist a Markov equilibrium.*

Proof: Same as for proposition 5.

Example 3

In this example we consider the parameter values of the benchmark economy presented in section 2, except for $\sigma = 0.2$ and $\gamma = 0.9$. This parametrization implies that assumptions 5 to 8 are not satisfied. We perform a computational exercise for this economy and find out that a continuous Markov equilibrium does exist since the computed equilibrium function is continuous and monotone. Under the selected parametrization this economy displays two stationary solutions, $u_1^* < u_2^*$. The first equilibrium at $u_1^* = 0.60$ is indeterminate and the second, at $u_2^* = 0.99$, is saddle path stable. In our exercise we compute the stable manifold of the stationary solution at u_2^*.

For the computation of the equilibrium we have used the method of reverse shooting. The results are presented in figure 3, there $m = K^{\varphi+\alpha-1}h^{1-\alpha+\phi+\varepsilon}L^{1-\varphi-\alpha}q^{1-\varepsilon}$ is the state variable of the economy and $\dot{m}$ is the law of motion. From the results in figure 3 it follows that the law of motion of the state variable is a continuous function of the states. It should be noted that a computational experiment is not a proof of existence of Markov equilibrium, however, when performing the experiment, we have have checked that the derivatives of the dynamical system do not change sign. This supports the idea that a continuous Markov equilibrium does exist for this economy.

4 Local Dynamics

In this section we complete the analysis of the dynamics of our model economy, focussing on the local dynamics of system (11) and considering all the possible parameter values included in section 2. In this local analysis we characterize the regions of the parameter space that yield determinate, indeterminate or unstable equilibria. Also, we describe the regions that display none, one or more than one BGP. Our analysis is restricted to the case of interior BGP's. Notice that the analysis carried out in this section comprises the cases considered in section 3 but it is not limited to them, since assumptions 5-8 are not considered here. We divide this section into two parts. First, we study the existence and multiplicity of the stationary solutions. Then, we investigate the stability properties of the stationary solutions.

4.1 Existence and Multiplicity of Stationary Solutions

To solve the BGP we impose the growth rates in system (11) to be zero. Then, substituting the first two equations in the third one we obtain the following expression,

$$\begin{aligned} 0 &= \frac{1-u^*}{(1-\gamma)u^*+(\varphi+\alpha)(1-u^*)}\{B(1-u^*)^{\gamma-1}(\frac{\xi\gamma}{1-\tau_l-\xi}+\gamma u^*) \\ &\quad +[B(1-u^*)^{\gamma}-\pi_h](\Psi+\varepsilon)(\sigma-1)+g(1-\varepsilon)\frac{(\varphi+\alpha)(\sigma-1)}{\varphi+\alpha-1}-\rho+n\} \end{aligned} \tag{13}$$

where $\Psi=\frac{1-\alpha+\phi+\varepsilon}{\varphi+\alpha-1}$.[5] Equation (13) only depends on variable u^*, hence, the number of roots of this equation gives the number of interior stationary solutions.

It can be easily checked that a corner solution for $u^*=0$ is not possible since this would entail that utility is not maximized. Besides, under assumption 4, $u^*=1$ is not possible either. Before stating the results on the number of roots of equation (13) we divide the parameter space into different regions. We first define two reference values for the inverse of the elasticity of intertemporal substitution.

$$\begin{aligned} \sigma_1^* &= \frac{\varphi+\phi+\varepsilon(\varphi+\alpha)}{1+\phi-\alpha+\varepsilon(\varphi+\alpha)} \\ \sigma_2^* &= 1+\frac{B\frac{\xi\gamma}{1-\tau_l-\xi}+n-\rho}{(\pi_h-B)\left[\frac{1-\alpha+\phi+\varepsilon}{\varphi+\alpha-1}+\varepsilon\right]-\frac{g(1-\varepsilon)}{\varphi+\alpha-1}(\varphi+\alpha)} \end{aligned}$$

Notice that under assumption 1, $\sigma_1^*\in(0,1)$. For notational convenience we define the parameter vector as $\theta\equiv\{\sigma,\rho,n,A,\phi,\alpha,\varphi,\varepsilon,g,\pi_K,\pi_h,B,\gamma,\tau_K,\tau_l,\xi\}$ and Θ as the space that

[5] Notice that $\Psi<0$ and also $|\Psi|>1$ whenever $\phi+\varepsilon>\varphi$.

contains all the possible values of θ. We divide Θ into the following subsets,

$$
\begin{aligned}
&\Theta_1 = \{\theta \in \Theta, \text{s.t. } \sigma > \max\{\sigma_1^*, \sigma_2^*\}\} \\
&\Theta_2 = \left\{\theta \in \Theta, \text{s.t. } \sigma_2^* \ < \sigma < \sigma_1^* \text{ and } \tfrac{(1-\gamma)\xi}{1-\tau_l-\xi} > [(\Psi+\varepsilon)(\sigma-1)-1] \ \right\} \\
&\Theta_3 = \left\{\theta \in \Theta, \text{s.t. } \gamma < 1,\, \sigma > \sigma_2^* \text{ and } \tfrac{(1-\gamma)\xi}{1-\tau_l-\xi} = [(\Psi+\varepsilon)(\sigma-1)-1] \ \right\} \\
&\Theta_4 = \left\{\theta \in \Theta, \text{s.t. } \gamma < 1\ ,\, \sigma \geq \sigma_2^*\ ,\, \sigma < \sigma_1^* \text{ and } \tfrac{(1-\gamma)\xi}{1-\tau_l-\xi} < [(\Psi+\varepsilon)(\sigma-1)-1] \ \right\} \\
&\Theta_5 = \{\theta \in \Theta, \text{s.t. } \gamma = 1\ ,\, \sigma < \min\{\sigma_1^*, \sigma_2^*\} \ \} \\
&\Theta_6 = \left\{\theta \in \Theta, \text{s.t. } \gamma < 1\ ,\, \sigma < \min\{\sigma_1^*, \sigma_2^*\}\ ,\, \tfrac{(1-\gamma)\xi}{1-\tau_l-\xi} < [(\Psi+\varepsilon)(\sigma-1)-1] \right. \\
&\qquad \left. \text{and } P(u^c) = 0\right\} \\
&\Theta_7 = \left\{\theta \in \Theta, \text{s.t. } \gamma < 1\ ,\, \sigma < \min\{\sigma_1^*, \sigma_2^*\}, \tfrac{(1-\gamma)\xi}{1-\tau_l-\xi} < [(\Psi+\varepsilon)(\sigma-1)-1] \right. \\
&\qquad \left. \text{and } P(u^c) < 0\right\}.
\end{aligned}
$$

$P(u)$ is defined by the second term of (13) and its expression is given in the appendix by equation (14). The roots of function $P(u)$ in the interval (0,1) give the stationary solutions u^*. In the appendix it is shown that $P(u)$ is strictly convex whenever $\gamma < 1$. u^c is the value of u that makes zero the derivative of $P(u)$. The next proposition determines the existence and the number of stationary solutions for each parameter vector in Θ.

PROPOSITION 8 : *Under assumptions 1-4 the existence and uniqueness of the BGP is determined by one of the following three possibilities:*

i) If $\theta \in \bigcup_{i=1}^{6} \Theta_i$ *there is one interior BGP.*

ii) If $\theta \in \Theta_7$ *there exist two interior BGP's* $u_1^* < u_2^*$.

iii) If $\theta \in \Theta \backslash \bigcup_{i=1}^{7} \Theta_i$ *there is no interior BGP.*

PROOF : See the appendix.

From proposition 8 and the derivative of $P(u)$, given by equation (15) in the appendix, we derive the expression of u^c,

$$
u^c = \frac{(\Psi+\varepsilon)(\sigma-1) - 1 - \frac{\xi(1-\gamma)}{1-\tau_l-\xi}}{(\Psi+\varepsilon)(\sigma-1) - \gamma}.
$$

This equation and proposition 8 imply, as in Alonso-Carrera [1], that there are three necessary conditions for the existence of two BGP's. The first is that the external effects of the average level of human capital must be positive ($\phi > 0$ or $\varepsilon > 0$) and high enough. The second is that there must be decreasing returns to scale in the effort devoted to schooling ($\gamma < 1$). The third condition is a sufficiently small value of σ. Other authors have obtained conditions under which there are multiple stationary solutions in growth models. For instance, Ladrón de Guevara, Ortigueira and Santos [20] and [21] show the existence of multiple stationary solutions in two-sector models of endogenous growth with leisure.

In figure 4 we report some computational experiments to clarify the implications of proposition 8. The parameter values considered in the experiment have been taken from the calibrated economy of section 2 except for σ, whose default value has been set to 0.5 in panel b. In panel a we allow to vary parameters γ and σ while in panel b ξ varies instead of σ. It follows from proposition 8 that there can not be multiple BGP's for values of σ greater than 1. This comes from the fact that $\sigma_1^* < 1$ and therefore θ can not be in subset Θ_7 if $\sigma > 1$. This is illustrated in panel a and b of figure 4, there it is possible the existence of two interior BGP's for values of σ and γ less than 1. It is also worth noticing that, for plausible parameter values, the region of the parameter space that yields no BGP is sizeable. Moreover, subsidies to education ξ play an important role in determining the number of stationary solutions. In panel b we can observe that for values of γ around 0.5, three different values of ξ in the interval $(0, 0.1)$ may imply three different situations. A value $\xi = 0.02$ entails a unique BGP, a higher value $\xi = 0.05$ means two BGP's and for the highest value of $\xi = 0.1$ no BGP does exist. Finally, taxes on labor τ_l play a similar role to ξ in determining the number of stationary solutions. This follows from equation (13), however, notice that if $\xi = 0$ the effect of taxes on labor disappears.

Function $P(u)$ is useful to understand the effect of taxes and subsidies on the growth rate of the economy. From equation (14) in the appendix we have that both $\partial(P(u^*))/\partial\xi$ and $\partial(P(u^*))/\partial\tau_l$ are positive. Using the sign of $P'(u^*)$ and the implicit function theorem we have the following result.

Corollary 9 : *The effects of taxes on labor τ_l and subsidies to education ξ on the value of u^* and therefore on the growth rate of the economy are determined by one of the following* *three possibilities:*

i) If $\gamma < 1$ and there is only one interior BGP, $\partial u^*/\partial \xi$ and $\partial u^*/\partial \tau_l$ are negative. Therefore, an increase (decrease) in τ_l and/or ξ decreases (increases) u^*, implying that the growth rate η_h at the BGP is higher (lower).

ii) If $\gamma < 1$ and there exist two interior BGP's ($u_1^* < u_2^*$), $\partial u_1^*/\partial \xi$ and $\partial u_1^*/\partial \tau_l$ are positive while $\partial u_2^*/\partial \xi$ and $\partial u_2^*/\partial \tau_l$ are negative. Therefore, an increase (decrease) in τ_l and/or ξ increases (decreases) u_1^* and decreases (increases) u_2^*. This implies that the growth rate η_h at the BGP is lower (higher) at u_1^* and higher (lower) at u_2^*.

iii) If $\gamma = 1$ and there is one interior BGP, the sign of $P'(u^*)$ is positive if and only if $\sigma > \sigma_1^*$ (see the appendix). Therefore, if $\sigma > \sigma_1^*$, $\partial u^*/\partial \xi$ and $\partial u^*/\partial \tau_l$ are negative. This implies that an increase (decrease) in τ_l and/or ξ decreases (increases) u^* and the growth rate η_h at the BGP is higher (lower). The opposite applies if $\sigma < \sigma_1^*$.

Corollary 9 implies that changes in taxes on labor and subsidies to education may have different effects on growth rates depending on the parameter values. If $\gamma = 1$ and the external effects are high enough (σ_1^* is high) values of σ above or below σ_1^* entail that the effects of the changes in τ_l and ξ work in opposite directions. If $\gamma < 1$ increases in τ_l and/or ξ have a positive effect on growth rates as long as there is only one BGP. If there are two BGP's the sign of the effect for the BGP with lower u^* is the opposite of the sign for the BGP with higher u^*. Finally notice that again changes in taxes on labor have no effect if subsidies to education are zero.

4.2 Stability of the Stationary Solutions

Once we have determined the number of stationary solutions for the different parameter vectors, we study the local stability of the stationary solutions of system (11). We ignore in our analysis set Θ_6 since at those points the Jacobian is singular. Moreover, those points have measure zero.

The competitive equilibrium solution is saddle path stable (i.e., the BGP is determinate) if the Jacobian matrix of the corresponding linear system evaluated at the steady state has two eigenvalues with positive real part and one with negative real part. If there are two eigenvalues with negative real part there is a continuum of equilibrium paths that converge to the stationary solution (i.e., the BGP is indeterminate). If the three eigenvalues have negative real part the steady state is a sink. Finally, if all the eigenvalues have positive real part the system is unstable. The Jacobian matrix of the corresponding linear system at the steady state is defined by matrix J^*,

$$J^* = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

where the elements of J^* are given in the appendix. The eigenvalues of J^* are the roots of the characteristic equation

$$-\lambda^3 + Tr(J^*)\lambda^2 - B(J^*)\lambda + Det(J^*) = 0,$$

where $B(J^*) = a_{11}a_{22} + a_{33}a_{22} + a_{11}a_{33} - (a_{13}a_{31} + a_{12}a_{21+}a_{32}a_{23})$. If we apply Routh's theorem (see Gantmacher [14]), we have that the number of roots of the characteristic equation with positive real part is equal to the number of variations of sign in the scheme,

$$-1, Tr(J^*), -B(J^*) + \frac{Det(J^*)}{Tr(J^*)}, Det(J^*).$$

The next proposition characterizes the signs of the eigenvalues of matrix J^* for the different values of the parameter vector θ. Before, we make the following two assumptions,

Assumption 9 : $\eta_h > 0$.

Assumption 10 : $2\alpha + \vartheta > \phi + \varepsilon$.

Assumption 9 requires that the growth rate of human capital in the BGP is positive. Assumption 10 requires that the size of the external effects on human capital is not bigger than twice the share of physical capital plus a positive constant ϑ, that is defined in the appendix. Both assumptions together guarantee that the trace of J^* is positive.[6]

Proposition 10 : *Under assumptions 1-4 and 9-10 the local stability of the stationary solutions is determined by one of the following three possibilities:*

[6] If one of these two assumptions is not satisfied and $Tr(J^*) < 0$ our results are still valid if $-B(J^*) +$

(i) *If* $\theta \in \bigcup_{i=1}^{4} \Theta_i$ *the Jacobian matrix has two eigenvalues with positive real part and one with negative real part. Therefore, the stationary solution is determinate and locally stable.*

(ii) *If* $\theta \in \Theta_5$ *the Jacobian matrix may have either three eigenvalues with positive real part or one with positive real part and two with negative real part. Therefore, in the first case the stationary solution is unstable and in the second the stationary solution is indeterminate and locally stable.*

(iii) *If* $\theta \in \Theta_7$ *the Jacobian matrix at* u_1^* *may have either three eigenvalues with positive real part or one eigenvalue with positive real part and two with negative real part. The Jacobian matrix at* u_2^* *has two eigenvalues with positive real part and one with negative real part. Therefore, the stationary solution at* u_1^* *is either unstable or indeterminate and locally stable. The stationary solution at* u_2^* *is determinate and locally stable.*

PROOF : See the appendix.

As already established in previous works, proposition 10 states that there is a unique, saddle path stable BGP for values of the elasticity of intertemporal substitution low enough (σ high enough). For values of $\gamma < 1$ the equilibrium is determinate as long as there is only one BGP. This result differs from Benhabib and Perli [7], in their work they assume $\gamma = 1$ [7] and find that it is possible to have only one stationary solution and indeterminate equilibria. Proposition 10 shows that their result is only possible if $\gamma = 1$. If there are two BGP's proposition 10 states that the BGP with lower u^* is either unstable or indeterminate while the BGP with higher u^* is determinate. It then follows that economies with two BGP's display

$\frac{Det(J^*)}{Tr(J^*)} > 0$. The following condition is sufficient for this to happen

$$\gamma B \left[\frac{1-\gamma}{1-u^*} \left(u^* + \frac{\xi}{1-\tau_l-\xi} \right) - (\phi + \varepsilon - \alpha) \right] (a_{11} + a^*) +$$
$$(1-\tau_K)\alpha \frac{a_{11}a^*}{u^*}(1-u^*)^{1-\gamma} - (\varphi+\alpha)a^*\gamma B\varepsilon + a_{11}a^*\alpha \frac{1-\tau_K}{\sigma} \frac{1-\gamma}{(1-u^*)^\gamma}$$
$$< -(\varphi + \tau_K\alpha)a_{11}\Psi B\gamma - (\varphi+\alpha)\frac{1}{\sigma}a^*\gamma B\varepsilon.$$

This condition implies that $B(J^*)$ is greater than zero and holds for sufficiently low values of τ_K, φ and ε. The definition of $B(J^*)$ is provided in the appendix.

[7] Notice that they also assume $\varphi = \varepsilon = \tau_K = \tau_l = \xi = 0$ which implies that their parametrization corresponds to sets Θ_1 and Θ_5 in our paper.

a poverty trap. Moreover, the results obtained in the computations of figure 4 show that all the economies computed with two BGP's are in the case in which the steady state with lower u^* exhibits indeterminacy. Subsidies to education play an important role in determining the existence of a poverty trap as panel b of figure 4 illustrates. There, it can be noticed that an increase of the value of ξ may move the economy from a unique, saddle path stable BGP to a region with two BGP's and a poverty trap. Equation (13) implies that this same result applies to taxes on labor τ_l.

In figure 5 we report four numerical experiments, assuming that $\gamma = 1$, in order to evaluate the possibility of indeterminate equilibria when there is only one BGP. We consider $B = 0.18$, the rest of the parameter values are the same as for figure 4. First, we should note that, as in figure 4, there is a wide region of the parameter space in which there is no BGP. This region includes values of $\sigma > 1$ (see panel a) if subsidies to education are high enough. For values of $\sigma < 1$ the size of this region increases (see panel b, c and d). Regarding the region where there exits a stationary solution, figure 5 shows where the BGP is unstable, saddle path stable or indeterminate. Saddle path stability is guaranteed for values of σ high enough. Instability seems to be related to values of the external effect of physical capital φ above 0.4 (see panels c and d). Finally, indeterminate equilibria seems to be the less likely event. Panel d shows that in order to have indeterminacy it is necessary to have values of the external effects ϕ and φ above 0.4 and values of σ around 0.5. We have checked in several computational experiments that if σ is above 0.7 there is no region displaying indeterminacy.

In summary, for values of the elasticity of intertemporal substitution high enough ($\sigma > 1.5$) and plausible parameter values there is a unique, saddle path stable stationary solution. For values of σ around 1, subsidies to education may determine whether the economy is in a region with a BGP or not. For values of σ less than 1, poverty traps coupled with indeterminacy are empirically plausible if there are decreasing returns to the effort devoted to schooling. However, when there are constant returns in this effort indeterminacy is only possible for large values of the external effects. In both cases a value of σ around 0.5 or lower is needed to have indeterminacy. Finally, our numerical experiments also suggest that the region in which no BGP does exist can not be excluded since its size is not negligible. These experiments also find that taxes on labor and subsidies to education may play a role in determining the number of BGP's and the existence of poverty traps.

These results differ from previous works. Benhabib and Perli [7] obtain indeterminacy only for values of σ close to zero. The works of Ben-Gad [2] and Raurich [31] obtain indeterminacy for values of σ greater that 1. In the first case this is due to sector-specific external effects, in the second because of the presence of unproductive government spending coupled with asymmetric taxation. The asymmetry in taxation or the sector specific-external effects may produce factor intensity reversals that entail either instability or indeterminacy. In our case, these factor intensity reversals are not present and the source of indeterminacy are economy-wide externalities. The effect of these economy-wide externalities on local dynamics is more complex and difficult to analyze and as we have shown imply different results on the existence of indeterminate equilibria.

Before we end this section we devote a few lines to discuss the role played by assumption 10. If it is not satisfied the stationary solution in set $\bigcup_{i=1}^{4} \Theta_i$ might be a sink (three eigenvalues with negative real part). However, it can be easily shown that for the model of Benhabib and Perli [7] the condition in footnote 6 is satisfied since then $\varphi = \tau_K = \varepsilon = 0$. We have also performed several computational experiments and found no case in which the stationary solution was a sink.

5 Concluding Remarks

In this paper we have studied the dynamics of growth models with physical and human capital accumulation, economy-wide external effects in the production function and in the law of motion of physical capital, and distortionary taxes on physical capital and labor. The analysis includes the study of both local and global dynamics. The study of global dynamics has focussed on establishing the monotonicity of the equilibrium path and the existence of Markov equilibrium. The competitive solution is monotone if the Jacobian matrix evaluated at the stationary solution has at least one eigenvalue with negative real part and the size of the external effects and taxes on physical capital is not too high. This monotonicity property ensures the existence of a continuous Markov equilibrium. The analysis also shows that if there are no external effects a continuous Markov equilibrium always exists for any feasible tax scheme.

In the examples presented in section 3 we study the dynamics of three economies that do not satisfy our sufficient conditions. We find that higher elasticity of intertemporal substitution and higher external effects may entail non-monotonicity. Furthermore, one of our examples shows that the existence of cycles cannot be ruled out.

The analysis of local dynamics in section 4 classifies the different regions of the parameter space depending on the existence, the number and the stability of the stationary solutions. We find that the local dynamics is determined by four sets of variables, the external effects, taxes on labor and subsidies to education, the productivity of the human capital sector and the elasticity of intertemporal substitution. The existence of two BGP and poverty traps is possible for sufficiently high values of the elasticity of intertemporal substitution (greater than 1), the concavity of the human capital production function and the external effects. Also, taxes on labor and subsidies to education may determine whether the economy displays one, two or none stationary solution. Indeterminate equilibria is empirically more plausible when there are two BGP and poverty traps. If there is only one BGP it is necessary higher values of the external effects and values of the elasticity of intertemporal substitution greater than one in order to have indeterminacy. These results differ from previous papers that include sector-specific external effects or asymmetric factor taxation. In these papers, indeterminacy is plausible for values of the elasticity of intertemporal substitution less than one. Finally, we find that taxes on physical capital have no influence on local dynamics.

6 Apppendix

6.1 System Obtained from the First Order Conditions of the Hamiltonian

$$
\begin{aligned}
\frac{\dot{c}_t}{c_t} &= \frac{1}{\sigma}\left[(1-\tau_K)\alpha A\left(\frac{K_t}{L_t u_t}\right)^{\varphi} K_t^{\alpha-1}(L_t u_t)^{1-\alpha} h_t^{1-\alpha+\phi+\varepsilon} q_t^{1-\varepsilon}\right. \\
&\quad \left. -\varepsilon[B(1-u_t)^{\gamma}-\pi_h]+(\varepsilon-1)g-\pi_K-\rho\right] \\
\frac{\dot{K}_t}{K_t} &= A\left(\frac{K_t}{L_t u_t}\right)^{\varphi} K_t^{\alpha-1}\left(L_t u_t\right)^{1-\alpha} h_t^{1-\alpha+\phi+\varepsilon} q_t^{1-\varepsilon} - \frac{L_t c_t q_t^{1-\varepsilon}\left(h_t\right)^{\varepsilon}}{K_t} - \pi_K \\
\frac{\dot{h}_t}{h_t} &= B\left(1-u_t\right)^{\gamma} - \pi_h \\
\frac{\dot{u}_t}{u_t} &= \frac{1-u_t}{\left(1-\gamma\right)u_t + \left(\varphi+\alpha\right)\left(1-u_t\right)}\left\{B\left(1-u_t\right)^{\gamma-1}\left[\left(1-\alpha+\phi+\varepsilon\right)\left(1-u_t\right)\right.\right. \\
&\quad \left. +\gamma u_t + \frac{\xi\gamma}{1-\tau_l-\xi}\right] + (\varphi+\tau_K\alpha)A\left(\frac{K_t}{L_t u_t}\right)^{\varphi} K_t^{\alpha-1}\left(L_t u_t\right)^{1-\alpha} h_t^{1-\alpha+\phi+\varepsilon} q_t^{1-\varepsilon} \\
&\quad + \left(1-\varepsilon\right)g - \left(\varphi+\alpha\right)\frac{L_t c_t q_t^{1-\varepsilon}\left(h_t\right)^{\varepsilon}}{K_t} - (1-\alpha+\phi+\varepsilon)\pi_h \\
&\quad \left. + \left(1-\alpha-\varphi\right)\left(\pi_K+n\right)\right\}
\end{aligned}
$$

6.2 System (12)

$$
\begin{aligned}
\Delta_x &= \frac{1}{\varphi+\alpha}[(1-\varepsilon)g+(1-\varphi-\alpha)(n+\pi_K)+(1-\alpha+\phi+\varepsilon)(B-\pi_h) \\
&\quad +(1-\varphi-\alpha)B\frac{\xi}{1-\tau_l-\xi}] \\
\Omega_x &= \frac{\alpha(1-\varphi-\alpha)(\tau_K-1)}{\varphi+\alpha}A \\
\Sigma_x &= -\frac{\varphi+\varepsilon+\phi}{\varphi+\alpha}B \\
\Delta_a &= n+(1-\frac{1}{\sigma})[(1-\varepsilon)g+\pi_K+\varepsilon(B-\pi_h)]-\frac{\rho}{\sigma} \\
\Omega_a &= [\frac{(1-\tau_K)\alpha}{\sigma}-1]A \\
\Sigma_a &= (\frac{1}{\sigma}-1)\varepsilon B \\
\Delta_u &= \frac{1}{\varphi+\alpha}[(1-\varepsilon)g+(1-\varphi-\alpha)(n+\pi_K)+(1-\alpha+\phi+\varepsilon)(B-\pi_h) \\
&\quad +B\frac{\xi}{1-\tau_l-\xi}]
\end{aligned}
$$

$$\Omega_u = \frac{\varphi + \alpha\tau_K}{\varphi + \alpha}A$$
$$\Sigma_u = \frac{\alpha - \varepsilon - \phi}{\varphi + \alpha}B$$

Under assumptions 1-4 one can determine the sign of some of the coefficients of the system. Δ_x, Δ_u and Ω_u are always positive while Ω_x and Σ_x are always negative. The remaining coefficients may have any sign depending on the value of the original parameters.

1. If $\sigma < 1 \Rightarrow \Delta_a < 0$, but if $\sigma > 1$, Δ_a may have any sign.
2. $\Omega_a > 0 \Leftrightarrow \sigma < \alpha(1 - \tau_K)$.
3. $\Sigma_a > 0 \Leftrightarrow \sigma < 1$.
4. $\Sigma_u > 0 \Leftrightarrow \alpha > \varepsilon + \phi$.

6.3 Proof of Theorem 1

The strategy of the proof of theorem 1 is to study the phase diagram of system (12). The main difficulty is that (12) is a 3-dimensional system and the dynamics cannot be represented in a single plane. Our method exploits the linearity of the growth rates of system (12) and studies the 2-dimensional phase diagrams orthogonal to each one of the three axis at the steady state solution. The study of these phase diagrams will allow us to rule out the possibility of non-monotone dynamics in the region of the parameter space considered in theorem 1.

In order to clarify the exposition, this proof is divided into two cases: First we study the case $\alpha(1 - \tau_K) < \sigma < 1$ and second $\sigma > 1$. The argument of the proof is similar in both cases.

CASE 1 : Here $\alpha(1 - \tau_K) < \sigma < 1$. The phase diagram orthogonal to the u-axis at $u = u^*$ has two different representations, the first is for $\sigma < \varphi + \alpha$ and the second is for $\sigma > \varphi + \alpha$. Figures 6 and 7 represent these two cases. In the phase diagrams we have drawn the lines at which variables x, a and u remain constant. The slopes of these lines can be obtained by setting the derivatives of system (12) equal to zero. Thus, in figures 6 and 7 the slope of the line at which $\dot{x} = 0$ is zero since variable a does not enter the first equation of (12). The slope of line $\dot{a} = 0$ is $-\frac{1}{\Omega_a} = \frac{\sigma}{(\sigma - (1-\tau_K)\alpha)A}$ and if $\sigma > \alpha(1 - \tau_K)$

it is greater than zero. Finally the slope of the line $\dot{u} = 0$ is $\frac{1}{\Omega_u} = \frac{\varphi+\alpha}{(\varphi+\alpha\tau_K)A}$, which is always positive. The difference between figure 6 and figure 7 is that in figure 7 the slope of line $\dot{u} = 0$ is greater than the slope of $\dot{a} = 0$. The necessary and sufficient condition for this to happen is $\sigma > \varphi + \alpha$. Outside lines $\dot{x} = 0$, $\dot{a} = 0$ and $\dot{u} = 0$ the arrows show the sense of motion. One can observe that the stable manifold can only approach the steady state through the area contained between the vertical line that crosses a^* and line $\dot{a} = 0$. Finally, for $\sigma > \varphi + \alpha$ (figure 7) the stable manifold may converge either from above or below line $\dot{u} = 0$.

In figure 8 we represent the phase diagram orthogonal to the x-axis at x^* when $\sigma < 1$. The slopes of lines $\dot{u} = 0$, $\dot{a} = 0$ and $\dot{x} = 0$ have been drawn as in figures 6 and 7. Assumption 7 guarantees that the slope of $\dot{u} = 0$ is positive while $\sigma < 1$ implies that the slope of $\dot{a} = 0$ is negative. The arrows show that the stable manifold cannot approach the steady state through any of the areas in this plane.

Figure 11 represents the plane orthogonal to the a-axis at $a = a^*$. One can notice that $\sigma \in (\alpha(1 - \tau_K), 1)$ implies that the slope of $\dot{a} = 0$ is positive and assumption 8 implies that the slope of $\dot{x} = 0$ is greater than the slope of $\dot{u} = 0$, which is negative due to assumption 7. It can also be observed that the equilibrium path can approach the steady state only through the area shown in figure 11.

In summary, the planes orthogonal to the u and a axes at the steady state define the areas through which the stable manifold can approach the stationary solution. There are two possibilities. First, if $\sigma < \varphi + \alpha$ figures 6 and 11 define the stable manifold and $\dot{x}$ and $\dot{a}$ have the same sign while $\dot{u}$ has the opposite. On the other hand, if $\sigma > \varphi + \alpha$ the equilibrium is defined by figures 7 and 11 and again the sign of $\dot{u}$ is the opposite of the sign of $\dot{x}$ and $\dot{a}$.

Finally, it remains to check that the stable manifold cannot approach the steady state through any other area not contained in the phase diagrams. The only possibility is the space contained between the three planes that define the stationary solution. This locus is not necessarily represented in any of the three phase diagrams drawn above. However, it can be verified that this cannot happen. If we study the plane orthogonal to the a-axis at any $a' < a^*$ we find that in the locus contained inside these three planes (namely $\dot{a} = 0$, $\dot{x} = 0$ and $\dot{u} = 0$) $\dot{a} < 0$. Since the phase diagrams are symmetric with respect to the steady state, $\dot{a} > 0$ occurs for the plane orthogonal to the a-axis at any $a'' > a^*$.

Case 2 : Now we consider $\sigma > 1$. The phase diagram orthogonal to the u-axis at the steady state is drawn in figure 7. The plane orthogonal to the x-axis at the steady state may have two different representations which are drawn in figures 9 and 10. The difference is that in figure 9 the slope of line $\dot{u} = 0$ is bigger than the slope of line $\dot{a} = 0$ while in figure 10 is the opposite. The stable manifold may approach the stationary solution through the area shown in figure 9, but this is not possible in figure 10. Finally, figure 12 represents the plane orthogonal to the a-axis at the stationary solution. The only difference with the case $\sigma < 1$ is that the slope of line $\dot{a} = 0$ is negative.

Summarizing, there are two different areas through which the stable manifold can approach the steady state. When the stable manifold is the one contained by figures 7 and 9 the area between the lines $\dot{u} = 0$ and $\dot{a} = 0$ in figure 7 contains the stable manifold, and $\dot{x}$, $\dot{a}$ and $\dot{u}$ have the same sign for every $t \geq 0$. The other possible path is the one contained by figures 7 and 12. In this equilibrium path, $\dot{u}$ has the opposite sign of $\dot{x}$ and $\dot{a}$.

To end the proof it has to be checked that the locus contained between the three planes that define the stationary solution cannot contain the equilibrium path. This can be done in a similar way to case 1.

The proof is now complete and under the assumptions of theorem 1 we conclude that the equilibrium path is monotone and the signs of $\dot{x} = 0$, $\dot{a} = 0$ and $\dot{u} = 0$ are constant for all $t \geq 0$.

6.4 Proof of Theorem 2

Given the strict monotonicity of x, a and u for every $t \geq 0$ shown in theorem 1, we can use the implicit function theorem to define variables a and u as functions of the state variable m. Moreover, given that $m = xu^{\varphi+\alpha-1}$ and that $x_t > 0$, $u_t > 0$ for every $t \geq 0$, $\dot{m} = f(m)$ is a continuous function.

6.5 Proof of Proposition 8

The second term of equation (13) defines function $P(u)$,

$$\begin{aligned} P(u) &= B(1-u)^{\gamma-1}[\frac{\xi\gamma}{1-\tau_l-\xi}+\gamma u]+\{B(1-u)^{\gamma}-\pi_h\}(\Psi+\varepsilon)(\sigma-1) \\ &+g(1-\varepsilon)\frac{(\varphi+\alpha)(\sigma-1)}{\varphi+\alpha-1}-\rho+n \end{aligned} \tag{14}$$

The roots of (13) determine the stationary solutions of the system. As previously discussed we are only interested in values $u^* \in (0,1)$. To clarify the exposition, we first consider the case $\gamma < 1$. We proceed in three steps.

STEP 1 : We compute the derivative of function $P(u)$ and study its possible values.

$$P'(u) = \frac{B\gamma}{(1-u)^{1-\gamma}}\left\{\frac{1-\gamma}{1-u}\left[u+\frac{\xi}{1-\tau_l-\xi}\right]-[(\Psi+\varepsilon)(\sigma-1)-1]\right\} \tag{15}$$

From (15) it follows that $P(u)$ is strictly convex in $(0,1)$. Besides, $P'(0)$ may take the following values,

- $P'(0) > 0 \Leftrightarrow \begin{cases} \sigma > \sigma_1^* \text{ or} \\ \sigma < \sigma_1^* \text{ and } \frac{(1-\gamma)\xi}{1-\tau_l-\xi} > [(\Psi+\varepsilon)(\sigma-1)-1] \end{cases}$
- $P'(0) = 0 \Leftrightarrow \frac{(1-\gamma)\xi}{1-\tau_l-\xi} = [(\Psi+\varepsilon)(\sigma-1)-1]$
- $P'(0) < 0 \Leftrightarrow \sigma < \sigma_1^*$ and $\frac{(1-\gamma)\xi}{1-\tau_l-\xi} < [(\Psi+\varepsilon)(\sigma-1)-1]$.

STEP 2 : We study the sign of $P(0)$.

$$\begin{aligned} P(0) &= B\left[\frac{\xi\gamma}{1-\tau_l-\xi}+(\Psi+\varepsilon)(\sigma-1)\right]-\pi_h[(\Psi+\varepsilon)(\sigma-1)] \\ &+g(1-\varepsilon)\frac{(\varphi+\alpha)(\sigma-1)}{\varphi+\alpha-1}-\rho+n \end{aligned}$$

Operating in the previous expression it is easy to check that $P(0) \gtreqless 0 \Leftrightarrow \sigma \lesseqgtr \sigma_2^*$.

STEP 3 : Finally we study the values of $P(u)$ for u close to 1. From (14) one obtains that $\lim_{u\to 1} P(u) = +\infty$. Regarding $P'(u)$, it can be checked from (15) that $\lim_{u\to 1} P'(u) = +\infty$.

Given the strict convexity of $P(u)$, if $P(0) < 0$ and $P'(0) \gtreqless 0$ there is only one BGP. The same happens if $P(0) = 0$ and $P'(0) < 0$. This two cases are included in subsets Θ_1 to Θ_4. If $P(0) > 0$ and $P'(0) < 0$ there may be three cases. If $P(u^c) > 0$ there is no BGP and we are in $\theta \in \Theta \backslash \bigcup_{i=1}^{7} \Theta_i$. If $P(u^c) = 0$ we are in Θ_6 and there is only one BGP. If $P(u^c) < 0$ there are two BGP and we are in Θ_7. Finally, if $P(0) \geq 0$ and $P(0) \geq 0$ we are in $\theta \in \Theta \backslash \bigcup_{i=1}^{7} \Theta_i$ and there is no BGP.

Now we consider $\gamma = 1$. Here, the slope of $P(u)$ has the sign of $-[(\Psi + \varepsilon)(\sigma - 1) - 1]$ and it is positive if and only if $\sigma > \sigma_1^*$. It follows that there can be at most one interior BGP and this happens in two cases. First, for $\sigma > \max\{\sigma_1^*, \sigma_2^*\}$, we are then in set Θ_1, and second for $\sigma < \min\{\sigma_1^*, \sigma_2^*\}$, set Θ_5. The value of u^* can be derived by imposing $\gamma = 1$ in equation (14).

6.6 Proof of Proposition 10

The first step is to compute the elements of the Jacobian matrix J^*.

$$
\begin{aligned}
a_{11} &= \left.\frac{\partial \dot{z}_t}{\partial z_t}\right|_{z^*,a^*,u^*} = (\varphi + \alpha - 1) A (z^*)^{\varphi+\alpha-1} (u^*)^{1-\varphi-\alpha} \\
a_{12} &= \left.\frac{\partial \dot{z}_t}{\partial a_t}\right|_{z^*,a^*,u^*} = -z^* \\
a_{13} &= \left.\frac{\partial \dot{z}_t}{\partial u_t}\right|_{z^*,a^*,u^*} = -\frac{z^*}{u^*}\left\{a_{11} + \Psi B \gamma (1-u^*)^{\gamma-1} u^*\right\} \\
a_{21} &= \left.\frac{\partial \dot{a}_t}{\partial z_t}\right|_{z^*,a^*,u^*} = \frac{a^*}{z^*} a_{11}\left[\frac{(1-\tau_k)\alpha}{\sigma} - 1\right] \\
a_{22} &= \left.\frac{\partial \dot{a}_t}{\partial a_t}\right|_{z^*,a^*,u^*} = a^* \\
a_{23} &= \left.\frac{\partial \dot{a}_t}{\partial u_t}\right|_{z^*,a^*,u^*} = a^*\left\{\left[1 - \frac{(1-\tau_k)\alpha}{\sigma}\right]\frac{a_{11}}{u^*} - \left(1 - \frac{1}{\sigma}\right)\gamma\varepsilon B (1-u^*)^{\gamma-1}\right\} \\
a_{31} &= \left.\frac{\partial \dot{u}_t}{\partial z_t}\right|_{z^*,a^*,u^*} = \frac{(1-u^*)\,u^*}{(1-\gamma)\,u^* + (\varphi+\alpha)(1-u^*)}(\varphi + \tau_k\alpha)\frac{a_{11}}{z^*} \\
a_{32} &= \left.\frac{\partial \dot{u}_t}{\partial a_t}\right|_{z^*,a^*,u^*} = -\frac{(1-u^*)\,u^*(\varphi+\alpha)}{(1-\gamma)\,u^* + (\varphi+\alpha)(1-u^*)} \\
a_{33} &= \left.\frac{\partial \dot{u}_t}{\partial u_t}\right|_{z^*,a^*,u^*} = \frac{(1-u^*)^{\gamma}\,u^*}{(1-\gamma)\,u^* + (\varphi+\alpha)(1-u^*)}\left\{\gamma B\left[\frac{(1-\gamma)}{1-u^*}\left[u^* + \right.\right.\right. \\
&\quad \left.\left.\left.\frac{\xi}{1-\tau_l-\xi}\right] - (\phi + \varepsilon - \alpha)\right] - (\varphi + \tau_k\alpha)\frac{a_{11}}{u^*}(1-u^*)^{1-\gamma}\right\}
\end{aligned}
$$

To apply Routh's theorem, the three key quantities $Tr(J^*)$, $B(J^*)$ and $Det(J^*)$ must be signed.

6.6.1 Sign of $Det(J^*)$

$Det\left(J^*\right)=a_{11}a_{22}a_{33}+a_{21}a_{32}a_{13}+a_{12}a_{23}a_{31}-\left(a_{13}a_{22}a_{31}+a_{12}a_{21}a_{33}+a_{23}a_{32}a_{11}\right)$. Substituting the corresponding values we obtain,

$$\begin{aligned} Det\left(J^*\right) &= \frac{\left(1-u^*\right)^{\gamma}u^* a_{11}a^*\left(1-\tau_k\right)\alpha}{\left[\left(1-\gamma\right)u^*+(\varphi+\alpha)(1-u^*)\right]\sigma}\left\{B\gamma\left[\frac{1-\gamma}{1-u^*}\left(u^*+\frac{\xi}{1-\tau_l-\xi}\right)\right.\right. \\ &\left.\left. +\left(\Psi+\varepsilon\right)\left(1-\sigma\right)+1\right]\right\}. \end{aligned}$$

Since $a_{11}<0$ equation (15) implies that $Det\left(J^*\right)$ has the opposite sign of $P'(u^*)$.

6.6.2 Sign of $Tr\left(J^*\right)$

To compute the trace of J^* we first compute a_{11}, a_{22}, and a_{33} in terms of u^*. The laws of motion of z and a in the steady state imply,

$$\begin{aligned} a_{11}+a_{22} &= \frac{\varphi+\tau_k\alpha}{\alpha(1-\tau_k)}\left\{-\frac{1-\varepsilon}{\varphi+\alpha-1}g+n+\pi_k-\Psi\left[B\left(1-u^*\right)^{\gamma}-\pi_h\right]\right\} \\ &+\frac{(\varphi+\alpha)\gamma B\left(1-u^*\right)^{\gamma-1}}{\alpha(1-\tau_k)}\left[u^*+\frac{\xi}{1-\tau_l-\xi}\right], \end{aligned}$$

if we add a_{33},

$$\begin{aligned} Tr\left(J^*\right) &= \frac{\left(1-u^*\right)^{\gamma}u^*B\gamma}{\left(1-\gamma\right)u^*+(\varphi+\alpha)(1-u^*)}\left\{\left[1+\frac{\varphi+\alpha}{\alpha(1-\tau_k)}\right]\frac{1-\gamma}{1-u^*}\left[u^*+\frac{\xi}{1-\tau_l-\xi}\right]\right. \\ &+\frac{\left(1-\gamma u^*\right)\left(1-u^*\right)^{-\gamma}\left(\varphi+\tau_k\alpha\right)}{u^*\gamma B\alpha\left(1-\tau_k\right)}\left[-\frac{1-\varepsilon}{\varphi+\alpha-1}g+n+\pi_k-\right. \\ &\left.\Psi\left[B\left(1-u^*\right)^{\gamma}-\pi_h\right]\right]+\frac{\xi\left((\varphi+\alpha)^2+\left(\varphi+\tau_k\alpha\right)\left(1-\varphi-\alpha\right)\right)}{\alpha u^*\left(1-\tau_k\right)\left(1-\tau_l-\xi\right)} \\ &\left.+\left(\alpha+\frac{(\varphi+\alpha)^2+(\varphi+\tau_k\alpha)(1-\varphi-\alpha)}{\alpha(1-\tau_k)}-\phi-\varepsilon\right)\right\}. \end{aligned} \tag{16}$$

If the growth rate of human capital is greater than zero, then $a_{11}+a_{22}$ is positive, and therefore $Tr\left(J^*\right)<0\Rightarrow a_{33}<0$. We define now constant ϑ in assumption 10.

$$
\begin{aligned}
\vartheta &= \left[1+\frac{\varphi+\alpha}{\alpha(1-\tau_k)}\right]\frac{1-\gamma}{1-u^*}\left[u^*+\frac{\xi}{1-\tau_l-\xi}\right]+\\
&\frac{(1-\gamma u^*)(1-u^*)^{-\gamma}(\varphi+\tau_k\alpha)}{u^*\gamma B\alpha(1-\tau_k)}\left[-\frac{1-\varepsilon}{\varphi+\alpha-1}g+n+\pi_k-\right.\\
&\left.\Psi\left[B(1-u^*)^{\gamma}-\pi_h\right]\right]+\frac{\xi\left((\varphi+\alpha)^2+(\varphi+\tau_k\alpha)(1-\varphi-\alpha)\right)}{\alpha u^*(1-\tau_k)(1-\tau_l-\xi)}\\
&+\left(\frac{(\varphi+\alpha)^2+(\varphi+\tau_k\alpha)(1-\varphi-\alpha)}{\alpha(1-\tau_k)}-\alpha\right).
\end{aligned}
$$

Notice that the last term is positive and that under assumption 9, ϑ is positive. Moreover, under assumption 10, $Tr(J^*)$ is also positive.

6.6.3 Sign of $B(J^*)$

Substituting the corresponding values in the definition of $B(J^*)$ we obtain,

$$
\begin{aligned}
B(J^*) &= \frac{(1-u^*)^{\gamma}u^*}{(1-\gamma)u^*+(\varphi+\alpha)(1-u^*)}\left\{(\varphi+\tau_k\alpha)a_{11}\Psi\gamma B-(\varphi+\alpha)\left(1-\frac{1}{\sigma}\right)a^*\gamma B\varepsilon\right.\\
&+\alpha a^*a_{11}\frac{(1-\tau_k)(1-\gamma)}{\sigma(1-u^*)^{\gamma}}+\gamma B\left[\frac{(1-\gamma)}{1-u^*}\left[u^*+\frac{\xi}{1-\tau_l-\xi}\right]\right.\\
&\left.\left.-(\phi+\varepsilon-\alpha)\right](a_{11}+a^*)+(1-\tau_k)\alpha\frac{a^*a_{11}}{u^*}(1-u^*)^{1-\gamma}\right\}.
\end{aligned}
$$

Now we can characterize the stability of the different stationary solutions:

- If $\theta\in\bigcup_{i=1}^{4}\Theta_i$ then $P'(u^*)>0\Rightarrow Det(J^*)<0$, therefore there are negative signs at the beginning and at the end of the sequence. Under assumptions 9 and 10, $Tr(J^*)>0$ which implies that there are two changes of sign. Hence, in this region of the parameter space there are always two eigenvalues with positive real part and one with negative real part. The steady state is a saddle point.

- If $\theta\in\Theta_5$ then $P'(u^*)<0\Rightarrow Det(J^*)>0$. If $Tr(J^*)>0$ then $B(J^*)$ may have any sign, and depending on its value there could be either one or three changes of sign. Hence, in this region of the parameter space there may be either one eigenvalue with positive real part and two with negative real part or three eigenvalues with positive real part. Therefore, there can be a continuum of equilibria converging to the steady state or global inestability.

- If $\theta \in \Theta_7$ there are two stationary solutions $u_1^* < u_2^*$. In u_2^* $Det\,(J^*) < 0$ and as in case 1 the equilibrium is a saddle point. In u_1^* $Det\,(J^*) > 0$ and as in case 2 we may have either a continuum of equilibria or global inestability.

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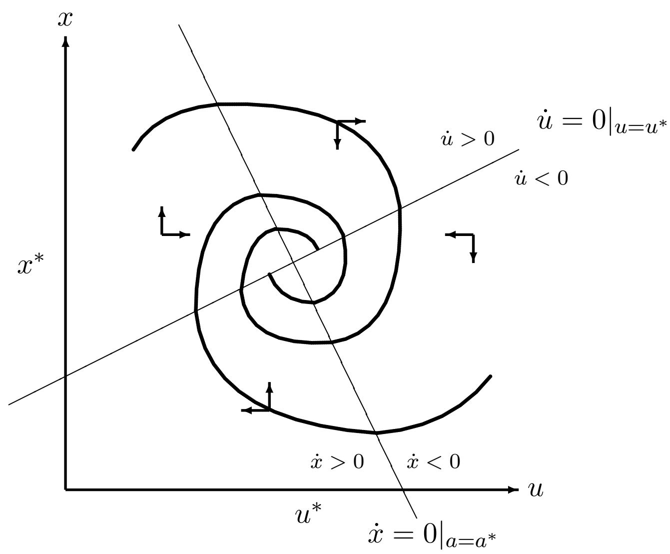

Figure 1: Example 1, $\gamma = 1$, $\sigma = \alpha(1 - \tau_K)$ and $\varepsilon = 0$.

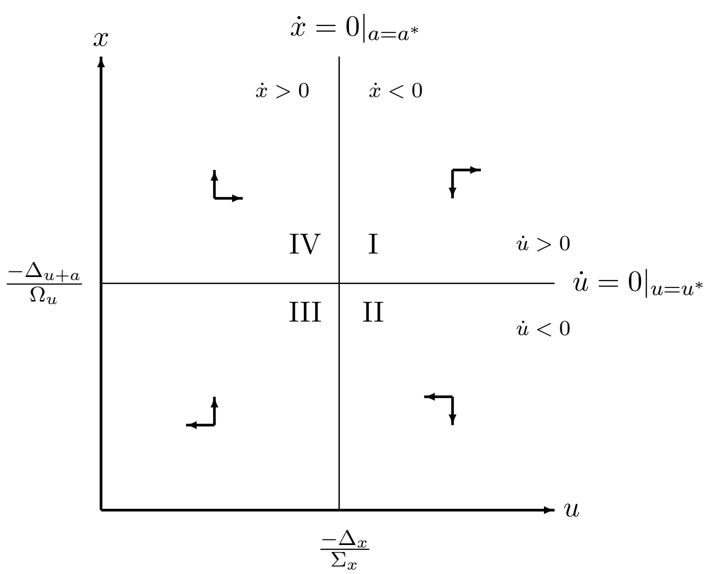

Figure 2: Example 2, $\gamma = 1$, $\sigma = \alpha(1 - \tau_K)$, $\varepsilon = 0$, $\alpha + \varphi = 1$ and $\alpha = \phi$.

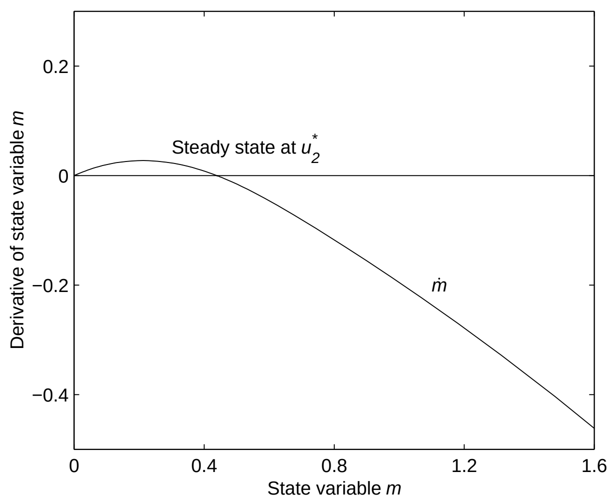

Figure 3: Example 3, stable manifold at u_2^* in terms of m.

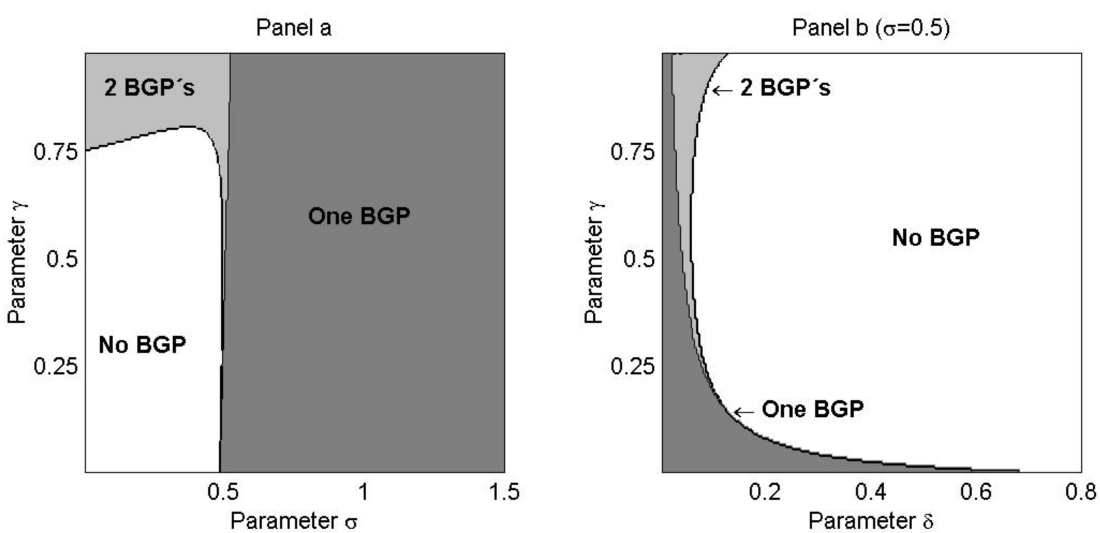

Figure 4: Number of BGP's.

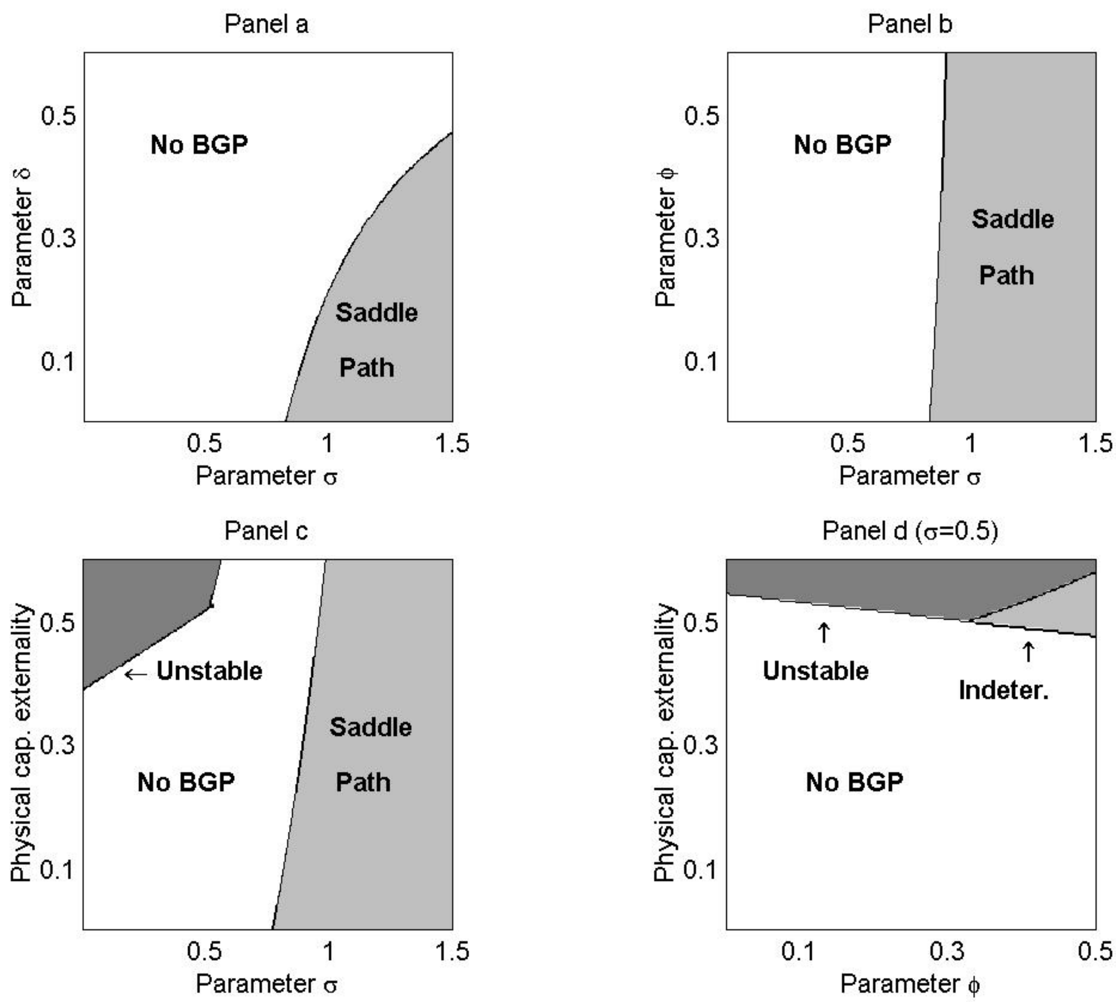

Figure 5: Stability of the BGP for $\gamma = 1$.

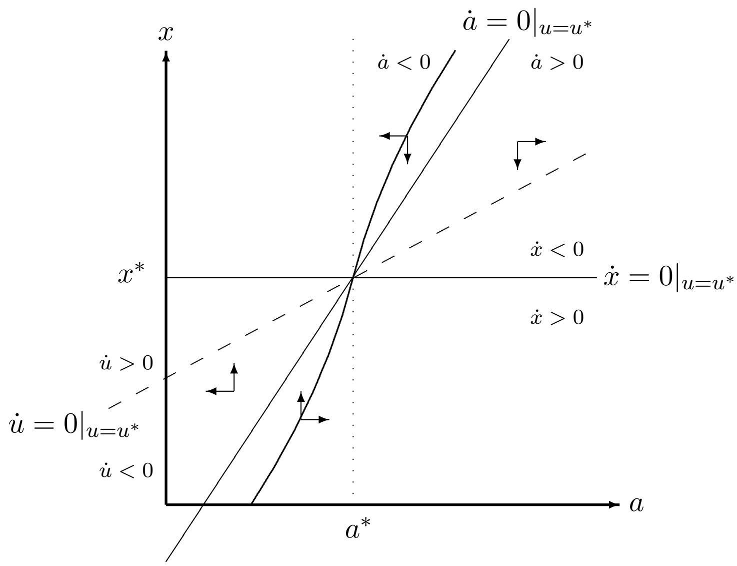

Figure 6: Equilibrium dynamics in $x - a$ plane for $u = u^*$ and $\alpha(1 - \tau_K) < \sigma < \varphi + \alpha$.

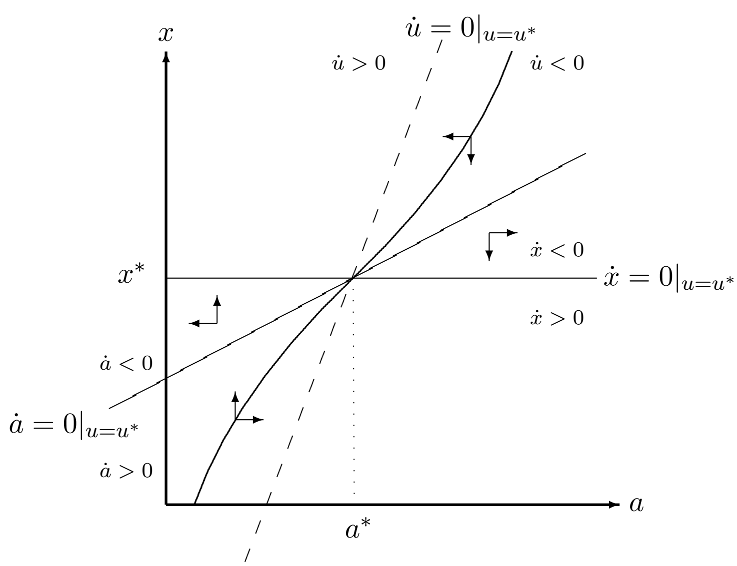

Figure 7: Equilibrium dynamics in $x - a$ plane for $u = u^*$ and $\sigma > \varphi + \alpha$.

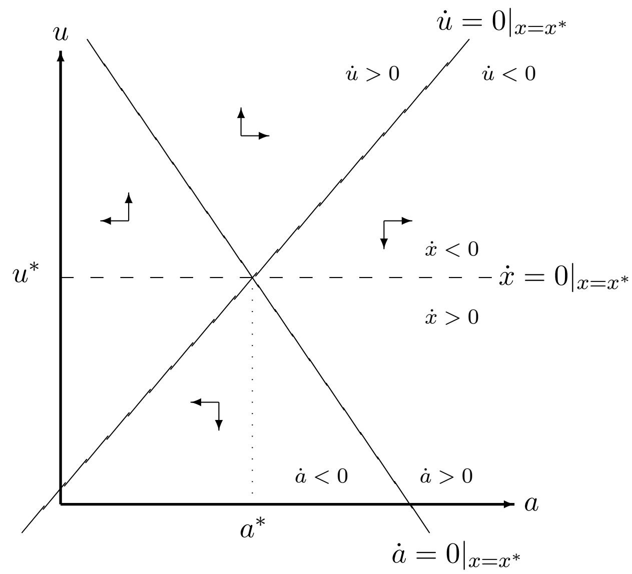

Figure 8: Equilibrium dynamics in $u - a$ plane for $x = x^*$ and $\sigma < 1$.

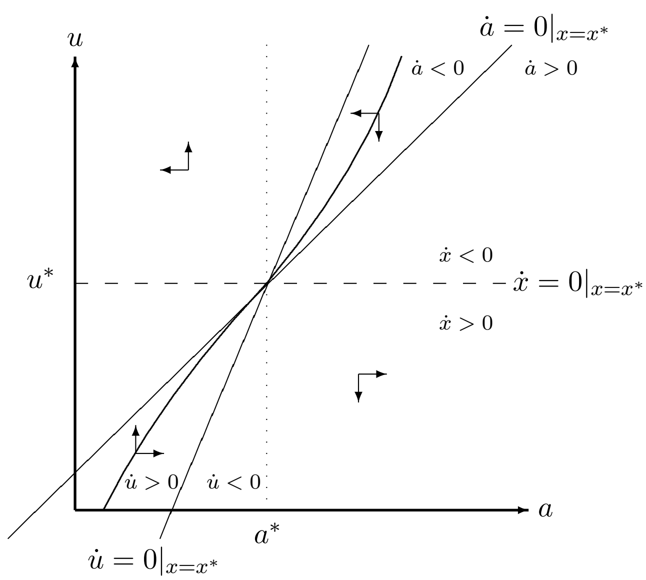

Figure 9: Equilibrium dynamics in $u - a$ plane for $x = x^*$ and $\sigma > 1$. Possibility 1.

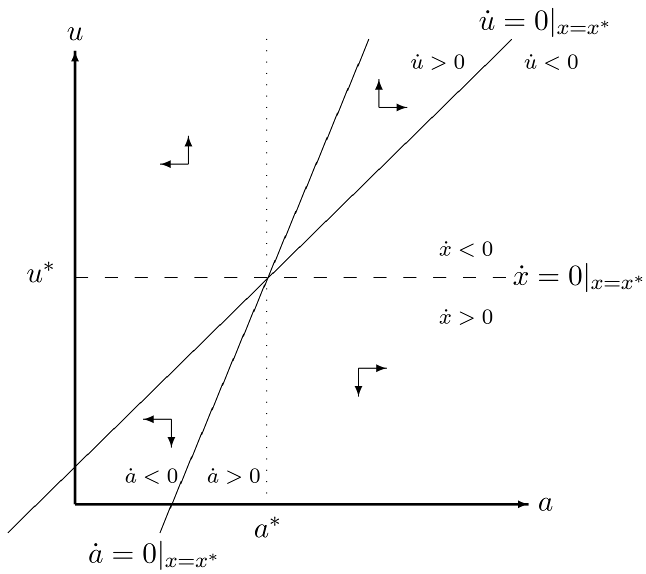

Figure 10: Equilibrium dynamics in $u - a$ plane for $x = x^*$ and $\sigma > 1$. Possibility 2.

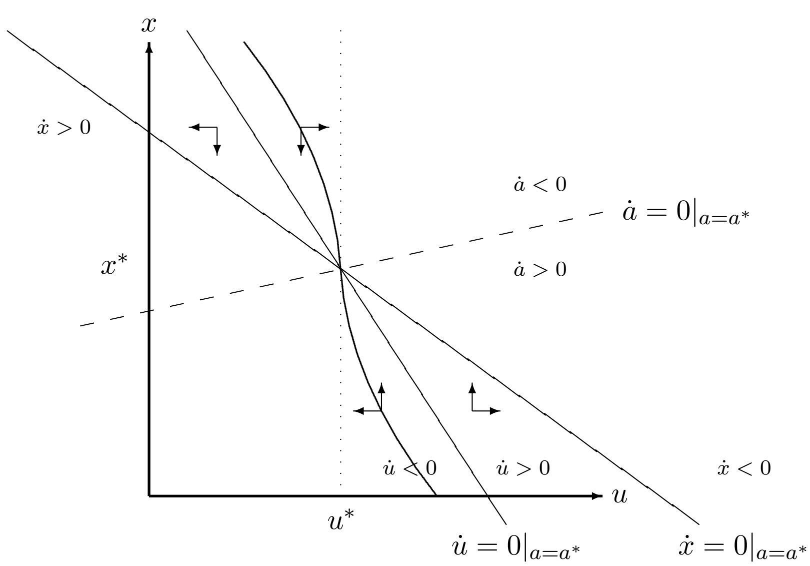

Figure 11: Equilibrium dynamics in $x - u$ plane for $a = a^*$ and $\alpha(1 - \tau_K) < \sigma < 1$.

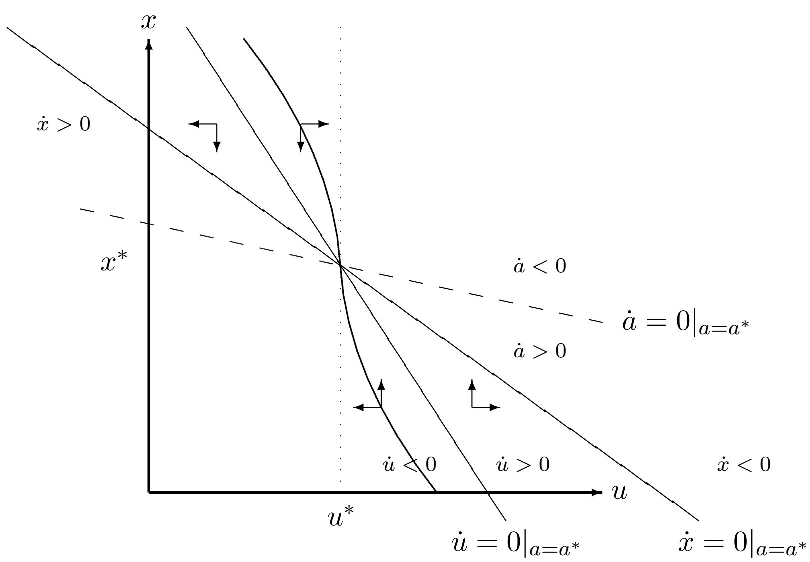

Figure 12: Equilibrium dynamics in $x - u$ plane for $a = a^*$ and $\sigma > 1$.