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MULTIPLIERS OF THE TRIGONOMETRIC SYSTEM

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Abstract
We study the system

$$\{\overline{L(t)}e^{int}\}_{n=-\infty}^{-1} \cup \{M(t)e^{int}\}_{n=0}^{\infty},$$

where L and M are boundary values of some outer functions defined in the unit disc. Necessary and sufficient conditions on functions L and M are found so that the system is a Schauder basis in $L^p(\mathbb{T}), 1 < p < \infty$.

Keywords Hardy spaces · Outer function · The trigonometric system · $M-$basis · Multiplier

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Introduction

We say that a system $\{z^m F(z)\}_{m=0}^{\infty}$ is a Beurling system if F is an outer function. In his fundamental work [3], Beurling particularly proved that if F is an outer function from $H^2(\mathbb{D})$, then the system $\{z^m F(z)\}_{m=0}^{\infty}$ is complete in the space $H^2(\mathbb{D})$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disc. This result can be easily extended for the spaces $H^p(\mathbb{D}), 1 \leq p < \infty$ (see [6]). In the previous paper [12], we studied questions of representations of functions from the spaces $H^p(\mathbb{D}), 1 \leq p < \infty$ by series with respect to Beurling systems. In the present paper, we study the following system of functions

$$\Psi_{L,M} = \{\overline{L(t)}e^{int}\}_{n=-\infty}^{-1} \cup \{M(t)e^{int}\}_{n=0}^{\infty}, \tag{1.1}$$

where L and M are boundary values of some outer functions defined in $\mathbb{D}$. We find necessary and sufficient conditions on functions L and M so that the system is a Schauder basis in $L^p(\mathbb{T}), 1 < p < \infty$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Completeness multipliers for a complete orthonormal system and for the trigonometric system have been studied in [4, 5]. Best references for the study of the trigonometric system are [2, 20]. The paper is divided into two parts. In the first part, we adopt some results for the spaces $H^p(\mathbb{T}), 1 \leq p < \infty$, and the second part will be dedicated to the study of the systems $e^{ikt}\Psi_{L,M}, k \in \mathbb{Z}$.

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Preliminary results, definitions and notations

Let

$$H^p(\mathbb{T}) = \{\phi \in L^p(\mathbb{T}) : \int_{\mathbb{T}} \phi(t)e^{int}dt = 0 \text{ for all } n \in \mathbb{N}\},$$

for $1 \leq p \leq \infty$. The spaces $H^p(\mathbb{T})$, $1 \leq p \leq \infty$ are Banach spaces of functions defined on $\mathbb{T}$. The convolution of functions $g, h \in L(\mathbb{T})$ is denoted by

$$g * h(t) = \frac{1}{2\pi} \int_{\mathbb{T}} g(\theta)h(t-\theta)d\theta.$$

We would like to define on $\mathbb{T}$ the analogue of an outer function on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. A holomorphic function F in $\mathbb{D}$ is an outer function if

$$F(re^{it}) = e^{i\alpha}e^{\phi * H_r(t)}, \quad \alpha \in \mathbb{T},$$

where ϕ is a real-valued integrable function defined on $\mathbb{T}$ [3] and

$$H_r(\theta) = \frac{1+re^{i\theta}}{1-re^{i\theta}} \quad (0 < r < 1, \theta \in \mathbb{T}).$$

The Cauchy and the Poisson kernels are defined as follows:

$$C_r(\theta) = \sum_{n=0}^{+\infty} r^n e^{in\theta} \quad 0 < r < 1, \theta \in \mathbb{T},$$

$$P_r(\theta) = \sum_{n=-\infty}^{+\infty} r^{|n|}e^{in\theta} = \frac{1-r^2}{1-2r\cos\theta + r^2}.$$

An important formula for representation of positive functions is due to G. Szegö [8, 19]. In his honour, the analogue of an outer function on $\mathbb{T}$ is called S–function. We say that a Lebesgue measurable function $\varphi : \mathbb{T} \to \mathbb{C}$ is an S–function if $\ln|\varphi| \in L^1(\mathbb{T})$ and

$$\varphi(t) = e^{i\alpha}|\varphi(t)|e^{il_{\varphi}(t)}, \quad \text{for some } \alpha \in \mathbb{T}, \tag{1.2}$$

where $l_{\varphi}(t) = \widetilde{\ln|\varphi|}(t)$ and the conjugate function of an integrable function g is denoted by $\tilde{g}$. For any measurable function $\psi : \mathbb{T} \to \mathbb{C}$ such that $\ln|\psi| \in L^1(\mathbb{T})$, we put

$$S(\psi)(t) = |\psi(t)|e^{il_{\psi}(t)}, \quad t \in \mathbb{T}.$$

The following statement follows from the definition.

Proposition 1.1 *Let φ be an S–function. Then, $S(\varphi)(t) = e^{i\alpha}\varphi(t)$ for some $\alpha \in \mathbb{T}$.*

It is easy to observe that the following properties also are true:

$$S(\phi\psi)(t) = S(\phi)(t) \cdot S(\psi)(t); \quad S(\frac{1}{\psi})(t) = \frac{1}{S(\psi)(t)}. \tag{1.3}$$

For a complex-valued integrable function g defined on $\mathbb{T}$, such that $\ln|g(t)|$ is integrable, we set

$$G_g(re^{it}) = e^{\ln|g| * H_r(t)}. \tag{1.4}$$

Evidently $G_g(z)$ is a non-zero holomorphic function in $z \in \mathbb{D}$, $G_g \in H^1(\mathbb{D})$ and

$$\lim_{r\to 1-} G_g(re^{it}) = S(g)(t) \quad \text{a.e. on } \mathbb{T}.$$

We also have that

$$\frac{1}{2\pi}\int_{\mathbb{T}} S(g)(t)dt = G_g(0) = e^{\frac{1}{2\pi}\int_{\mathbb{T}} \ln|g(t)|dt} \neq 0. \tag{1.5}$$

The class of S–functions in $H^p(\mathbb{T}), 1 \le p \le \infty$ is denoted by $H^p_s(\mathbb{T})$.
There is a vast literature on this topic (see, e.g. [6, 9, 13, 20] and others). It is well known (see, e.g. [9]) that $\ln|f(t)| \in L^1(\mathbb{T})$ if $f \in H^1(\mathbb{T})$. The following fact is an easy consequence of well-known results that can be found in the literature which we have mentioned earlier.

Proposition 1.2 *For any* $f \in H^1(\mathbb{T})$,

$$f(t) = F(t)\cdot S(f)(t),$$

where $F \in H^\infty(\mathbb{T})$ *and* $|F(t)| = 1$ *a.e. on* $\mathbb{T}$.

Given $f \in H^p_s(\mathbb{T})$, we can recover a non-zero holomorphic outer function in $\mathbb{D}$ by the Poisson integral. If φ is a non-negative integrable function such that $\ln\varphi \in L^1(\mathbb{T})$, then φ is the modulus of a function from $H^1(\mathbb{T})$ (see, e.g. [9], p.53). If $1 < p < \infty$, we use the previous assertion for $\varphi^{1/p}$ to deduce a similar proposition for a function in $H^p(\mathbb{T})$. The case $p = \infty$ is studied in [9]. Thus, the following statement holds.

Proposition 1.3 *Let g be a Lebesgue measurable function $g : \mathbb{T} \to \mathbb{C}$ such that $S(g) \in L^p(\mathbb{T}), 1 \le p \le \infty$. Then, $S(g) \in H^p(\mathbb{T})$.*

Set

$$\mathbf{N}(\mathbb{T}) = \left\{ \frac{g}{S(h)} \,:\, g, h \in H^\infty(\mathbb{T}) \right\}.$$

It is clear that $\ln|f(t)| \in L^1(\mathbb{T})$ if $f \in \mathbf{N}(\mathbb{T})$. The analogue of the following result is well known in $\mathbb{D}$. We give the proof because it is short.

Theorem 1.1 $H^1(\mathbb{T}) \subset \mathbf{N}(\mathbb{T})$.

Proof Let $f \in H^1(\mathbb{T})$. Set

$$f_0(t) = \begin{cases} \frac{1}{f(t)} & \text{if} \quad \frac{1}{|f(t)|} \le 1 \\ 1 & \text{if} \quad |f(t)| < 1. \end{cases}$$

By Proposition 1.3, we have that $S(f_0) \in H^\infty_s(\mathbb{T})$. On the other hand, $f\cdot S(f_0) \in H^\infty(\mathbb{T})$. Hence, $f \in \mathbf{N}(\mathbb{T})$. □

By the above theorem, we establish

Proposition 1.4 *Let $\phi \in H^1(\mathbb{T})$ and let $\psi \in H^1_s(\mathbb{T})$. Then, $\frac{\phi}{\psi} \in \mathbf{N}(\mathbb{T})$.*

Proof By Theorem 1.1, we have that

$$\phi = \frac{\phi_1}{\phi_2}; \quad \psi = \frac{\psi_1}{\psi_2},$$

where $\phi_1, \psi_1 \in H^\infty(\mathbb{T})$ and $\phi_2, \psi_2 \in H^\infty_s(\mathbb{T})$. On the other hand, by Proposition 1.1 and (1.3), we deduce that for some $\beta_1 \in \mathbb{T}$

$$e^{i\beta_1}\psi = S(\frac{\psi_1}{\psi_2}) = \frac{S(\psi_1)}{S(\psi_2)}.$$

Thus,

$$\begin{aligned}\frac{\phi}{\psi} =& e^{i\beta_1}\frac{\phi}{S(\psi)} = e^{i\beta_1}\frac{\phi_1 S(\psi_2)}{\phi_2 S(\psi_1)}\\ =& e^{i\beta_2}\frac{\phi_1 S(\psi_2)}{S(\phi_2)S(\psi_1)} = e^{i\beta_2}\frac{\phi_1 S(\psi_2)}{S(\phi_2\psi_1)},\end{aligned}$$

where $\beta_2 \in \mathbb{T}$. □

We also have

Theorem 1.2 *For any $p, 1 \le p \le \infty$*

$$L^p(\mathbb{T}) \cap \mathbf{N}(\mathbb{T}) = H^p(\mathbb{T}).$$

Proof Let $\phi \in L^p(\mathbb{T})$ and

$$\phi = \frac{g}{S(h)}, \quad \text{where } g, h \in H^\infty(\mathbb{T})$$

By Proposition 1.2, we have that $g(t) = G(t)S(g)(t)$, where $G \in H^\infty(\mathbb{T})$ and $|G(t)| = 1$ a.e. on $\mathbb{T}$. Hence, by (1.3), we deduce

$$\phi(t) = \frac{g(t)}{S(h)(t)} = \frac{G(t)S(g)(t)}{S(h)(t)} = G(t)S(\frac{g}{h})(t).$$

Thus, $S(\frac{g}{h}) \in L^p(\mathbb{T})$ and by Proposition 1.2 it follows that $\phi \in H^p(\mathbb{T})$. The inclusion $H^p(\mathbb{T}) \subseteq L^p(\mathbb{T}) \cap \mathbf{N}(\mathbb{T})$ holds by Theorem 1.1. □

The closed linear span in a separable Banach space $\mathbf{B}$ of a system of elements $X = \{x_k\}_{x=0}^\infty \subset \mathbf{B}$ is denoted by $\overline{\mathrm{span}}_{\mathbf{B}}(X)$. A system $X = \{x_k\}_{x=0}^\infty$ is complete in $\mathbf{B}$ if $\overline{\mathrm{span}}_{\mathbf{B}}(X) = \mathbf{B}$.
Beurling's approximation theorem [3, 6] in the $H^p(\mathbb{T}), 1 \le p < \infty$ spaces have a simple formulation. We give its proof for the completeness of the exposition.

Theorem 1.3 *Let $p \in [1, +\infty)$ and let $f \in H^p_s(\mathbb{T})$. Then, the system $\{f(t)e^{int}\}_{n=0}^\infty$ is complete in $H^p(\mathbb{T})$.*

Proof Consider the case $p > 1$. We assume that there exists $g \in H^{p'}(\mathbb{T}), 1/p + 1/p' = 1$ such that

$$\int_{\mathbb{T}} f(t)e^{int}\overline{g(t)}dt = 0 \quad \text{for all } n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}. \tag{1.6}$$

If we put $\varphi(t) = f(t) \cdot \overline{g(t)}$, then $\varphi \in H^1(\mathbb{T})$. By Proposition 1.4 and Theorem 1.2, it follows that

$$\overline{g(t)} = \frac{\varphi(t)}{f(t)} \in H^{p'}(\mathbb{T}).$$

Hence, g is a constant function. This means that $\int_{\mathbb{T}} f(t)dt = 0$, which contradicts the condition (1.5). If $p = 1$, then there exists $g \in L^\infty(\mathbb{T})$ such that (1.6) holds and the following condition

$$\int_{\mathbb{T}} e^{ikt} g(t)dt = 0 \quad \text{for all } k \in \mathbb{N}.$$

is not true. The rest of the proof is similar to the case $p > 1$ and we skip it. □

The following results were obtained by the author in the recent paper.
A system $X = \{x_k\}_{k=0}^{\infty} \subset \mathbf{B}$ is called minimal, if there exists a system $X^* = \{\phi_n\}_{n=0}^{\infty} \subset \mathbf{B}^*$, such that

$$\phi_n(x_k) = \delta_{nk} \quad (n, k \in \mathbb{N}),$$

where δ_{nk} is the Kronecker symbol ($\delta_{nk} = 0$ if $n \neq k$ and $\delta_{kk} = 1$). The system X^* is called dual to X. If X is a complete and minimal system in $\mathbf{B}$, then the dual system X^* is unique [14]. A set $\Psi \subset \mathbf{B}^*$ is called total if

$$\eta(x) = 0 \quad \text{for all} \quad \eta \in \Psi$$

if and only if $x = \mathbf{0}$. A system $X = \{x_k\}_{k=0}^{\infty} \subset \mathbf{B}$ is an $M-$basis in $\mathbf{B}$ if X is complete and minimal in $\mathbf{B}$ and its dual system X^* is total.
The following theorems were proved in [12]

Theorem 1.4 *Let $p \in [1, +\infty)$ and let $f \in H^p_{\mathrm{s}}(\mathbb{T})$. The system $\{e^{int} f(t)\}_{n=0}^{\infty}$ is an $M-$basis in $H^p(\mathbb{T})$.*

That the system $\{e^{int}\}_{n=0}^{\infty}$ is minimal in $H^p(\mathbb{T}, w), 1 \leq p < \infty$ was mentioned in [11] without proof. A complete and minimal system $X = \{x_k\}_{k=0}^{\infty} \subset \mathbf{B}$ with the dual system $X^* = \{\phi_k\}_{k=0}^{\infty} \subset \mathbf{B}^*$ is uniformly minimal if there exists $C > 0$ such that

$$\|x_k\|_{\mathbf{B}} \|\phi_k\|_{\mathbf{B}^*} \leq C \qquad \text{for all} \quad k \in \mathbb{N}_0.$$

Theorem 1.5 *Let $f \in H^p_{\mathrm{s}}(\mathbb{T})$ for some $1 < p < \infty$. Then, the system $\{e^{int} f(t)\}_{n=0}^{\infty}$ is uniformly minimal in $H^p(\mathbb{T})$, if and only if $[f]^{-1} \in H^{p'}(\mathbb{T})$.*

Theorem 1.6 *Let $f \in H^1_{\mathrm{s}}(\mathbb{T})$. If the system $\{e^{int} f(t)\}_{n=0}^{\infty}$ is uniformly minimal in $H^1(\mathbb{T})$, then $[f]^{-1} \in H^{\infty}(\mathbb{T})$. If $[f]^{-1} \in H^{\infty}(\mathbb{T})$ and the partial sums of its Fourier series are uniformly bounded in the $C(\mathbb{T})$ norm, then the system $\{e^{int} F^*(t)\}_{n=0}^{\infty}$ is uniformly minimal in $H^1(\mathbb{T})$.*

For our study, we need to know the dual system to $\{e^{int} f(t)\}_{n=0}^{\infty}$ in Theorem 1.5. By $S[\varphi](t)$, we denote the Fourier series of a function $\varphi \in L^1(\mathbb{T})$. By (1.5), we know that $c_0(f) \neq 0$, if $f \in H^1_{\mathrm{s}}(\mathbb{T})$. For any $m \in \mathbb{N}$

$$S_m[\varphi](t) = \sum_{j=-m}^{m} c_j(\varphi) e^{ijt}, \qquad c_j(\varphi) = \frac{1}{2\pi} \int_{\mathbb{T}} \varphi(\theta) e^{-ij\theta} d\theta.$$

Let $f \in H^1_{\mathrm{s}}(\mathbb{T})$ and assume without loss of generality that $c_0(f) = 1$. We set

$$F_f(z) = \sum_{j=0}^{\infty} c_j(f) z^j \qquad z \in \mathbb{D},$$

which is a non-zero holomorphic function. Hence, $[F_f(z)]^{-1}$ is also a holomorphic function in $\mathbb{D}$:

$$[F_f(z)]^{-1} = \sum_{j=0}^{\infty} b_{f,j} z^j \qquad z \in \mathbb{D}.$$

We have that

$$\sum_{j=0}^{k} c_j(f) b_{f,k-j} = 0 \quad \text{for all} \quad k \in \mathbb{N} \quad \text{and} \quad b_{f,0} = 1. \tag{1.7}$$

Set $T_0(f,t) \equiv 1$ and for $n \in \mathbb{N}$

$$T_n(f,t) = e^{int} + \sum_{\nu=0}^{n-1} \bar{b}_{f,n-\nu} e^{i\nu t} \qquad t \in \mathbb{T}. \tag{1.8}$$

By (1.7), we obtain that if $j \in \mathbb{N}_0$ and $j \le n$

$$\frac{1}{2\pi} \int_{\mathbb{T}} e^{ijt} f(t) \overline{T_n(f,t)} dt = \frac{1}{2\pi} \int_{\mathbb{T}} \sum_{k=0}^{n-j} c_k(f) e^{i(j+k)t} \overline{T_n(f,t)} dt$$

$$= \sum_{k=0}^{n-j} c_k(f) b_{f,n-j-k} = \delta_{jn}. \tag{1.9}$$

It is clear that the above integral is equal to zero if $j > n$.
An integrable non-negative function on $\mathbb{T}$ is called a weight function. We say that a weight function w is in the class $\mathcal{A}_p(\mathbb{T}), p \ge 1$ if there exists $C_p > 0$ such that for any interval $I \subset \mathbb{T}$

$$\frac{1}{|I|} \int_I w(t) dt \left[\frac{1}{|I|} \int_I w(t)^{-\frac{1}{p-1}} dt \right]^{p-1} \le C_p.$$

Sometimes it is called Muckenhoupt's condition [15].
If $f \in H^p_s(\mathbb{T})$ for some $1 < p < \infty$ and $\varphi \in H^p(\mathbb{T})$, as it was shown in [12], the partial sums of the expansion of φ with respect to the system $\{e^{int} f(t)\}_{n=0}^{\infty}$ can be represented by the formulae

$$\sigma_m[\varphi](t) = f(t) S_m[\varphi f^{-1}](t) \qquad m \in \mathbb{N}_0. \tag{1.10}$$

A function $g \in L^p(\mathbb{T}, w), 1 \le p < \infty$ if $g : \mathbb{T} \to \mathbb{C}$ is measurable on $\mathbb{T}$ and the norm is defined by

$$\|g\|_{L^p(\mathbb{T},w)} = \left(\int_{\mathbb{T}} |g(t)|^p w(t) dt \right)^{\frac{1}{p}} < +\infty.$$

The trigonometric system with different multipliers

The sufficiency of the following theorem was proved in [12].

Theorem 2.1 *Let $1 < p < \infty$ and let $f \in H^p_s(\mathbb{T})$. Then, the system $\{e^{int} f(t)\}_{n=0}^{\infty}$ is a Schauder basis in $H^p(\mathbb{T})$ if and only if $|f|^p \in \mathcal{A}_p(\mathbb{T})$.*

Proof We give only the proof of the necessity. Assume that the system $\{e^{int} f(t)\}_{n=0}^{\infty}$ is a Schauder basis in $H^p(\mathbb{T})$. Then, it is uniformly minimal in $H^p(\mathbb{T})$ and by Theorem 1.5 it follows that $[f]^{-1} \in H^{p'}(\mathbb{T})$. Thus, for some $C_p > 0$ independent of φ

$$\|f \cdot S_m[\varphi f^{-1}]\|_{L^p} = \|\sigma_m[\varphi]\|_{H^p} \le C_p \|\varphi\|_{H^p} \qquad \forall m \in \mathbb{N}_0.$$

The last inequality yields (see, e.g. [17], p.40)

$$\|f \cdot (\varphi f^{-1}) * P_r\|_{L^p} \le C_p \|\varphi\|_{H^p} \qquad \text{for } 0 < r < 1.$$

□

We need the following lemma for the proof.

Lemma 2.1 *Let $g \in L^p(\mathbb{T})$ and $h \in H^{p'}(\mathbb{T})$, where $1 \le p < \infty$. Then for $0 < r < 1, t \in \mathbb{T}$*

$$\frac{1}{2\pi}\int_{\mathbb{T}} g(\theta)h(\theta)P_r(t-\theta)d\theta = \frac{1}{2\pi}\int_{\mathbb{T}} g(\theta)P_r(t-\theta)d\theta \frac{1}{2\pi}\int_{\mathbb{T}} h(\theta)P_r(t-\theta)d\theta.$$

Proof For $z = re^{it}$, $|z| < 1$ and any $k \in \mathbb{N}_0$, we have

$$\frac{1}{2\pi}\int_{\mathbb{T}} g(\theta)e^{ik\theta}P_r(t-\theta)d\theta = \sum_{j=-\infty}^{+\infty} c_j(g)z^{j+k} = z^k \sum_{j=-\infty}^{+\infty} c_j(g)z^j.$$

Hence, for any $m \in \mathbb{N}$

$$\frac{1}{2\pi}\int_{\mathbb{T}} g(\theta)\sum_{k=0}^{m} c_k(h)e^{ik\theta}P_r(t-\theta)d\theta = \sum_{k=0}^{m} c_k(h)z^k \sum_{j=-\infty}^{+\infty} c_j(g)z^j.$$

Which yields

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{T}} g(\theta)h(\theta)P_r(t-\theta)d\theta &= \lim_{m\to+\infty}\frac{1}{2\pi}\int_{\mathbb{T}} g(\theta)\sum_{k=0}^{m} c_k(h)e^{ik\theta}P_r(t-\theta)d\theta \\ &= \sum_{k=0}^{+\infty} c_k(h)z^k \sum_{j=-\infty}^{+\infty} c_j(g)z^j = h * P_r(t)\cdot g * P_r(t).\end{aligned}$$

Let $\psi(\theta) = \mathrm{Re}\varphi(\theta)$. By Lemma 2.1, we easily check that

$$|(\psi f^{-1}) * P_r(t)| \le |(\varphi f^{-1}) * P_r(t)| \qquad 0 < r < 1, t \in \mathbb{T}.$$

Thus, for any real-valued $\psi \in L^p(\mathbb{T})$

$$\|f\cdot(\psi f^{-1}) * P_r\|_{L^p} \le B_p\|\psi\|_{L^p} \qquad \text{for } 0 < r < 1,$$

where $B_p > 0$ is independent of ψ. Afterwards, it is easy to see that the same inequality holds for any $\psi \in L^p(\mathbb{T})$. If we set $\omega(t) = |f(t)|^p$, then it follows that for some $B_p > 0$ independent of $g \in L^p(\mathbb{T}, \omega)$

$$\|g * P_r\|_{L^p(\mathbb{T},\omega)} \le B_p\|g\|_{L^p(\mathbb{T},\omega)} \qquad \text{for } 0 < r < 1,$$

By [7, 17], we finish the proof. □

Let $L, M \in H^p_s(\mathbb{T})$, $1 \le p < \infty$. Consider the system of functions (1.1). From Theorem 1.5, we deduce

Theorem 2.2 *Let $1 \le p < \infty$ and let $L, M \in H^p_s(\mathbb{T})$. Then, the system $\Psi_{L,M}$ is an $M-$basis in $L^p(\mathbb{T})$.*

Proof Suppose that the system $\Psi_{L,M}$ is not complete in $L^p(\mathbb{T})$. Then in the dual space $L^{p'}(\mathbb{T})$, there exists a non-trivial $\phi \in L^{p'}(\mathbb{T})$ such that

$$\int_{\mathbb{T}} \overline{L(t)}e^{-int}\overline{\phi(t)}dt = 0 \quad n \in \mathbb{N}$$

and

$$\int_{\mathbb{T}} M(t)e^{int}\overline{\phi(t)}dt = 0 \quad n \in \mathbb{N}_0.$$

From the above relations, it follows that $L\phi \in H^1(\mathbb{T})$ and $M\overline{\phi}$ is not a constant and is a $H^1(\mathbb{T})$ function. Afterwards, by Proposition 1.4 and Theorem 1.2, we obtain that $\phi \in H^{p'}(\mathbb{T})$ and $\overline{\phi} \in H^{p'}(\mathbb{T})$. Which can happen if and only if $\phi \equiv const$. Thus, $M \equiv const$, which contradicts the condition that $M\overline{\phi}$ is not a constant function.

Let us show that the system

$$\{\overline{T_{-n}(L,t)}\}_{n=-\infty}^{-1} \cup \{T_n(M,t)\}_{n=0}^{\infty} \tag{2.1}$$

is dual to $\Psi_{L,M}$. We have that for any $j \in \mathbb{N}$

$$\int_{\mathbb{T}} e^{ijt}M(t)e^{int}dt = 0 \quad \text{for all } n \in \mathbb{N}_0.$$

Thus from (1.7), it follows that for any $k \in \mathbb{N}$

$$\int_{\mathbb{T}} T_k(L,t)M(t)e^{int}dt = 0 \quad \text{for all } n \in \mathbb{N}_0.$$

For any $j \in \mathbb{N}$ by (1.9), we obtain that

$$\frac{1}{2\pi}\int_{\mathbb{T}} e^{-ijt}\overline{L(t)}T_n(L,t)dt = \frac{1}{2\pi}\overline{\int_{\mathbb{T}} e^{ijt}L(t)\overline{T_n(L,t)}dt} = \delta_{jn} \quad \forall n \in \mathbb{N}.$$

The rest of the proof of the minimality of the system $\Psi_{L,M}$ is clear. It remains to check that the system (2.1) is total. Assume that it is not true. Then for some non-trivial $\psi \in L^p(\mathbb{T})$ such that

$$\int_{\mathbb{T}} \psi(t)T_n(L,t)dt = 0 \quad n \in \mathbb{N}$$

and

$$\int_{\mathbb{T}} \psi(t)\overline{T_n(M,t)}dt = 0 \quad n \in \mathbb{N}_0.$$

The relations on the last line are equivalent to the following conditions: $\int_{\mathbb{T}} \psi(t)e^{-ijt}dt = 0$ for all $j \in \mathbb{N}_0$. Using the fact that $\int_{\mathbb{T}} \psi(t)dt = 0$, we deduce that $\int_{\mathbb{T}} \psi(t)e^{ikt}dt = 0 \quad \forall k \in \mathbb{N}$. Hence, $\psi(t) = 0$ a.e. on $\mathbb{T}$. □

By the above result and Theorem 1.5, we obtain

Theorem 2.3 *Let $1 < p < \infty$ and let $L, M \in H^p_s(\mathbb{T})$. Then, the system $\Psi_{L,M}$ is uniformly minimal in $L^p(\mathbb{T})$, if and only if the functions $[L]^{-1}, [M]^{-1} \in H^{p'}(\mathbb{T})$.*

From Theorems 2.2 and 2.3, we easily obtain

Theorem 2.4 *Let $1 \le p < \infty$, $k \in \mathbb{Z}$ and let $L, M \in H^p_s(\mathbb{T})$. Then, the system*

$$e^{ikt}\Psi_{L,M} = \{\overline{L(t)}e^{i(n+k)t}\}_{n=-\infty}^{-1} \cup \{M(t)e^{i(n+k)t}\}_{n=0}^{\infty}$$

is an M–basis in $L^p(\mathbb{T})$. Moreover, if $1 \le p < \infty$, the system $e^{ikt}\Psi_{L,M}$ is uniformly minimal in $L^p(\mathbb{T})$, if and only if the functions $[L]^{-1}, [M]^{-1} \in H^{p'}(\mathbb{T})$.

When $p = 1$ by Theorem 1.6, we have

Theorem 2.5 *Let $L, M \in H^1_s(\mathbb{T})$ and let $k \in \mathbb{Z}$. If the system $e^{ikt}\Psi_{L,M}$ is uniformly minimal in $L^1(\mathbb{T})$, then $[L]^{-1}, [M]^{-1} \in H^\infty(\mathbb{T})$. Moreover, if $[L]^{-1}, [M]^{-1} \in H^\infty(\mathbb{T})$ and the partial sums of the Fourier series of the functions $[L]^{-1}, [M]^{-1}$ are uniformly bounded in the $C(\mathbb{T})$ norm, then the system $e^{ikt}\Psi_{L,M}$ is uniformly minimal in $L^1(\mathbb{T})$.*

The following lemma was proved in [12]

Lemma 2.2 *Let $f \in H^p(\mathbb{T})$ and $g \in H^{p'}(\mathbb{T})$, where $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, $p' = \infty$ if $p = 1$. Then, $S_n[fg](t) = \sum_{j=0}^{n} c_j(f)e^{ijt}S_{n-j}[g](t)$ for any $n \in \mathbb{N}_0$.*

Theorem 2.6 *Let $1 < p < \infty$, $k \in \mathbb{Z}$ and let $L, M \in H^p_s(\mathbb{T})$. Then the system $e^{ikt}\Psi_{L,M}$ is a Schauder basis in $L^p(\mathbb{T})$ if and only if $|L|^p, |M|^p \in \mathcal{A}_p(\mathbb{T})$.*

Proof By Theorem 2.4, we know that the system $e^{ikt}\Psi_{L,M}$ is complete and minimal in $L^p(\mathbb{T})$. For any $n \in \mathbb{N}$, we set

$$
\begin{aligned}
B^*_{L,n}(t,\theta) =& \overline{L(t)} \sum_{k=-n}^{-1} e^{ikt} T_{-k}(L,\theta) \\
=& \overline{L(t)} \sum_{k=1}^{n} e^{-ikt} \sum_{j=0}^{k} \overline{b_{L,k-j}} e^{ij\theta} \\
=& \overline{L(t)} \sum_{k=1}^{n} \overline{b_{L,k}} e^{-ikt} \\
&+ \overline{L(t)} \sum_{j=1}^{n} e^{ij\theta} \sum_{k=j}^{n} \overline{b_{L,k-j}} e^{-ikt} \\
=& \overline{L(t)}\, \overline{S_n[L^{-1}-1](t)} \\
&+ \overline{L(t)} \sum_{j=1}^{n} e^{ij(\theta-t)} \overline{S_{n-j}[L^{-1}](t)}
\end{aligned}
$$

Let $g \in L^p(\mathbb{T})$ be a real-valued function and let g_H be its projection on $H^p(\mathbb{T})$. By Theorem 2.2, it follows that

$$\int_{\mathbb{T}} g_H(\theta) B^*_{L,n}(t,\theta)d\theta = 0 \quad n \in \mathbb{N}.$$

Hence, for any $n \in \mathbb{N}$

$$\Lambda^*_{L,n}[g](t) = \frac{1}{2\pi} \int_{\mathbb{T}} g(\theta) B^*_{L,n}(t,\theta)d\theta = \Lambda^*_{L,n}[g - g_H](t).$$

We have that

$$
\begin{aligned}
\Lambda^*_{L,n}[g-g_H](t) =& \overline{L(t)} \sum_{j=1}^{n} c_{-j}(g) e^{-ijt} \overline{S_{n-j}[L^{-1}](t)} \\
=& \overline{L(t)} \sum_{j=1}^{n} \overline{c_j(g)} e^{-ijt} \overline{S_{n-j}[L^{-1}](t)} \\
=& \overline{L(t)} \sum_{j=1}^{n} \overline{c_j(\overline{g-g_H})} e^{-ijt} \overline{S_{n-j}[L^{-1}](t)} \\
=& \overline{L(t)}\, \overline{S_n[(\overline{g-g_H})L^{-1}](t)}.
\end{aligned}
$$

The last equation is obtained by Lemma 2.2. In a similar way, we deduce (see also [12])

$$\begin{aligned}
B_{M,n}(t,\theta) =& M(t)\sum_{k=0}^{n} e^{ikt}\overline{T_k(M,\theta)} \\
=& M(t)\sum_{k=0}^{n} e^{ikt}\sum_{j=0}^{k} b_{M,k-j}e^{-ij\theta} \\
=& M(t)\sum_{j=0}^{n} e^{-ij\theta}\sum_{k=j}^{n} b_{M,k-j}e^{ikt} \\
=& M(t)\sum_{j=0}^{n} e^{ij(t-\theta)}S_{n-j}[M^{-1}](t).
\end{aligned}$$

We set

$$\begin{aligned}
\Lambda_{M,n}[g](t) =& \frac{1}{2\pi}\int_{\mathbb{T}} g(\theta)B_{M,n}(t,\theta)d\theta \\
=& M(t)\sum_{j=0}^{n} c_j(g_H)e^{ijt}S_{n-j}[M^{-1}](t) \\
=& M(t)S_n[g_H M^{-1}](t) = M(t)S_n[g_H M^{-1}](t),
\end{aligned}$$

Let $u(t) = |L(t)|^p$, and let $v(t) = |M(t)|^p$, then by a well-known weighted norm inequality [7] (see also [10]) we finish the proof of sufficiency because the conditions of Banach's theorem [1] hold in our case. We will give the proof of the inequality

$$\int_{\mathbb{T}} |\Lambda^*_{L,n}[g](t)|^p dt \le C_p \int_{\mathbb{T}} |g(t)|^p dt,$$

where $C_p > 0$ is independent of g. Indeed,

$$\begin{aligned}
\int_{\mathbb{T}} |\Lambda^*_{L,n}[g](t)|^p dt =& \int_{\mathbb{T}} |\Lambda^*_{L,n}[g-g_H](t)|^p dt \\
=& \int_{\mathbb{T}} |L(t)\, S_n[\overline{(g-g_H)}L^{-1}](t)|^p dt \\
=& \int_{\mathbb{T}} |S_n[\overline{(g-g_H)}L^{-1}](t)|^p u(t)dt \\
\le& A_p\int_{\mathbb{T}} |[g(t)-g_H(t)][L(t)]^{-1}|^p u(t)dt \\
=& A_p\int_{\mathbb{T}} |g(t)-g_H(t)|^p dt \\
\le& C_p\int_{\mathbb{T}} |g(t)|^p dt.
\end{aligned}$$

The proof of the necessity follows by Theorem 2.1. ◻

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Declarations

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