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A Fourier restriction theorem for a perturbed hyperbolic paraboloid: polynomial partitioning

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Abstract

We consider a surface with negative curvature in $\mathbb{R}^3$, which is a cubic perturbation of the saddle. For this surface, we prove a new restriction theorem, analogous to the theorem for paraboloids proved by L. Guth in 2016. This specific perturbation has turned out to be of fundamental importance also to the understanding of more general classes of one-variate perturbations, and we hope that the present paper will further help to pave the way for the study of general perturbations of the saddle by means of the polynomial partitioning method.

Keywords Hyperbolic hypersurface · Fourier restriction · Polynomial partitioning

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1 Introduction

Let $S \subset \mathbb{R}^n$ be a smooth hypersurface. The Fourier restriction problem, introduced by E. M. Stein in the seventies (for general submanifolds), asks for the range of exponents $\tilde{p}$ and $\tilde{q}$ for which an a priori estimate of the form

$$\left(\int_S |\widehat{f}|^{\tilde{q}}\, d\sigma \right)^{1/\tilde{q}} \leq C \|f\|_{L^{\tilde{p}}(\mathbb{R}^n)}$$

holds true for every Schwartz function $f \in \mathcal{S}(\mathbb{R}^n)$, with a constant C independent of f. Here, $d\sigma$ denotes the surface measure on S.

The sharp range in dimension $n = 2$ for curves with non-vanishing curvature was determined through work by Fefferman et al. [13,38]. In higher dimension, the sharp $L^{\tilde{p}} - L^2$ result for hypersurfaces with non-vanishing Gaussian curvature was obtained by Stein and Tomas [27,34] (see also Strichartz [29]). Some more general classes of surfaces were treated by Greenleaf [16]. In work by Ikromov, Kempe and Müller [18] and Ikromov and Müller [19,20], the sharp range of Stein-Tomas type $L^{\tilde{p}} - L^2$ restriction estimates has been determined for a large class of smooth, finite-type hypersurfaces, including all analytic hypersurfaces.

The question about general $L^{\tilde{p}} - L^{\tilde{q}}$ restriction estimates is nevertheless still wide open. Fourier restriction to hypersurfaces with non-negative principal curvatures has been studied intensively by many authors. Major progress was due to J. Bourgain in the nineties [4–6]. At the end of that decade the bilinear method was introduced [23–25,30–33,37]. A new impulse to the problem has been given with the multilinear method [3,7]. The best results up to date have been obtained with the polynomial partitioning method, developed by Guth [14,15] (see also [17,36] for recent improvements).

For the case of hypersurfaces of non-vanishing Gaussian curvature but principal curvatures of different signs, besides Tomas-Stein type Fourier restriction estimates, until recently the only case which had been studied successfully was the case of the hyperbolic paraboloid (or "saddle") in $\mathbb{R}^3$: in 2015, independently Lee [22] and Vargas [35] established results analogous to Tao's theorem [30] on elliptic surfaces (such as the 2 -sphere), with the exception of the end-point, by means of the bilinear method. Recently, Stovall [28] was able to include also the end-point case. Moreover, Cho and Lee [12], and Kim [21], improved the range by adapting ideas by Guth [14,15] which are based on the polynomial partitioning method. Results on higher dimensional hyperbolic paraboloids have just been reported by Barron [1].

In our previous paper [9], we considered a one variable perturbation of the hyperbolic paraboloid, and applied the bilinear method, obtaining results analogous to [22,35]. Further results for more general classes of one-variate finite type, respectively flat, perturbations based on the bilinear method were obtained in [10,11]. Bilinear estimates are also key elements in the results obtained with the polynomial partitioning method for the non–negative curvature case. With the base of our previous bilinear results, we explore in this article the application of that method to our model surfaces. We obtain the analogous result to [14] for our class of hyperbolic surfaces.

More precisely, we consider the family of functions

$$\phi_\gamma(x, y) = xy + \frac{\gamma}{3}y^3 \quad \text{for } -1 \le \gamma \le 1,$$

defined on $\Sigma := [0, 1] \times [0, 1]$, and the corresponding surfaces

$$S_\gamma = \{(x, y, \phi_\gamma(x, y)) : (x, y) \in \Sigma\}.$$

The associated adjoints to the corresponding Fourier restriction operators are the *extension* operators given by

$$\mathcal{E}_\gamma(\xi) := \int_\Sigma f(x, y)\, e^{i[\xi_1 x + \xi_2 y + \xi_3 \phi_\gamma(x, y)]}\, dx\, dy, \quad \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3.$$

Our main result will be the following analogue of the results by Bassam Shayya for elliptic surfaces with weighted norms [26] and by Jongchon Kim for the unperturbed hyperbolic paraboloid [21]:

Theorem 1.1 *For any $p > 3.25$ with $p > 2q'$, there is a constant $C_{p,q}$ which is independent of $\gamma \in [-1, 1]$ such that*

$$\|\mathcal{E}_\gamma f\|_{L^p(\mathbb{R}^3)} \le C_{p,q}\|f\|_{L^q(\Sigma)}$$

for all $f \in L^q(\Sigma)$.

> **Remark 1.2** (i) Note that in this result and the corresponding Fourier restriction estimate we can replace the domain $\Sigma := [0, 1] \times [0, 1]$ by the larger neighborhood $[-1, 1] \times [-1, 1]$ of the origin, simply by dividing the latter into four sectors of angle $\pi/2$ and reducing the corresponding estimates in each of these sectors to the estimate given in the theorem by means of symmetry considerations.
>
> ii) Our arguments in this paper easily extend to more general perturbations of xy of cubic type in the sense of [10] in place of the perturbation $\frac{\gamma}{3}y^3$, and the same reasoning as in [10] then allows to prove Fourier restriction to surfaces given as the graph of $\phi(x, y) := xy + h(y)$, where the function h is smooth and of finite type at the origin, in the same range of p's and q's as in Theorem 1.1.

To simplify the understanding of this paper, we will closely follow the notation and structure of the paper [14], which makes use of induction on scales arguments.

Denote by B_R the cube $B_R := [-R, R]^3$, $R \ge 0$. For technical reasons that will become clear soon we shall not be able to induct on an $L^\infty \to L^{3.25}$ estimate for $\mathcal{E}_\gamma$ as in [14] (Theorem 2.2). Instead, we shall induct on the following statement:

Theorem 1.3 *For any $\epsilon > 0$, there is a constant C_ϵ such that for any $\gamma \in [-1, 1]$ and for any $R \ge 1$*

$$\|\mathcal{E}_\gamma f\|_{L^{3.25}(B_R)} \le C_\epsilon R^\epsilon \|f\|_{L^2(\Sigma)}^{2/q} \|f\|_{L^\infty(\Sigma)}^{1-2/q},$$

for all $3.25 \ge q > 2.6$ and all $f \in L^\infty(\Sigma)$.

Applying this estimate to characteristic functions, we obtain the estimate

$$\|\mathcal{E}_\gamma f\|_{L^{3.25}(B_R)} \le C_\epsilon R^\epsilon \|f\|_{L^{q,1}(\Sigma)},$$

for all $q > 2.6$. Real interpolation with the trivial $L^1 \to L^\infty$ estimate for the extension operator then gives

$$\|\mathcal{E}_\gamma f\|_{L^p(B_R)} \le C_\epsilon R^\epsilon \|f\|_{L^q(\Sigma)},$$

for all $p > 3.25$, $p > 2q'$. Finally, an ϵ-removal theorem (Theorem 5.3 in [21]) gives Theorem 1.1.

2 Broad points

Definition 2.1 Fix $K \gg 1$ to be a large dyadic number. We introduce four different partitions of the square $\Sigma = [0, 1] \times [0, 1]$:

We divide Σ into K^2 disjoint squares (called *caps*) τ of sidelength K^{-1}. For a cap τ, we denote by $f_\tau := f\chi_\tau$. This *basic decomposition* into caps will play a fundamental role in many places of our subsequent arguments, as in [14]. However, in contrast to [14], it will play no role in the definition of α-broadness given below. For the latter notion, the next three decompositions will be relevant:

We divide Σ into $K^{1/4}$ disjoint *long horizontal strips* L of dimensions $1 \times K^{-1/4}$, we divide Σ into $K^{1/2}$ disjoint *long vertical strips* L of dimensions $K^{-1/2} \times 1$ and, finally, we divide Σ into $K^{3/4}$ disjoint *short vertical strips* L of dimensions $K^{-1/2} \times K^{-1/4}$, by looking at all intersections of a long horizontal with a long vertical strip. For a strip L, we denote by $f_L := f\chi_L$.

Let $\alpha \in (0, 1)$. Given the function f, $\gamma \in [-1, 1]$ and K, we say that the point $\xi \in \mathbb{R}^3$ is *α-broad for $\mathcal{E}_\gamma f$* if

$$\max_L |\mathcal{E}_\gamma f_L(\xi)| \leq \alpha |\mathcal{E}_\gamma f(\xi)|,$$

where the $\max_L$ is taken over all

(a) horizontal strips as above if $|\gamma| K^{1/2} \geq 1$, or
(b) horizontal and vertical strips as above if $|\gamma| K^{1/2} < 1$.

We define $Br_\alpha \mathcal{E}_\gamma f(\xi)$ to be $|\mathcal{E}_\gamma f(\xi)|$ if ξ is α-broad, and zero otherwise.

Note: In contrast to [14], we shall here consider the functions f to be defined on the square Σ, which will have slight technical advantages, whereas Guth views them as functions on the surface S_γ. Of course, we can as well identify our functions f with the corresponding functions $(x, y, \phi_\gamma(x, y)) \mapsto f(x, y)$ on S_γ. Accordingly, one can identify our "caps" τ and strips L with the corresponding subsets of the surface S_γ that are the graphs of ϕ_γ over these sets. This explain why we still like to call the sets τ "caps" .

We will prove the following analogue to Theorem 2.4. in [14]:

Theorem 2.1 *For any $0 < \epsilon < 10^{-10}$, there are constants $K = K(\epsilon) \gg 1$ and C_ϵ such that for any radius $R \geq 1$ and for any $|\gamma| \leq 1$*

$$\|Br_{K^{-\epsilon}} \mathcal{E}_\gamma f\|_{L^{3.25}(B_R)} \leq C_\epsilon R^\epsilon \|f\|_{L^2(\Sigma)}^{12/13} \|f\|_{L^\infty(\Sigma)}^{1/13}$$

for all $f \in L^\infty(\Sigma)$. Moreover $K(\epsilon) \to \infty$ as $\epsilon \to 0$.

Note that Theorem 1.3 follows from this theorem by arguments that are similar to those in [14]. To show this, let us put $p := 3.25$.

We divide the domain of integration B_R in (1.3) into four subsets:

$A := \{\xi \in B_R : \xi \text{ is } K^{-\epsilon} - \text{broad for } \mathcal{E}_\gamma f\}$,

$B := \{\xi \in B_R : |\mathcal{E}_\gamma f_L(\xi)| > K^{-\epsilon} |\mathcal{E}_\gamma f(\xi)| \text{ for some long horizontal strip } L\}$,

$C := \{\xi \in B_R \setminus B : |\mathcal{E}_\gamma f_L(\xi)| > K^{-\epsilon} |\mathcal{E}_\gamma f(\xi)| \text{ for some long vertical strip } L\}$

$$D := \{\xi \in B_R \setminus (B \cup C) : |\mathcal{E}_\gamma f_L(\xi)| > K^{-\epsilon}|\mathcal{E}_\gamma f(\xi)| \text{ for some short vertical strip } L\}.$$

By the definition of broad points, $B_R = A \cup B \cup C \cup D$. Notice also that if $|\gamma|K^{1/2} \geq 1$, then $C = D = \emptyset$ by construction.

If $\xi \in A$, then $|\mathcal{E}_\gamma f(\xi)| = Br_{K^{-\epsilon}}\mathcal{E}_\gamma f(\xi)$, so that the contribution of A can be controlled using Theorem 2.1. Notice that

$$\|f\|_{L^2(\Sigma)}^{12/13} \|f\|_{L^\infty(\Sigma)}^{1/13} \leq \|f\|_{L^2(\Sigma)}^{2/q} \|f\|_{L^\infty(\Sigma)}^{1-2/q},$$

since $q > 2.6 > 13/6$.

For the other parts, we induct on the size of R.

For $\xi \in B$ we estimate

$$|\mathcal{E}_\gamma f(\xi)| < K^\epsilon \sup_L |\mathcal{E}_\gamma f_L(\xi)| \leq K^\epsilon \left(\sum_L |\mathcal{E}_\gamma f_L(\xi)|^p\right)^{1/p}, \tag{1}$$

where here the supremum and sum are taken over all long horizontal strips L.

If $L = [0, 1] \times [b, b + K^{-1/4}]$ is any of these long horizontal strips, we scale and translate $y = b + K^{-1/4}y'$. Then

$$K^{1/4}\phi_\gamma(x, y) = (x + \gamma K^{-1/4}by')y' + \frac{\gamma}{3K^{1/2}}y'^3 + K^{1/4}bx + b^2\gamma y' + \text{ constant}.$$

By applying the linear change of coordinates $x' = x + \gamma K^{-1/4}by'$, we obtain

$$K^{1/4}\phi_\gamma(x, y) = \phi_{\gamma/K^{1/2}}(x', y') + K^{1/4}bx' + \text{ constant}.$$

Then

$$|\mathcal{E}_\gamma f_L(\xi)| = K^{-1/4}\left|\mathcal{E}_{\gamma/K^{1/2}}f^L\left(\xi_1 + b\xi_3, (\xi_2 - b\gamma\xi_1)K^{-1/4}, \xi_3 K^{-1/4}\right)\right|, \tag{2}$$

where we have defined f^L by $f^L(x', y') := f_L(x, y)$, so that $\|f^L\|_2 = K^{1/8}\|f_L\|_2$ and $\|f^L\|_\infty \leq \|f\|_\infty$. Note that we have $y' \in [0, 1]$ and $x' \in [-1, 2]$, since $|\gamma bK^{-1/4}| \leq 1$, and that the function $\mathcal{E}_{\gamma/K^{1/2}}f^L$ is supported in a box of dimensions $2R \times \frac{2R}{K^{1/4}} \times \frac{2R}{K^{1/4}}$. What is crucial here is that, compared to B_R, this box is shorter by the factor $2K^{-1/4} \leq 1/2$ in the ξ_3-direction, for K sufficiently large.

A problem more of technical nature is that in ξ_1-direction it is still of the same size as R. However, as we shall show in Lemma 5.1, we can automatically pass from an estimate on a box $B_{R'}$ to a corresponding estimate on the whole "plate" $P_{R'} := \mathbb{R}^2 \times [0, R']$ containing $B_{R'}$. Applying this in the present situation, with $R' := 2K^{-1/4}R \leq R/2$, by our induction hypothesis we may then assume that the following estimate holds true:

$$\|\mathcal{E}_{\gamma/K^{1/2}}f^L\|_{L^{3.25}(P_{R'})} \leq C_\epsilon R'^\epsilon \|f^L\|_{L^2(\Sigma)}^{2/q} \|f^L\|_{L^\infty(\Sigma)}^{1-2/q}.$$

Thus, by (1) and (2), we see that

$$\begin{aligned}
\|\mathcal{E}_\gamma f\|_{L^p(B)} &\leq K^{1/2p-1/4+\epsilon}\left(\sum_L \|\mathcal{E}_{\gamma/K^{1/2}}f^L\|_{L^p\left(P_{2R/K^{1/4}}\right)}^p\right)^{1/p} \\
&\leq CC_\epsilon R^\epsilon K^{1/2p-1/4q'+3\epsilon/4}\|f\|_2^{2/q} \|f\|_\infty^{1-2/q} \\
&\leq \frac{1}{10}C_\epsilon R^\epsilon \|f\|_2^{2/q} \|f\|_\infty^{1-2/q},
\end{aligned}$$

since $p > 2q'$.

For $\xi \in C$, i.e., in the case of long vertical strips, we need to be a bit more careful. The natural change of coordinates is now $x = a + K^{-1/2}x'$, if the long vertical strip L is given by $L = [a, a + K^{-1/2}] \times [0, 1]$. Then

$$K^{1/2}\phi_\gamma(x, y) = x'y + K^{1/2}\frac{\gamma}{3}y^3 + aK^{1/2}y = \phi_{\gamma K^{1/2}}(x', y) + aK^{1/2}y,$$

so to fit into our scheme, we need that $|\gamma K^{1/2}| \leq 1$. This is the reason why we consider this type of strips only when $|\gamma K^{1/2}| \leq 1$. Then we find that

$$|\mathcal{E}_\gamma f_L(\xi)| = K^{-1/2}|\mathcal{E}_{\gamma K^{1/2}}f^L(\xi_1 K^{-1/2}, \xi_2 + a\xi_3, \xi_3 K^{-1/2})|,$$

where f^L is now defined by $f^L(x', y) := f_L(x, y)$, and can argue in a similar way as in the preceding case.

As for D, if $L = [a, a + K^{-1/2}] \times [b, b + K^{-1/4}]$ is any of the short vertical strips, then we scale and translate $x = a + K^{-1/2}x'$, $y = b + K^{-1/4}y'$. Then

$$K^{3/4}\phi_\gamma(x, y) = (x' + \gamma K^{1/4}by')(y' + K^{1/4}b) + \frac{\gamma}{3}y'^3 + K^{1/2}ay' + \text{constant}.$$

By applying the linear change of coordinates $x'' = x' + \gamma K^{1/4}by'$, $y'' = y'$ (note that, since $|\gamma|K^{1/2} \leq 1$, we have that $|\gamma|bK^{1/4} \leq 1$), we obtain

$$K^{3/4}\phi_\gamma(x, y) = \phi_\gamma(x'', y'') + K^{1/4}bx'' + K^{1/2}ay'' + \text{constant}.$$

Then, if $\xi \in E$,

$$|\mathcal{E}_\gamma f_L(\xi)| = K^{-3/4}|\mathcal{E}_\gamma f^L(K^{-1/2}(\xi_1 + b\xi_3), K^{-1/4}(\xi_2 + a\xi_3 - b\gamma\xi_1), K^{-3/4}\xi_3)|,$$

where we have defined f^L by $f^L(x'', y'') := f_L(x, y)$. From here on , we argue in a similar way as before. $\qquad\square$

3 Reduction of Theorem 2.1 to a setup allowing for inductive arguments

Following Section 3 in [14], we shall next devise a setup and formulate a more general statement in Theorem 3.2 which will become amenable to inductive arguments. As in that paper, we change and extend our previous notation slightly. We introduce a "multiplicity" $\mu \geq 1$, and choose accordingly *caps* τ which now are allowed to be squares of possibly larger side length $r_\tau \in [K^{-1}, \mu^{1/2}K^{-1}]$ than before. It can then happen that such a cap τ is no longer contained in Σ; in that case, we truncate it by replacing it with its intersection with Σ.

We assume that we are given a family of such caps τ covering $\Sigma = [0, 1] \times [0, 1]$ such that their centers are K^{-1}- separated. Hence, at any point there will be at most μ of these caps which overlap at that point. Notice also that there are at most K^2 caps τ in the family. We also assume that we have a decomposition

$$f = \sum_\tau f_\tau, \tag{3}$$

where $\text{supp} f_\tau \subset \tau$.

Given the family of caps, we define recursively a fixed family of *ragged long horizontal strips* S_ℓ, $(\ell = 1, 2, \ldots, [\mu^{-1/2}K^{1/4}])$, of "widths" $\sim \mu^{1/2}K^{-1/4}$, in the following way:

$$\mathcal{F}_1 := \{\tau : \tau^0 \cap ([0, 1] \times [0, \mu^{1/2}K^{-1/4}]) \neq \emptyset\} \quad \text{and} \quad S_1 := \bigcup_{\tau \in \mathcal{F}_1} \tau,$$

$$\mathcal{F}_2 := \{\tau \notin \mathcal{F}_1 : \tau^0 \cap ([0, 1] \times [\mu^{1/2}K^{-1/4}, 2\mu^{1/2}K^{-1/4}]) \neq \emptyset\} \quad \text{and} \quad S_2 := \bigcup_{\tau \in \mathcal{F}_2} \tau,$$

$$\vdots$$

$$\mathcal{F}_\ell := \{\tau \notin \cup_{j=1}^{\ell-1}\mathcal{F}_j : \tau^0 \cap ([0, 1] \times [(\ell-1)\mu^{1/2}K^{-1/4}, \ell\mu^{1/2}K^{-1/4}]) \neq \emptyset\} \quad \text{and} \quad S_\ell := \bigcup_{\tau \in \mathcal{F}_\ell} \tau,$$

$$\vdots$$

Here, τ^0 denotes the open interior of τ. Note that the families $\mathcal{F}_\ell$ are pairwise disjoint. Define $fs_\ell := \sum_{\tau \in \mathcal{F}_\ell} f_\tau$, so that $f = \sum_\ell fs_\ell$.

When $|\gamma|K^{1/2} \leq 1$, we also define a family of pairwise in measure disjoint *ragged long vertical strips* of "widths" $\sim \mu^{1/2}K^{-1/2}$ in an analogous way, and a family of pairwise in measure disjoint *ragged short vertical strips* of dimensions $\sim \mu^{1/2}K^{-1/2} \times \mu^{1/2}K^{-1/4}$ given by all intersections of a long horizontal and a long vertical strip, and add them to our set of ragged strips by denoting them by S_ℓ, $\ell = [\mu^{-1/2}K^{1/4}] + 1, \ldots$, and put as before $fs_\ell := \sum_{\tau \in \mathcal{F}_\ell} f_\tau$.

Given a family of caps τ as above, and given the corresponding ragged strips S_ℓ and functions f_τ and fs_ℓ as before, we say that a point $\xi \in \mathbb{R}^3$ is α-*broad for* $\mathcal{E}_\gamma f$ *and the given family of caps*, if

$$\max_{S_\ell} |\mathcal{E}_\gamma fs_\ell(\xi)| \leq \alpha|\mathcal{E}_\gamma f(\xi)|,$$

where the maximum is taken over the set of all ragged strips S_ℓ as defined above (recall that this set depends on the size of $|\gamma|K^{1/2}$).

We also define $Br_\alpha\mathcal{E}_\gamma f(\xi) := |\mathcal{E}_\gamma f(\xi)|$ if ξ is α-broad, and zero otherwise.

Remark 3.1 Note that when $\mu = 1$, then ragged strips are indeed strips in the sense of Definitions 2.1, and our present definition of broadness of points coincides in this case with the one given before.

The key result will be the following analogue to Theorem 3.1 in [14]:

Theorem 3.2 *For any* $0 < \epsilon < 10^{-10}$, *there are constants* $K = K(\epsilon)$ *and* C_ϵ, *independent of* $\gamma \in [-1, 1]$, *such that for any family of caps* τ *with multiplicity at most* μ *covering* Σ *as above and the associated family of ragged strips* S_ℓ *and associated functions* f_τ *and* fs_ℓ *as defined above which decompose* f, *for any length* $R \geq 1$, *any* $\alpha \geq K^{-\epsilon}$ *and for any* $\gamma \in [-1, 1]$, *the following holds true:*

If for every $\omega \in \Sigma$, *and every cap* τ *as above,*

$$\fint_{B(\omega, R^{-1/2})} |f_\tau|^2 \leq 1, \tag{4}$$

then,

$$\int_{B_R} (Br_\alpha\mathcal{E}_\gamma f)^{3.25} \leq C_\epsilon R^\epsilon \left(\sum_\tau \int |f_\tau|^2\right)^{3/2+\epsilon} R^{\delta_{trans} \log(K^\epsilon\alpha\mu)}, \tag{5}$$

where $\delta_{trans} := \epsilon^6$. *Moreover* $K(\epsilon) \to \infty$ *as* $\epsilon \to 0$.

Here, in $\mathbb{R}^n$, by $B(\omega, r)$ we denote the Euclidean ball of radius $r > 0$ and center ω, and by $\fint_A f := \frac{1}{|A|} \int_A f$ we denote the mean value f over the measurable set A of volume $|A| > 0$.

We can easily recover Theorem 2.1 by applying Theorem 3.2 with $\mu = 1$, $\epsilon < 10^{-10}$ and $\alpha = K^{-\epsilon}$, in the same way as Guth shows how Theorem 2.4 follows from Theorem 3.1 in [14]. Keep here Remark 3.1 in mind, and note that for these choices of μ, ϵ and α, we have $\delta_{trans} \log(2K^\epsilon \alpha \mu) \leq 10\delta_{trans} \leq \epsilon$.

4 Proof of Theorem 3.2

Recall that we had put $\delta_{trans} := \epsilon^6$, so that, if we define $\delta_{deg} := \epsilon^4$ and $\delta := \epsilon^2$, then

$$\delta_{trans} \ll \delta_{deg} \ll \delta \ll \epsilon < 10^{-10}.$$

We also set, for given $R \geq 1$,

$$K = K(\epsilon) := e^{\epsilon^{-10}} \quad \text{and} \quad D = D(\epsilon) := R^{\delta_{deg}} = R^{\epsilon^4}.$$

Remark 4.1 a) It is enough to consider the case where $\alpha \mu \leq 10^{-5}$, because in the other case, the exponent $\delta_{trans} \log(K^\epsilon \alpha \mu)$ is very large and the estimate (5) trivially holds true. Henceforth, we shall therefore always assume that $\alpha \mu \leq 10^{-5}$.

b) It is then also enough to consider the case where $R \geq 1000 \, e^{e^{\epsilon^{-12}}}$.

To justify the last claim, notice first that our assumption (4) implies that $\|f_\tau\|_2 \leq 1$. Since there are at most $K(\epsilon)^2$ caps τ, we have $\sum_\tau \|f_\tau\|_2 \leq K(\epsilon)^2$. Therefore, we trivially even obtain that when $R \leq 1000 \, e^{e^{\epsilon^{-12}}}$, then

$$\begin{aligned}
\int_{B_R} |\mathcal{E}_\gamma f|^{3.25} &\leq R^3 \|f\|_1^{3.25} \leq R^3 \left(\sum_\tau \|f_\tau\|_1\right)^{3.25} \leq R^3 \left(\sum_\tau \|f_\tau\|_2\right)^{3.25} \\
&\leq R^3 K(\epsilon)^{2(1/4 - 2\epsilon)} \left(\sum_\tau \|f_\tau\|_2\right)^{3+2\epsilon} \\
&\leq R^3 K(\epsilon)^{1/2 - 4\epsilon} K(\epsilon)^{2(3/2+\epsilon)} \left(\sum_\tau \|f_\tau\|_2^2\right)^{3/2+\epsilon} \\
&\leq C_1(\epsilon) \left(\sum_\tau \|f_\tau\|_2^2\right)^{3/2+\epsilon},
\end{aligned}$$

with $C_1(\epsilon) := (1000 \, e^{e^{\epsilon^{-12}}})^3 K(\epsilon)^{7/2 - 2\epsilon}$, hence (5).

As usual, we will work with wave packet decompositions of the functions f defined on S_γ. Following [14], we decompose Σ into squares ("caps") θ of side length $R^{-1/2}$. By ω_θ we shall denote the center of θ, and by $\nu(\theta)$ the "outer" unit normal to S_γ at the point $(\omega_\theta, \phi_\gamma(\omega_\theta)) \in S_\gamma$, which points into the direction of $(-\nabla \phi_\gamma(\omega_\theta), -1)$. $\mathbb{T}(\theta)$ will denote a set of $R^{1/2}$-separated tubes T of radius $R^{1/2+\delta}$ and length R, which are all parallel to $\nu(\theta)$ and for which the corresponding thinner tubes of radius $R^{1/2}$ with the same axes cover B_R. We will write $\nu(T) := \nu(\theta)$ when $T \in \mathbb{T}(\theta)$.

Note that for each θ, every point $\xi \in B_R$ lies in $O(R^{2\delta})$ tubes $T \in \mathbb{T}(\theta)$. We put $\mathbb{T} := \bigcup_\theta \mathbb{T}(\theta)$. Arguing in the same way as in [14], Proposition 2.6, we arrive at the following approximate wave packet decomposition:

Proposition 4.2 *Assume that R is sufficiently large (depending on δ). Then, for any $\gamma \in [-1, 1]$, given $f \in L^2(\Sigma)$, we may associate to each tube $T \in \mathbb{T}$ a function f_T such that the following hold true:*

    a) If $T \in \mathbb{T}(\theta)$, then $\operatorname{supp} f_T \subset 3\theta$.
    b) If $\xi \in B_R \setminus T$, then $|\mathcal{E}_\gamma f_T(\xi)| \leq R^{-1000} \|f\|_2$.
    c) For any $x \in B_R$, we have $|\mathcal{E}_\gamma f(x) - \sum_{T \in \mathbb{T}} \mathcal{E}_\gamma f_T(x)| \leq R^{-1000} \|f\|_2$.
    d) (Essential orthogonality) If $T_1, T_2 \in \mathbb{T}(\theta)$ are disjoint, then
        $\left| \int f_{T_1} \overline{f_{T_2}} \right| \leq R^{-1000} \int_{3\theta} |f|^2.$
    e) $\sum_{T \in \mathbb{T}(\theta)} \int_\Sigma |f_T|^2 \leq C \int_{3\theta} |f|^2.$

Remark 4.3 Note that since $|\gamma| \leq 1$, in this proposition we have bounds that are uniform in γ. Moreover, note that the same argument as in Remark 4.1 b) shows that, in order to prove Theorem 3.2 it is enough to consider the case where R is sufficiently large (depending on δ, i.e., depending on ϵ).

We next recall the version of the polynomial ham sandwich theorem with non-singular polynomials from [14]. If P is a real polynomial on $\mathbb{R}^n$, we denote by $Z(P) := \{\xi \in \mathbb{R}^n : P(\xi) = 0\}$ its null variety. P is said to be *non-singular* if $\nabla P(\xi) \neq 0$ for every point $\xi \in Z(P)$.

Then, by Corollary 1.7 in [14] there is a non-zero polynomial P of degree at most D which is a product of non-singular polynomials such that the set $\mathbb{R}^3 \setminus Z(P)$ is a disjoint union of $\sim D^3$ cells O_i such that, for every i,

$$\int_{O_i \cap B_R} (Br_\alpha \mathcal{E}_\gamma f)^{3.25} \sim D^{-3} \int_{B_R} (Br_\alpha \mathcal{E}_\gamma f)^{3.25}. \tag{6}$$

We next define W as the $R^{1/2+\delta}$ neighborhood of $Z(P)$ and put $O_i' := (O_i \cap B_R) \setminus W$.

Moreover, note that if we apply Proposition 4.2 to f_τ in place of f (what we shall usually do), then by property (a) in Proposition 4.2, for every tube $T \in \mathbb{T}$ the function $f_{\tau,T}$ is supported in an $O(R^{-1/2})$ neighborhood of τ. Following Guth, we define

$$\mathbb{T}_i := \{T \in \mathbb{T} : T \cap O_i' \neq \emptyset\}, \quad f_{\tau,i} := \sum_{T \in \mathbb{T}_i} f_{\tau,T}, \quad f_{S_\ell,i} := \sum_{\tau \in \mathcal{F}_\ell} f_{\tau,i} \quad \text{and} \quad f_i := \sum_\tau f_{\tau,i}.$$

Then we can use the following analogue to Lemma 3.2 in [14]:

Lemma 4.4 *Each tube $T \in \mathbb{T}$ lies in at most $D + 1$ of the sets $\mathbb{T}_i$.*

We cover B_R with $\sim R^{3\delta}$ balls B_j of radius $R^{1-\delta}$. Recall Definitions 3.3 and 3.4 from [14]:

Definition 4.1 a) We define $\mathbb{T}_{j,tang}$ as the set of all tubes $T \in \mathbb{T}$ that satisfy the following conditions:

$$T \cap W \cap B_j \neq \emptyset,$$

and if $\xi \in Z(P)$ is any nonsingular point (i.e., $\nabla P(\xi) \neq 0$) lying in $2B_j \cap 10T$, then

$$\operatorname{angle}(\nu(T), T_\xi Z(P)) \leq R^{-1/2+2\delta}.$$

Here, $T_\xi Z(P)$ denotes the tangent space to $Z(P)$ at ξ, and we recall that $\nu(T)$ denotes the unit vector in direction of T. Accordingly, we define

$$f_{\tau,j,tang} := \sum_{T \in \mathbb{T}_{j,tang}} f_{\tau,T} \quad \text{and} \quad f_{j,tang} := \sum_\tau f_{\tau,j,tang}.$$

b) We define $\mathbb{T}_{j,trans}$ as the set of all tubes $T \in \mathbb{T}$ that satisfy the following conditions:

$$T \cap W \cap B_j \neq \emptyset,$$

and there exists a nonsingular point $\zeta \in Z(P)$ lying in $2B_j \cap 10T$, so that

$$\text{angle}(\nu(T), T_\zeta Z(P)) > R^{-1/2+2\delta}.$$

Accordingly, we define

$$f_{\tau,j,trans} := \sum_{T \in \mathbb{T}_{j,trans}} f_{\tau,T} \quad \text{and} \quad f_{j,trans} := \sum_{\tau} f_{\tau,j,trans}.$$

We also recall Lemmas 3.5 and 3.6 in [14]:

Lemma 4.5 *Each tube $T \in \mathbb{T}$ belongs to at most* $\text{Poly}(D) = R^{O(\delta_{deg})}$ *different sets* $\mathbb{T}_{j,trans}$.

Lemma 4.6 *For each j, the number of different θ so that* $\mathbb{T}_{j,tang} \cap \mathbb{T}(\theta) \neq \emptyset$ *is at most* $R^{1/2+O(\delta)}$.

Note that the previous lemma makes use of the fact that the Gaussian curvature does not vanish on the surface S_γ so that the Gauß map is a diffeomorphism onto its image.

To motivate the next lemma, suppose we have a point ξ contained in a cell O'_i. Then it is not hard to see that in the wave packet decomposition of $\mathcal{E}_\gamma f(\xi)$ essentially only those tubes T should matter which intersect the cell O'_i, that is, $T \in \mathbb{T}_i$. It is thus natural to expect that we may replace $\mathcal{E}_\gamma f(\xi)$ by $\mathcal{E}_\gamma f_i(\xi)$ with only a small error. An analogous statement holds true even for the corresponding broad parts, as the following analogue to Lemma 3.7 in [14] shows:

Lemma 4.7 *If $\xi \in O'_i$. Then, given our assumptions on R from Remarks 4.1, we have*

$$Br_\alpha \mathcal{E}_\gamma f(\xi) \leq Br_{2\alpha} \mathcal{E}_\gamma f_i(\xi) + R^{-900} \sum_{\tau} \|f_\tau\|_2.$$

Proof Let $\xi \in O'_i$. By Proposition 4.2 c), we have

$$\mathcal{E}_\gamma f_\tau(\xi) = \sum_{T \in \mathbb{T}} \mathcal{E}_\gamma f_{\tau,T}(\xi) + O(R^{-1000} \|f_\tau\|_2).$$

If $\xi \in T$, then, $T \cap O'_i \neq \emptyset$, i.e., $T \in \mathbb{T}_i$. If $\xi \notin T$, then Proposition 4.2 b) shows that $|\mathcal{E}_\gamma f_{\tau,T}(\xi)| \leq R^{-1000} \|f_\tau\|_2$. The contribution of these T's is thus negligible.

Using the short hand notation "neglig" for terms which are much smaller than $R^{-940} \sum_{\tau} \|f_\tau\|_2$ (and "neglig$_\tau$" for terms which are much smaller than $R^{-950} \|f_\tau\|_2$), we thus have

$$\mathcal{E}_\gamma f_\tau(\xi) = \mathcal{E}_\gamma f_{\tau,i}(\xi) + \text{neglig}_\tau, \tag{7}$$

and summing in τ,

$$\mathcal{E}_\gamma f(\xi) = \mathcal{E}_\gamma f_i(\xi) + \text{neglig}. \tag{8}$$

We can assume that ξ is α-broad for $\mathcal{E}_\gamma f$ and that

$$|\mathcal{E}_\gamma f(\xi)| \geq R^{-900} \sum_{\tau} \|f_\tau\|_2. \tag{9}$$

Hence,

$$|\mathcal{E}_\gamma f_i(\xi)| \geq |\mathcal{E}_\gamma f(\xi)| - \text{neglig} \geq \frac{1}{2} R^{-900} \sum_\tau \|f_\tau\|_2. \tag{10}$$

Now assume that S_ℓ is any of the ragged strips used in the definition of α-broadness. Then we have accordingly

$$\mathcal{E}_\gamma f_{S_\ell}(\xi) = \sum_{\tau \in \mathcal{F}_\ell} \mathcal{E}_\gamma f_\tau(\xi) = \sum_{\tau \in \mathcal{F}_\ell} \mathcal{E}_\gamma f_{\tau,i}(\xi) + \text{neglig} = \mathcal{E}_\gamma f_{S_\ell,i}(\xi) + \text{neglig}. \tag{11}$$

Since ξ is α-broad for $\mathcal{E}_\gamma f$, (11) shows that

$$|\mathcal{E}_\gamma f_{S_\ell,i}(\xi)| \leq |\mathcal{E}_\gamma f_{S_\ell}(\xi)| + \text{neglig} \leq \alpha|\mathcal{E}_\gamma f(\xi)| + \text{neglig}.$$

Notice also that by Remarks 4.1, $10^{-5} \geq \alpha \gg K^{-\epsilon} \gg K^{-100} \gg R^{-1}$. In combination with (8), and (10), we then obtain that

$$|\mathcal{E}_\gamma f_{S_\ell,i}(\xi)| \leq \alpha|\mathcal{E}_\gamma f_i(\xi)| + \text{neglig} \leq 2\alpha|\mathcal{E}_\gamma f_i(\xi)| \tag{12}$$

for every ragged strip S_ℓ. This estimate shows that ξ is 2α-broad for $\mathcal{E}_\gamma f_i$, and thus the claimed estimate in the lemma follows from (8) and the assumptions that we made subsequently. $\square$

Our definition of broadness of points was chosen differently from Guth's, since we shall also need a different notion of "non-adjacent" caps. This will be related to the validity of certain bilinear Fourier extension estimates which will be needed in the proof and which will be established later. In order to prepare those, let us review some notions and results concerning such bilinear estimates.

4.1 Transversality for bilinear estimates

We shall be brief here and refer for more details to the corresponding literature dealing with bilinear estimates, for instance [22,23,35], or [2].

Following in particular and more specifically our discussions in [8,9], we first recall that according to Theorem 1.1 in [22], given two open subsets $U_1, U_2 \subset [0, 1] \times [0, 1]$, the proper type of transversality for bilinear estimates is achieved if the modulus of the following quantity

$$\Gamma_z^\gamma(z_1, z_2, z_1', z_2') := \left\langle (H\phi_\gamma)^{-1}(z)(\nabla\phi_\gamma(z_2) - \nabla\phi_\gamma(z_1)), \nabla\phi_\gamma(z_2') - \nabla\phi_\gamma(z_1') \right\rangle \tag{13}$$

is bounded from below for any $z_i = (x_i, y_i)$, $z_i' = (x_i', y_i') \in U_i$, $i = 1, 2$, and $z = (x, y) \in U_1 \cup U_2$, $H\phi_\gamma$ denoting the Hessian of ϕ. If such an inequality holds, then we do have bilinear estimates with constants C that depend only on lower bounds of (the modulus of) in (13), and on upper bounds for the derivatives of ϕ_γ. Note that those upper bounds are independent of $\gamma \in [-1, 1]$; we will be more precise about this later. If U_1 and U_2 are sufficiently small (with sizes depending on upper bounds of the first and second order derivatives of ϕ_γ and a lower bound for the determinant of $H\phi_\gamma$) this condition reduces to the estimate

$$|\Gamma_z^\gamma(z_1, z_2)| \geq c > 0, \tag{14}$$

for $z_i = (x_i, y_i) \in U_i$, $i = 1, 2$, $z = (x, y) \in U_1 \cup U_2$, where

$$\Gamma_z^\gamma(z_1, z_2) := \left\langle (H\phi_\gamma)^{-1}(z)(\nabla\phi_\gamma(z_2) - \nabla\phi_\gamma(z_1)), \nabla\phi_\gamma(z_2) - \nabla\phi_\gamma(z_1) \right\rangle. \tag{15}$$

The bounds in the corresponding bilinear estimates will then depend on the lower bound c in (14). In contrast to [8,9], where we had to devise quite specific "admissible pairs" of sets U_1, U_2 for our bilinear estimates, we shall here only have to consider caps τ_1, τ_2, and the required bilinear estimates will be a of somewhat different nature. Nevertheless, the geometric transversality conditions that we need here will be the same.

It is easy to check that we explicitly have

$$\Gamma_z^\gamma(z_1, z_2) = 2(y_2 - y_1)[x_2 - x_1 + \gamma(y_1 + y_2 - y)(y_2 - y_1)]$$
$$=: 2(y_2 - y_1)\, \mathbf{t}_z^\gamma(z_1, z_2). \tag{16}$$

Since $z = (x, y) \in U_1 \cup U_2$, it will be particularly important to look at the expression (16) when $z = z_1 \in U_1$, and $z = z_2 \in U_2$. As above, if U_1 and U_2 are sufficiently small, we can actually reduce to this case. We then see that for our perturbed saddle, still the difference $y_2 - y_1$ in the y-coordinates plays an important role as for the unperturbed saddle, but in place of the difference $x_2 - x_1$ in the x-coordinates now the quantities

$$\mathbf{t}_{z_1}^\gamma(z_1, z_2) := x_2 - x_1 + \gamma y_2(y_2 - y_1) \tag{17}$$
$$\mathbf{t}_{z_2}^\gamma(z_1, z_2) := x_2 - x_1 + \gamma y_1(y_2 - y_1) \tag{18}$$

become relevant. Observe also that

$$\mathbf{t}_z^\gamma(z_1, z_2) = -\mathbf{t}_z^\gamma(z_2, z_1). \tag{19}$$

This definition of transversality motivates the following

Definition 4.2 a) We say that two caps τ_1, τ_2 are *strongly separated* if

$$\min\{|y_2^c - y_1^c|, \max\{|\mathbf{t}_{z_1^c}^\gamma(z_1^c, z_2^c)|, |\mathbf{t}_{z_2^c}^\gamma(z_1^c, z_2^c)|\}\} \geq 10\mu^{1/2}K^{-1},$$

where $z_1^c = (x_1^c, y_1^c)$ denotes the center of τ_1 and $z_2^c = (x_2^c, y_2^c)$ the center of τ_2.
b) Following from here again [14], we define

$$\text{Bil}(\mathcal{E}_\gamma f_{j,tang}) := \sum_{\tau_1, \tau_2 \text{ strongly separated}} |\mathcal{E}_\gamma f_{\tau_1, j, tang}|^{1/2} |E_\gamma f_{\tau_2, j, tang}|^{1/2}.$$

Remark 4.8 If the caps τ_1 and τ_2 are strongly separated, so that, say, $|y_2^c - y_1^c| \geq 10\mu^{1/2}K^{-1}$ and $|\mathbf{t}_{z_2^c}^\gamma(z_1^c, z_2^c)|\} \geq 10\mu^{1/2}K^{-1}$, then by (16) we have

$$|\Gamma_z^\gamma(z_1, z_2, z_1', z_2')| \geq 4\mu K^{-2} \quad \text{for all} \quad z_1, z_1' \in \tau_1, \ z, z_2, z_2' \in \tau_2. \tag{20}$$

Indeed, one computes that

$$\begin{aligned}
\Gamma_z^\gamma(z_1, z_2, z_1', z_2') &= -2\gamma y(y_2 - y_1)(y_2' - y_1') \\
&\quad + (y_2' - y_1')(x_2 - x_1 + \gamma(y_2^2 - y_1^2)) \\
&\quad + (y_2 - y_1)(x_2' - x_1' + \gamma((y_2')^2 - (y_1')^2)) \\
&= (y_2' - y_1')\mathbf{t}_z^\gamma(z_1, z_2) + (y_2 - y_1)\mathbf{t}_z^\gamma(z_1', z_2'),
\end{aligned}$$

with $\mathbf{t}_z^\gamma(z_1, z_2)$ defined in (16).

Now, by (18), $\mathbf{t}_{z_2^c}^\gamma(z_1^c, z_2^c) = x_2^c - x_1^c + \gamma(y_2^c + y_1^c - y_2^c)(y_2^c - y_1^c)$, where $|\mathbf{t}_{z_2^c}^\gamma(z_1^c, z_2^c)| \geq 10\mu^{1/2}K^{-1}$. Since the caps τ_1, τ_2 have side lengths $\leq \mu^{1/2}K^{-1}$, it is easily seen that $|\mathbf{t}_{z_2}^\gamma(z_1, z_2) - \mathbf{t}_{z_2^c}^\gamma(z_1^c, z_2^c)| \leq 8\mu^{1/2}K^{-1}$, so that $\mathbf{t}_z^\gamma(z_1, z_2)$ and $\mathbf{t}_{z_2^c}^\gamma(z_1^c, z_2^c)$ have the same

sign and $|\mathbf{t}_{\tilde{z}}^{\gamma}(z_1, z_2)| \geq 2\mu^{1/2}K^{-1}$, and analogously we find that $\mathbf{t}_{\tilde{z}}^{\gamma}(z'_1, z'_2)$ and $\mathbf{t}_{\tilde{z}_2^c}^{\gamma}(z_1^c, z_2^c)$ have the same sign, and that $|\mathbf{t}_{\tilde{z}}^{\gamma}(z'_1, z'_2)| \geq 2\mu^{1/2}K^{-1}$. In a similar way, we see that $(y'_2 - y'_1)$ and $(y_2 - y_1)$ have the same sign as $(y_2^c - y_1^c)$, and that $\min\{|y'_2 - y'_1|, |y_2 - y_1|\} \geq 2\mu^{1/2}K^{-1}$, since $|y_2^c - y_1^c| \geq 10\mu^{1/2}K^{-1}$. Therefore, $|\Gamma_{\tilde{z}}^{\gamma}(z_1, z_2, z'_1, z'_2)| \geq 2(2\mu^{1/2}K^{-1})^2$.

For any subset I of the family of caps τ, we define

$$f_{I,j,trans} := \sum_{\tau \in I} f_{\tau,j,trans}.$$

The remaining part of this subsection will be devoted to the proof of the following crucial analogue to the key Lemma 3.8 in [14]:

Lemma 4.9 *If $\xi \in B_j \cap W$ and $\alpha\mu \leq 10^{-5}$, then*

$$Br_\alpha \mathcal{E}_\gamma f(\xi) \leq 2\left(\sum_I Br_{60\alpha} \mathcal{E}_\gamma f_{I,j,trans}(\xi) + K^{100}\mathrm{Bil}(\mathcal{E}_\gamma f_{j,tang})(\xi) + R^{-900} \sum_\tau \|f_\tau\|_2 \right), \quad (21)$$

where the first sum is over all possible subsets I of the given family of caps τ.

Remark 4.10 The splitting into a "transversal" and "tangential" part here is as such not surprising. The crucial point is the presence of the bilinear term. In short, and oversimplified, a given family of caps τ will either contain two strongly separated caps, which gives rise to the bilinear term, or otherwise we will see by the Geometric Lemma 4.11 that the family cannot contain too many caps, and their contributions can be "bootstrapped" by means of Lemma 4.5. For the last point, broadness will be of utmost importance (compare (25)).

Proof Let $\xi \in B_j \cap W$. We may assume that ξ is α-broad for $\mathcal{E}_\gamma f$ and that $|\mathcal{E}_\gamma f(\xi)| \geq R^{-900} \sum_\tau \|f\|_2$. Let

$$I := \{\tau : |\mathcal{E}_\gamma f_{\tau,j,tang}(\xi)| \leq K^{-100}|\mathcal{E}_\gamma f(\xi)|\}. \quad (22)$$

We consider two possible cases:

Case 1: I^c contains two strongly separated caps τ_1 and τ_2. Then trivially

$$|\mathcal{E}_\gamma f(\xi)| \leq K^{100}|\mathcal{E}_\gamma f_{\tau_1,j,tang}(\xi)|^{1/2}|\mathcal{E}_\gamma f_{\tau_2,j,tang}(\xi)|^{1/2} \leq K^{100}\mathrm{Bil}(\mathcal{E}_\gamma f_{j,tan})(\xi), \quad (23)$$

hence (21).

Case 2: I^c does not contain two strongly separated caps.

In this case, we shall make use of the following lemma whose proof will be postponed to Subsection 4.3. Recall the fixed family $\{S_\ell\}_\ell$ of of ragged strips that was associated to our given family of caps τ (covering Σ) in Section 3. □

Lemma 4.11 (The Geometric Lemma) *Assume that $K \geq 20$, and let I^c be any subfamily of the given family of caps which does not contain two strongly separated caps.*

a) If $|\gamma|K^{1/2} > 1$, then all of the caps of I^c belong to the union of at most 40 of the families $\mathcal{F}_m$ associated to long horizontal ragged strips S_m of width $\mu^{1/2}K^{-1/4}$.

b) If $|\gamma|K^{1/2} \leq 1$, then either all of the caps of I^c belong to the union of at most 3 of families $\mathcal{F}_m$ associated to long horizontal ragged strips S_m of width $\mu^{1/2}K^{-1/4}$, or all belong to the union of at most 40 of the families $\mathcal{F}_m$ associated to long vertical ragged strips S_m of width $\mu^{1/2}K^{-1/2}$.

Remark 4.12 Note that the two cases in a) and b) basically match with the corresponding distinction of cases in our definition of α-broad points. For our subsequent argument this distinction will, however, not be relevant.

Using the Geometric Lemma we finish the proof of Lemma 4.9 as follows. We denote by $\{S_m\}_{m \in \mathcal{M}}$ the subset of at most 40 long ragged strips given by the Geometric Lemma. By

$$J := \bigcup_{m \in \mathcal{M}} \mathcal{F}_m$$

we denote the corresponding subset of caps τ. Then $I^c \subset J$, i.e., $J^c \subset I$. We write

$$f = \sum_{m \in \mathcal{M}} \sum_{\tau \in \mathcal{F}_m} f_\tau + \sum_{\tau \in J^c} f_\tau. \tag{24}$$

Hence,

$$|\mathcal{E}_\gamma f(\xi)| \leq \sum_{m \in \mathcal{M}} |\mathcal{E}_\gamma f_{S_m}(\xi)| + |\sum_{\tau \in J^c} \mathcal{E}_\gamma f_\tau(\xi)|.$$

Since ξ is α-broad,

$$\sum_{m \in \mathcal{M}} |\mathcal{E}_\gamma f_{S_m}(\xi)| \leq \sum_{m \in \mathcal{M}} \alpha |\mathcal{E}_\gamma f(\xi)| \leq 40\alpha |\mathcal{E}_\gamma f(\xi)| \leq \frac{1}{10} |\mathcal{E}_\gamma f(\xi)|, \tag{25}$$

where the last inequality holds because we are assuming that $\alpha \leq 10^{-5}$ (compare Remark 4.1 a)). Thus,

$$|\mathcal{E}_\gamma f(\xi)| \leq \frac{1}{10} |\mathcal{E}_\gamma f(\xi)| + |\sum_{\tau \in J^c} \mathcal{E}_\gamma f_\tau(\xi)|,$$

and therefore

$$|\mathcal{E}_\gamma f(\xi)| \leq \frac{10}{9} |\sum_{\tau \in J^c} \mathcal{E}_\gamma f_\tau(\xi)|.$$

Since $\xi \in B_j \cap W$, by Proposition 4.2,

$$\mathcal{E}_\gamma f_\tau(\xi) = \mathcal{E}_\gamma f_{\tau,j,trans}(\xi) + \mathcal{E}_\gamma f_{\tau,j,tang}(\xi) + O(R^{-1000}) \|f_\tau\|_2. \tag{26}$$

Moreover, since $J^c \subset I$, and since there are at most K^2 caps τ,

$$\sum_{\tau \in J^c} |\mathcal{E}_\gamma f_{\tau,j,tang}(\xi)| \leq \sum_{\tau \in I} |\mathcal{E}_\gamma f_{\tau,j,tang}(\xi)| \leq K^{-100} \sum_{\tau \in I} |\mathcal{E}_\gamma f(\xi)| \leq K^{-98} |\mathcal{E}_\gamma f(\xi)|, \tag{27}$$
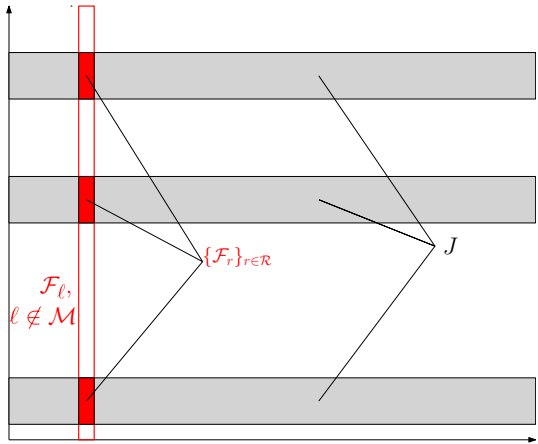
where the second inequality is a consequence of the definition of I. Thus,

$$\frac{9}{10} |\mathcal{E}_\gamma f(\xi)| \leq |\sum_{\tau \in J^c} \mathcal{E}_\gamma f_{\tau,j,trans}(\xi)| + K^{-98} |\mathcal{E}_\gamma f(\xi)| + \sum_\tau R^{-1000} \|f_\tau\|_2$$

$$= |\mathcal{E}_\gamma f_{J^c,j,trans}(\xi)| + K^{-98} |\mathcal{E}_\gamma f(\xi)| + \sum_\tau R^{-1000} \|f_\tau\|_2,$$

and hence, since $|\mathcal{E}_\gamma f(\xi)| \geq R^{-900} \sum \|f_\tau\|_2$,

$$|\mathcal{E}_\gamma f(\xi)| \leq \frac{11}{9} |\mathcal{E}_\gamma f_{J^c,j,trans}(\xi)|. \tag{28}$$

Fig. 1 Intersection of long horizontal and long vertical strips (color figure online)

It will then finally suffice to show that ξ is 60α-broad for $\mathcal{E}_\gamma g$, where $g := f_{J^c, j, trans}$. To this end let us set $g_\tau := f_{\tau, j, trans}$, if $\tau \in J^c$, and zero otherwise, so that

$$g = \sum g_\tau.$$

Observe first that by (26)

$$|\mathcal{E}_\gamma f_{\tau, j, trans}(\xi)| \leq |\mathcal{E}_\gamma f_\tau(\xi)| + |\mathcal{E}_\gamma f_{\tau, j, tang}(\xi)| + \text{neglig},$$

so that if $\tau \in J^c \subset I$, then by the definition of I,

$$|\mathcal{E}_\gamma f_{\tau, j, trans}(\xi)| \leq |\mathcal{E}_\gamma f_\tau(\xi)| + K^{-100}|\mathcal{E}_\gamma f(\xi)| + \text{neglig}. \tag{29}$$

We have to show that

$$|\mathcal{E}_\gamma g_{S_\ell}(\xi)| \leq 60\alpha |\mathcal{E}_\gamma g(\xi)| \tag{30}$$

for all ragged strips S_ℓ. But, $g_{S_\ell} = \sum_{\tau \in \mathcal{F}_\ell \cap J^c} f_{\tau, j, trans}$, and therefore the following two cases can arise:

(i) If $\ell \in \mathcal{M}$, then $\mathcal{F}_\ell \cap J^c = \emptyset$.
(ii) If $\ell \notin \mathcal{M}$, then by our construction of the set J there is a collection $\{\mathcal{F}_r\}_{r \in \mathcal{R}}$ of at most 40 families (possibly empty) associated to short vertical ragged strips so that $\mathcal{F}_\ell \cap J = \cup_{\mathcal{R}} \mathcal{F}_r$ (cf. Fig. 1).

In case (i), (30) is trivial. In case (ii), observe first that by summing (26) over all $\tau \in \mathcal{F}_\ell \cap J^c$ we obtain

$$|\mathcal{E}_\gamma g_{S_\ell}(\xi)| \leq |\sum_{\tau \in \mathcal{F}_\ell \cap J^c} \mathcal{E}_\gamma f_\tau(\xi)| + \sum_{\tau \in \mathcal{F}_\ell \cap J^c} |\mathcal{E}_\gamma f_{\tau, j, tang}(\xi)| + \text{neglig}.$$

By (27), the second term can again be estimated by

$$\sum_{\tau \in \mathcal{F}_\ell \cap J^c} |\mathcal{E}_\gamma f_{\tau, j, tang}(\xi)| \leq K^{-98}|\mathcal{E}_\gamma f(\xi)|.$$

We write

$$\sum_{\tau \in \mathcal{F}_\ell \cap J^c} \mathcal{E}_\gamma f_\tau(\xi) = \mathcal{E}_\gamma f_{S_\ell}(\xi) - \sum_{r \in \mathcal{R}} \mathcal{E}_\gamma f_{S_r}(\xi).$$

Since ξ is α-broad for $\mathcal{E}_\gamma f$, both terms are estimated using again broadness:

$$| \sum_{\tau \in \mathcal{F}_\ell \cap J^c} \mathcal{E}_\gamma f_\tau(\xi)| \leq 41\alpha|\mathcal{E}_\gamma f(\xi)|.$$

Since $\alpha \geq K^{-\epsilon} \gg 10K^{-98}$, in combination with (28) we conclude that

$$|\mathcal{E}_\gamma g_{S_\ell}(\xi)| \leq 41\alpha|\mathcal{E}_\gamma f(\xi)| + K^{-98}|\mathcal{E}_\gamma f(\xi)| + \text{neglig} \leq (41 + 1/2)\alpha|\mathcal{E}_\gamma f(\xi)|$$
$$\leq 60\alpha|\mathcal{E}_\gamma f_{J^c,j,trans}(\xi)| = 60\alpha|\mathcal{E}_\gamma g(\xi)|.$$

This completes the proof of Lemma 4.9.   □

The contribution by the bilinear term in (21) will be controlled by means of the following analogue to Proposition 3.9 in [14]:

Proposition 4.13 *We have*

$$\int_{B_j \cap W} \text{Bil}(\mathcal{E}_\gamma f_{j,tang})^{3.25} \leq C_\epsilon R^{O(\delta)+\epsilon/2} \left(\sum_\tau \int |f_\tau|^2 \right)^{3/2+\epsilon}.$$

With Proposition 4.13 at hand, the rest of the proof of Theorem 3.2, which we shall detail in the next subsection, will be a literal copy of the arguments in pages 396-398 of [14].

The proof of this proposition can easily be reduced to the following analogue to Lemma 3.10 in [14]. We shall give some details below. It is in this lemma where we shall need the full thrust of the strong separation condition between caps τ_1 and τ_2. Suppose we have covered $B_j \cap W$ with a minimal number of cubes Q of side length $R^{1/2}$, and denote by $\mathbb{T}_{j,tang,Q}$ the set of all tubes T in $\mathbb{T}_{j,tang}$ such that $10T$ intersects Q.

Lemma 4.14 *Fix j, i.e., a ball B_j. If τ_1, τ_2 are strongly separated caps, then for any of the cubes Q we have*

$$\int_Q |\mathcal{E}_\gamma f_{\tau_1,j,tang}|^2 |\mathcal{E}_\gamma f_{\tau_2,j,tang}|^2$$

$$\leq R^{O(\delta)} R^{-1/2} \left(\sum_{T_1 \in \mathbb{T}_{j,tang,Q}} \|f_{\tau_1,T_1}\|_2^2 \right) \left(\sum_{T_2 \in \mathbb{T}_{j,tang,Q}} \|f_{\tau_2,T_2}\|_2^2 \right) + \text{neglig}.$$

Indeed, the main ingredient in Guth's argument that needs to be checked here is the following geometric property (compare p. 402 in [14]):

Lemma 4.15 *If τ_1 and τ_2 are two strongly separated caps, then, for any two points $z_1 = (x_1, y_1) \in \tau_1$ and $z_2 = (x_2, y_2) \in \tau_2$ the angle between the normals to S_γ at the corresponding points on S_γ is $\gtrsim K^{-1}$.*

Proof By $N_\gamma(x, y)$ we denote the following normal to our surface S_γ at $(x, y, \phi_\gamma(x, y)) \in S_\gamma$:

$$N_\gamma(x, y) := \begin{pmatrix} {}^t\nabla\phi_\gamma(x, y) \\ -1 \end{pmatrix}. \tag{31}$$

Note that these normal vectors are of size $|N_\gamma(x, y)| \sim 1$. Since $\nabla\phi_\gamma(x, y) = (y, x + \gamma y^2)$, we see that

$$|\nabla\phi_\gamma(x_2, y_2) - \nabla\phi_\gamma(x_1, y_1)| \geq |y_2 - y_1| \gtrsim 10K^{-1} \tag{32}$$

since τ_1 and τ_2 are strongly separated. This implies the claim about the angle.   □

With this at hand, we can follow Guth to deduce from Lemma 4.14 the following L^4 estimate

$$\|\mathrm{Bil}(\mathcal{E}_\gamma f_{j,tang})\|_{L^4(B_j \cap W)} \leq R^{O(\delta)} R^{-1/8} (\sum_\tau \|f_{\tau,j,tan}\|_2^2)^{1/2} + \text{neglig}, \qquad (33)$$

which corresponds to inequality (43) in [14]. Indeed, we can use the standard estimate

$$\|\mathcal{E}_\gamma f\|_{L^2(B_R)} \lesssim R^{1/2} \|f\|_2$$

to deduce that

$$\|\mathrm{Bil}(\mathcal{E}_\gamma f_{j,tang})\|_{L^2(B_j \cap W)} \leq R^{1/2} (\sum_\tau \|f_{\tau,j,tan}\|_2^2)^{1/2}.$$

From this and (33), by Hölder's inequality we get for $2 \leq p \leq 4$,

$$\int_{B_j \cap W} \mathrm{Bil}(\mathcal{E}_\gamma f_{j,tang})^p \lesssim R^{O(\delta)} R^{\frac{5}{2} - \frac{3}{4}p} \left(\sum_\tau \|f_{\tau,j,tang}\|_2^2 \right)^{p/2} + \text{neglig}.$$

Lemma 4.6 tells us that $\mathbb{T}_{j,tang}$ contains tubes in only $R^{O(\delta)} R^{1/2}$ directions. Hence, each function $f_{\tau,j,tang}$ is supported on at most $R^{O(\delta)} R^{1/2}$ caps θ. By Proposition 4.2,

$$\fint_\theta |f_{\tau,j,tang}|^2 \lesssim \fint_{10\theta} |f_\tau|^2 \lesssim 1.$$

Adding the contribution of $R^{O(\delta)} R^{1/2}$ caps θ, we get $\int |f_{\tau,j,tang}|^2 \leq C R^{O(\delta)} R^{-1/2}$. Since there are $K^2 \ll R^{O(\delta)}$ caps τ, this implies that $\sum_\tau \|f_{\tau,j,tang}\|_2^2 \leq C R^{O(\delta)} R^{-1/2}$. Hence, we get, for $p > 3$, $\epsilon < 2(p-3)$,

$$\int_{B_j \cap W} \mathrm{Bil}(\mathcal{E}_\gamma f_{j,tang})^p \lesssim R^{O(\delta)} R^{\frac{5}{2} - \frac{3}{4}p - \frac{1}{2}(\frac{p}{2} - \frac{3}{2} - \epsilon)} \left(\sum_\tau \|f_{\tau,j,tang}\|_2^2 \right)^{3/2 + \epsilon}.$$

This finishes the proof of Proposition 4.13, for $p = 3.25 = \frac{13}{4}$.

Proof of Lemma 4.14. Let τ_1 and τ_2 be two strongly separated caps, and assume without loss of generality that $\min\{|y_2^c - y_1^c|, |\mathfrak{t}_{z_2^c}^\gamma(z_1^c, z_2^c)|\} \geq 10\mu^{1/2}K^{-1}$, where $z_1^c = (x_1^c, y_1^c)$ denotes the center of τ_1 and $z_2^c = (x_2^c, y_2^c)$ the center of τ_2.

Following in a first step a standard argument as in [14] based on Plancherel's theorem, and making use of Proposition 4.2 we see that

$$\int_Q |\mathcal{E}_\gamma f_{\tau_1,j,tang}|^2 |\mathcal{E}_\gamma f_{\tau_2,j,tang}|^2$$

$$\leq \sum_{T_1, T_1', T_2, T_2' \in \mathbb{T}_{j,tang,Q}} \int \mathcal{E}_\gamma f_{\tau_1,T_1} \mathcal{E}_\gamma f_{\tau_2,T_2} \overline{\mathcal{E}_\gamma f_{\tau_1',T_1'} \mathcal{E}_\gamma f_{\tau_2',T_2'}} + \text{neglig}$$

$$= \sum_{T_1, T_1', T_2, T_2' \in \mathbb{T}_{j,tang,Q}} (f_{\tau_1,T_1} \, d\sigma_\gamma * f_{\tau_2,T_2} \, d\sigma_\gamma) \overline{(f_{\tau_1',T_1'} \, d\sigma_\gamma * f_{\tau_2',T_2'} \, d\sigma_\gamma)} + \text{neglig}.$$

$$(34)$$

Here σ_γ denotes the surface carried measure on S_γ chosen so that $\mathcal{E}_\gamma f = \widetilde{f d\sigma_\gamma}$, if we set $\tilde{f}(z, \phi_\gamma(z)) := f(z)$.

For each tube T, we denote by $\theta(T)$ the cap θ so that $T \in \mathbb{T}(\theta)$, and let $\omega(T)$ be the center of $\theta(T)$. By $\tilde{\omega}(T) := (\omega(T), \phi_\gamma(\omega(T)))$ we denote the corresponding point on S_γ.

A given term in the first sum is not negligible only if there are four points $z_1, z_1' \in \tau_1$, $z_2, z_2' \in \tau_2$ that satisfy

$$(z_1, \phi_\gamma(z_1)) + (z_2, \phi_\gamma(z_2)) = (z_1', \phi_\gamma(z_1')) + (z_2', \phi_\gamma(z_2')) \tag{35}$$

and

$$(z_i, \phi_\gamma(z_i)) = \tilde{\omega}(T_i) + O(R^{-1/2+\delta}), \quad (z_i', \phi_\gamma(z_i')) = \tilde{\omega}(T_i') + O(R^{-1/2+\delta}), \quad i = 1, 2. \tag{36}$$

Let us denote by $\mathbf{S}_i$ the piece of the surface S_γ corresponding to τ_i, $i = 1, 2$ (which are "genuine" caps). Since the caps τ_1 and τ_2 are strongly separated, by Lemma 4.15 these two subsurfaces are transversal, so that we can locally define the *intersection curve*

$$\Pi_{z_1, z_2'} := [\mathbf{S}_1 + (z_2', \phi(z_2'))] \cap [\mathbf{S}_2 + (z_1, \phi(z_1))].$$

Note that by (35)

$$(z_1, \phi_\gamma(z_1)) + (z_2, \phi_\gamma(z_2)) = (z_1', \phi_\gamma(z_1')) + (z_2', \phi_\gamma(z_2')) \in \Pi_{z_1, z_2'}.$$

Set $\psi(z) := \phi_\gamma(z - z_1) + \phi_\gamma(z_1) - \phi_\gamma(z - z_2') - \phi_\gamma(z_2')$. Then, the orthogonal projection of the curve $\Pi_{z_1, z_2'}$ on the z - plane is the curve given by $\{z : \psi(z) = 0\}$ (just consider $z := z_1 + z_2 = z_1' + z_2'$ for z when (35) is satisfied).

We introduce a parametrization by arc length $z(t)$, $t \in J$, of this curve, where t is from an open interval J. Notice that this curve $z(t)$ depends on the choices of the points z_1 and z_2'. By

$$z_1'(t) := z(t) - z_2' \quad \text{and} \quad z_2(t) := z(t) - z_1$$

we denote the corresponding curves on $\mathbf{S}_1$ and $\mathbf{S}_2$, respectively. We may assume that $0 \in J$ and $z_1'(0) = z_1', z_2(0) = z_2$. Then, for z_1, z_2' fixed, the pairs $(z_1'(t), z_2(t)), t \in J$, locally provide all solutions (z_1', z_2) to (35).

Note that $\psi(z(t)) \equiv 0$ implies that

$$\langle \nabla \phi_\gamma(z_2(t)) - \nabla \phi_\gamma(z_1'(t)), \frac{dz}{dt}(t) \rangle = 0 \quad \text{for every} \quad t \in J. \tag{37}$$

Note also that $(z_1', z_2) := (z_1, z_2')$ is a solution of (35), so that we may assume that there is some $t_2 \in J$ such that $z_2' = z_2(t_2)$. Recall also that $z_2(0) = z_2$.

Recall the normal $N_\gamma(x, y)$ to the surface S_γ at the point $(x, y, \phi_\gamma(x, y)) \in S_\gamma$ from (31), and note that the angle between the tube T_i and $N_\gamma(z_i)$, $i = 1, 2$, is bounded by $R^{-1/2}$.

Since T_1, T_2, T_1', T_2' lie in $\mathbb{T}_{j, tang, Q}$, we then obviously have

$$R^{-1/2+2\delta} \geq |\det(N_\gamma(z_1), N_\gamma(z_2), N_\gamma(z_2'))| = |\det(N_\gamma(z_1), N_\gamma(z_2), N_\gamma(z_2') - N_\gamma(z_2))|$$

$$= \left| \int_0^{t_2} \det(N(z_1), N(z_2), \frac{dN_\gamma(z_2(t))}{dt}) \, dt \right|. \tag{38}$$

For a given t,

$$\det\left(N_\gamma(z_1), N_\gamma(z_2), \frac{dN_\gamma(z_2(t))}{dt}\right) = \det\begin{pmatrix} {}^t\nabla\phi_\gamma(z_1) & {}^t\nabla\phi_\gamma(z_2) & H\phi_\gamma(z_2(t)) \cdot {}^t(\frac{dz}{dt}(t)) \\ -1 & -1 & 0 \end{pmatrix}$$

$$= \det\left({}^t\nabla\phi_\gamma(z_1) - {}^t\nabla\phi_\gamma(z_2), H\phi_\gamma(z_2(t)) \cdot {}^t(\frac{dz}{dt}(t)) \right)$$

$$= \det H\phi_\gamma(z_2(t)) \det\left(H\phi_\gamma(z_2(t))^{-1} \cdot \left({}^t\nabla\phi_\gamma(z_1) - {}^t\nabla\phi_\gamma(z_2) \right), {}^t(\frac{dz}{dt}(t)) \right).$$

Since $|\frac{dz}{dt}(t))| = 1$ and $\det H\phi_\gamma(z(t)) = 1$, in combination with (37) we thus see that

$$
\left| \det \left(N_\gamma(z_1), N_\gamma(z_2), \frac{dN_\gamma(z_2(t))}{dt} \right) \right|
$$
$$
= \frac{\left| \langle (H\phi_\gamma(z_2(t)))^{-1} \left({}^t\nabla\phi(z_1) - {}^t\nabla\phi(z_2) \right), \nabla\phi_\gamma(z_2(t)) - \nabla\phi_\gamma(z_1'(t)) \rangle \right|}{|\nabla\phi_\gamma(z_2(t)) - \nabla\phi_\gamma(z_1'(t))|}
$$
$$
= \frac{\left| \Gamma^\gamma_{z_2(t)}(z_1, z_2, z_1'(t), z_2(t)) \right|}{|\nabla\phi_\gamma(z_2(t)) - \nabla\phi_\gamma(z_1'(t))|}.
$$

Note that here $|\nabla\phi_\gamma(z_2(t)) - \nabla\phi_\gamma(z_1'(t))| \leq 4$. Moreover, by our assumptions and Remark 4.8, we have $|\Gamma^\gamma_{z_2(t)}(z_1, z_2, z_1'(t), z_2(t))| \geq 4\mu K^{-2}$.

Therefore

$$
\left| \det \left(N_\gamma(z_1), N_\gamma(z_2), \frac{dN_\gamma(z_2(t))}{dt} \right) \right| \geq \mu K^{-2},
$$

and since the integrand in (38) has constant sign, we see that

$$
R^{-1/2+2\delta} \geq \left| \int_0^{t_2} \mu K^{-2} \, dt \right|.
$$

Hence, $|t_2| \leq K^2 R^{-1/2+2\delta}$ and, since the curve $t \mapsto z_2(t)$ is parametrized by arc length, we find that $|z_2 - z_2'| \lesssim K^2 R^{-1/2+2\delta}$. Since $z_1 - z_1' = z_2' - z_2$ by (35), we also get $|z_1 - z_1'| \lesssim K^2 R^{-1/2+2\delta}$.

In a similar way, we see that $|z_1 - z_1'| \leq K^2 R^{-1/2+2\delta}$.

Hence, given T_1 and T_2, there are at most $R^{O(\delta)}$ possible tubes T_1', T_2' which give a non-negligible contribution to (34), and by Schur's lemma this implies that

$$
\int_Q |\mathcal{E}_\gamma f_{\tau_1, j, tang}|^2 |\mathcal{E}_\gamma f_{\tau_2, j, tang}|^2 \leq R^{O(\delta)} \sum_{T_1, T_2 \in \mathbb{T}_{j, tang, Q}} \int |f_{\tau_1, T_1} \, d\sigma_\gamma * f_{\tau_2, T_2} \, d\sigma_\gamma|^2 + \text{neglig.}
$$

Finally, note that Lemma 4.15 implies that $T_1 \cap T_2$ is contained in a cube of side length $K R^{1/2+\delta}$. Hence, the same reasoning used to prove inequality (38) in [14] leads to

$$
\int |f_{\tau_1, T_1} \, d\sigma_\gamma * f_{\tau_2, T_2} \, d\sigma_\gamma|^2 \leq R^{-1/2} \|f_{\tau_1, T_1}\|_2^2 \|f_{\tau_2, T_2}\|_2^2,
$$

and combining these two estimates we complete the proof of Lemma 4.14. □

4.2 Completing the proof of Theorem 3.2

Following [14], pp. 396–398, we use induction on the size of R, the radius of B_R. Moreover, for given R, we also induct on the size of $\sum_\tau \int |f_\tau|^2$. Here we understand that a positive quantity is *of size* 2^k, $k \in \mathbb{Z}$, if it lies in the interval $(2^{k-1}, 2^k]$.

Bases of induction. a) We recall from Remark 4.1 b) that for $1 \leq R \leq 1000 \, e^{e^{\epsilon^{-12}}}$

$$
\int_{B_R} |\mathcal{E}_\gamma f|^{3.25} \leq C_4(\epsilon) \left(\sum_\tau \|f_\tau\|_2^2 \right)^{3/2+\epsilon},
$$

so that the estimate in Theorem 3.2 holds true for this range of R's.

b) Also, if $\sum_\tau \int |f_\tau|^2 \leq R^{-1000}$, then the estimate in Theorem 3.2 holds trivially, since

$$\int_{B_R} |\mathcal{E}_\gamma f|^{3.25} \leq R^3 \|f\|_1^{3.25} \leq R^3 \|f\|_2^{3.25} \leq R^{-100} \|f\|_2^{3+2\epsilon}$$

$$\leq K(\epsilon)^2 R^{-100} \Big(\sum_\tau \int |f_\tau|^2\Big)^{3/2+\epsilon}.$$

In the induction procedure, it will thus suffice to show that in each step we can reduce to situations where either R, or $\sum_\tau \int |f_\tau|^2$, becomes smaller by a factor $\leq 1/2$, until we go below one of the thresholds described in a), or b).

We shall show that inequality (5) of Theorem 3.2 will then hold with the constant $C_\epsilon := \max\{K(\epsilon)^2, C_4(\epsilon)\}$.

Induction hypotheses. Assume that Theorem 3.2 holds for all radii $\leq R/2$, or, given R, for all functions g in place of f such that $\sum_\tau \int |g_\tau|^2 \leq \frac{1}{2} \sum_\tau \int |f_\tau|^2$ and every $\mu \geq 1$.

Write

$$\int_{B_R} (Br_\alpha \mathcal{E}_\gamma f)^{3.25} = \sum_i \int_{B_R \cap O_i'} (Br_\alpha \mathcal{E}_\gamma f)^{3.25} + \int_{B_R \cap W} (Br_\alpha \mathcal{E}_\gamma f)^{3.25}. \qquad (39)$$

Case 1. Assume that the first term (cellular term) dominates (39). In this case, by (6) there will be $\sim D^3$ cells O_i', and for each of them

$$\int_{B_R \cap O_i'} Br_\alpha \mathcal{E}_\gamma f^{3.25} \sim D^{-3} \int_{B_R} Br_\alpha \mathcal{E}_\gamma f^{3.25}.$$

In combination with Lemma 4.7, then, for every i,

$$\int_{B_R} (Br_\alpha \mathcal{E}_\gamma f)^{3.25} \sim D^3 \int_{B_R \cap O_i'} (Br_\alpha \mathcal{E}_\gamma f)^{3.25}$$

$$\lesssim D^3 \int_{B_R \cap O_i'} (Br_{2\alpha} \mathcal{E}_\gamma f_i)^{3.25} + R^{-900} \Big(\sum_\tau \|f_\tau\|_2\Big)^{3.25}. \qquad (40)$$

If the second term in (40) dominates, then, since $\alpha \geq K^\epsilon$, $R^{\delta_{trans} \log(K^\epsilon \alpha \mu)} \geq R^{\delta_{trans}} \geq 1$, and that finishes the proof.

If the first term in (40) dominates, we use Lemma 4.4 and the following immediate analogue to Lemma 2.7 in [14]:

Lemma 4.16 *Consider some subsets $\mathbb{T}_i \subset \mathbb{T}$ indexed by $i \in \mathcal{I}$. If each tube T belongs to at most κ of the subsets $\{\mathbb{T}_i\}_{i \in \mathcal{I}}$, then, for every θ,*

$$\sum_{i \in \mathcal{I}} \int_{3\theta} |f_{\tau,i}|^2 \leq \kappa \int_{10\theta} |f_\tau|^2,$$

and

$$\sum_{i \in \mathcal{I}} \int |f_{\tau,i}|^2 \leq \kappa \int |f_\tau|^2.$$

Applying this lemma in combination with Lemma 4.4, we see that for each τ,

$$\sum_i \int |f_{\tau,i}|^2 \leq (D+1) \int |f_\tau|^2,$$

and therefore

$$\sum_i \sum_\tau \int |f_{\tau,i}|^2 \leq (D+1) \sum_\tau \int |f_\tau|^2.$$

Now, recall that there are $\sim D^3$ indices i. Thus we can choose and fix an index i_0 such that

$$\sum_\tau \int |f_{\tau,i_0}|^2 \lesssim D\, D^{-3} \sum_\tau \int |f_\tau|^2 = D^{-2} \sum_\tau \int |f_\tau|^2 \ll \frac{1}{2} \sum_\tau \int |f_\tau|^2. \quad (41)$$

We finish this case by applying the induction hypothesis (on the size of $\sum_\tau \int |g_\tau|^2$) to the function $f_{i_0} := \sum_\tau f_{\tau,i_0}$. Note that the support of f_{τ,i_0} is a tiny neighborhood of τ. For this reason we need μ in the statement of Theorem 3.2, so that here we can apply the induction hypothesis with 2μ in place of μ.

To this end, note also that

$$\fint_{B(\omega,R^{-1/2})} |f_{\tau,i_0}|^2 \leq C \fint_{B(\omega,10R^{-1/2})} |f_\tau|^2 \leq C,$$

where the first inequality is a consequence of the following immediate analogue to Lemma 2.8 in [14]:

Lemma 4.17 *If* $\mathbb{T}_i \subset \mathbb{T}$, *then for any cap* θ, *and any* τ,

$$\int_{3\theta} |f_{\tau,i}|^2 \leq \int_{10\theta} |f_\tau|^2.$$

We then apply our induction hypothesis to $\frac{1}{\sqrt{2C}} f_{i_0} = \frac{1}{\sqrt{2C}} \sum_\tau f_{\tau,i_0}$. Since we assume that the first term in (40) dominates, this yields

$$\int_{B_R} (Br_\alpha \mathcal{E}_\gamma f)^{3.25} \lesssim D^3 \int_{B_R \cap O'_{i_0}} (Br_{2\alpha} \mathcal{E}_\gamma f_{i_0})^{3.25}$$

$$\leq (2C)^{1/8-\epsilon} D^3 C_\epsilon R^\epsilon \left(\sum_\tau \int |f_{\tau,i_0}|^2 \right)^{3/2+\epsilon} R^{\delta_{trans}} \log(K^\epsilon 2\alpha 2\mu),$$

and thus by (41)

$$\int_{B_R} (Br_\alpha \mathcal{E}_\gamma f)^{3.25} \leq C_1 D^3 C_\epsilon R^\epsilon \left(D^{-2} \sum_\tau \int |f_\tau|^2 \right)^{3/2+\epsilon} R^{\delta_{trans}} \log(K^\epsilon \alpha \mu) R^{c\delta_{trans}}$$

$$= C_1 C_\epsilon R^\epsilon \left(\sum_\tau \int |f_\tau|^2 \right)^{3/2+\epsilon} R^{\delta_{trans}} \log(K^\epsilon \alpha \mu) D^{-2\epsilon} R^{c\delta_{trans}}$$

$$\leq C_\epsilon R^\epsilon \left(\sum_\tau \int |f_\tau|^2 \right)^{3/2+\epsilon} R^{\delta_{trans}} \log(K^\epsilon \alpha \mu),$$

closing the induction.

Case 2. Assume that the second term (wall term) dominates (39). In this case we apply Lemma 4.9 to obtain

$$\int_{B_R} (Br_\alpha \mathcal{E}_\gamma f)^{3.25} \leq C_\epsilon \sum_j \int_{B_j \cap W} \sum_I (Br_{60\alpha} \mathcal{E}_\gamma f_{I,j,trans})^{3.25}$$

$$+ C K^{325} \sum_j \int_{B_j \cap W} \text{Bil}(\mathcal{E}_\gamma f_{j,tang})^{3.25} + C \left(R^{-900} \sum_\tau \|f_\tau\|_2 \right)^{3.25}$$

$$(42)$$

(note that the number of all possible subsets I of the given family of caps is only a constant depending on ϵ).

Again, if the third term of this last sum dominates, the proof is easily finished.

If the second term dominates, then by Proposition 4.13, since $K \ll R$,

$$\int_{B_R} (Br_\alpha \mathcal{E}_\gamma f)^{3.25} \lesssim C_\epsilon K^{325} \sum_j \int_{B_j \cap W} \text{Bil}(\mathcal{E}_\gamma f_{j,tang})^{3.25}$$

$$\leq C_\epsilon K^{325} R^{O(\delta)+\epsilon/2} \left(\sum_\tau \int |f_\tau|^2 \right)^{3/2+\epsilon} \leq C R^\epsilon \left(\sum_\tau \int |f_\tau|^2 \right)^{3/2+\epsilon}.$$

This finishes the proof in this case.

Finally, assume that the first term in (42) dominates. Then, since the ball B_j has radius $R^{1-\delta} < \frac{R}{2}$, we shall induct on the size of R. Note also that $f_{\tau,j,trans,I}$ is supported in a tiny neighborhood of τ, so we shall again apply the induction hypothesis with 2μ in place of μ.

By Lemma 4.17,

$$\fint_{B(\omega,R^{-1/2})} |f_{I,j,trans,\tau}|^2 \leq \fint_{B(\omega,R^{-1/2})} |f_\tau|^2 \leq C,$$

which implies the same kind of control over larger balls of radius $(R^{1-\delta})^{-1/2}$. Thus, $\frac{1}{C} f_{I,j,trans}$ satisfies the induction hypothesis of Theorem 3.2, and therefore

$$\int_{B_j \cap W} (Br_{60\alpha} \mathcal{E}_\gamma f_{I,j,trans})^{3.25} \leq C_\epsilon R^{\epsilon(1-\delta)} \left(\sum_{\tau \in I} \int |f_{\tau,j,trans}|^2 \right)^{3/2+\epsilon} R^{\delta_{trans}(1-\delta)\log(60K^\epsilon \alpha \mu)}.$$

By Lemma 4.5,

$$\sum_j \int |f_{\tau,j,trans}|^2 \leq \text{Poly}(D) \int |f_\tau|^2.$$

Moreover,

$$\sum_{I,j} \left(\sum_{\tau \in I} \int |f_{\tau,j,trans}|^2 \right)^{3/2+\epsilon} \leq \sum_I \left(\sum_j \sum_{\tau \in I} \int |f_{\tau,j,trans}|^2 \right)^{3/2+\epsilon}.$$

Since there are at most M_ϵ families I, combining these estimates we see that

$$\sum_{I,j} \left(\sum_{\tau \in I} \int |f_{\tau,j,trans}|^2 \right)^{3/2+\epsilon} \leq M_\epsilon \text{Poly}(D) \left(\sum_\tau \int |f_\tau|^2 \right)^{3/2+\epsilon},$$

and thus finally

$$\int_{B_R} (Br_\alpha \mathcal{E}_\gamma f)^{3.25} \leq M_\epsilon C_\epsilon \text{Poly}(D) R^{\epsilon(1-\delta)} \left(\sum_\tau \int |f_\tau|^2 \right)^{3/2+\epsilon} R^{\delta_{trans}(1-\delta) \log(60K^\epsilon \alpha \mu)}$$

$$\leq (M_\epsilon \text{Poly}(D) R^{-\delta\epsilon + c\delta_{trans}}) C_\epsilon R^\epsilon \left(\sum_\tau \int |f_\tau|^2 \right)^{3/2+\epsilon} R^{\delta_{trans} \log(K^\epsilon \alpha \mu)}.$$

By our choices of δ and δ_{trans}, since we assume that R is sufficiently large, we find that the first factor in parentheses is bounded by 1, and thus

$$\int_{B_R} (Br_\alpha \mathcal{E}_\gamma f)^{3.25} \leq C_\epsilon R^\epsilon \left(\sum_\tau \int |f_\tau|^2 \right)^{3/2+\epsilon} R^{\delta_{trans} \log(K^\epsilon \alpha \mu)}.$$

This closes the induction and thus completes the proof of Theorem 3.2.

4.3 Proof of the Geometric Lemma

In this subsection we prove Lemma 4.11. Assume that we are given a family of caps $\{\tau_k\}$ such that for any k, m with $k \neq m$ we have

$$\min\{|y_m^c - y_k^c|, \max\{|\mathbf{t}_{z_m^c}^\gamma (z_m^c, z_k^c)|, |\mathbf{t}_{z_k^c}^\gamma (z_m^c, z_k^c)|\}\} \leq 10\mu^{1/2} K^{-1}, \qquad (43)$$

where we denote by $z_k^c = (x_k^c, y_k^c)$ the center of the cap τ_k.

Case 1. For all k, m we have $|y_m^c - y_k^c| \leq 10\mu^{1/2} K^{-1}$. Then, all caps are contained in a horizontal strip of width $10\mu^{1/2} K^{-1} \leq \mu^{1/2} K^{-1/4}$, since $K \geq 30$.

Case 2. There are two caps, say τ_1, τ_2, such that $|y_1^c - y_2^c| > 10\mu^{1/2} K^{-1}$. We may assume that $y_2^c - y_1^c = \max_{j \neq k} |y_j^c - y_k^c|$. Then, for all k,

$$y_1^c \leq y_k^c \leq y_2^c.$$

Since τ_1 and τ_2 are not strongly separated, $|\mathbf{t}_{z_1^c}^\gamma (z_1^c, z_2^c)| \leq 10\mu^{1/2} K^{-1}$ and $|\mathbf{t}_{z_2^c}^\gamma (z_1^c, z_2^c)| \leq 10\mu^{1/2} K^{-1}$. Therefore, by (18),

$$|\gamma| \, |y_2^c - y_1^c|^2 = |\mathbf{t}_{z_1^c}^\gamma (z_1^c, z_2^c) - \mathbf{t}_{z_2^c}^\gamma (z_1^c, z_2^c)| \leq 20\mu^{1/2} K^{-1},$$

and since $|y_k^c - y_1^c| \leq |y_2^c - y_1^c|$, we see that for all k

$$|y_k^c - y_1^c| \leq (20\mu^{1/2}|\gamma|^{-1} K^{-1})^{\frac{1}{2}}. \qquad (44)$$

In combination with (18), this also implies that

$$\begin{aligned} |x_k^c - x_1^c| &\leq |\mathbf{t}_{z_1^c}^\gamma (z_1^c, z_k^c)| + |\gamma| |y_k^c| |y_k^c - y_1^c| \\ &\leq 10\mu^{1/2} K^{-1} + |\gamma|(20\mu^{1/2}|\gamma|^{-1} K^{-1})^{\frac{1}{2}} \\ &\leq 15\mu^{1/2} K^{-1/2}, \end{aligned} \qquad (45)$$

since $|\gamma| \leq 1$, $\mu \geq 1$ and $K \geq 1$.

When $|\gamma| K^{1/2} > 1$, we conclude from (44) that $|y_k^c - y_1^c| \leq (20\mu^{1/2} K^{-1/2})^{1/2}$ for all k. Hence, all the caps are contained in a horizontal strip of width $10\mu^{1/2} K^{-1/4}$. Decomposing these further into horizontal strips of width $0.5\mu^{1/2} K^{-1/4}$, each of which is contained in one of the ragged strips S_ℓ that have been fixed in Section 3, and distributing the caps τ_k of our family over these ragged strips, we arrive at at most 40 horizontal ragged strips of

width $\mu^{1/2} K^{-1/4}$ which contain all the caps considered in Case 2. Note that by our passage to ragged strips the width does not increase by more than $2\mu^{1/2} K^{-1} \leq 0.5\mu^{1/2} K^{-1/4}$, since $K \geq 20$.

When $|\gamma| K^{1/2} \leq 1$, we conclude from (45) that all the caps are contained in a vertical strip of width $20\mu^{1/2} K^{-1/2}$, and arguing as before we can conclude the proof of Lemma 4.11 also in this case. □

5 Passing from extension estimates on cubes to estimates on plates: an orthogonality lemma

We will here finally discuss an auxilary lemma that we needed in Section 2.

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and $\phi : \Omega \to \mathbb{R}$ any phase function such that $|\nabla \phi(x)| \lesssim 1$ for all $x \in \Omega$. Assume further that $\rho \in C^\infty(\Omega)$, and consider the Fourier extension operator

$$\mathcal{E}f(\xi) := \int_\Omega f(x) \, e^{-i[\xi' \cdot x + \xi_{n+1}\phi(x)]} \rho(x) \, dx,$$

where $\xi = (\xi', \xi_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$ (for convenience, we have chosen here a different sign in the phase than in the definition of $\mathcal{E}_\gamma f$).

Lemma 5.1 *Let* $2 \leq q \leq p$. *Assume that for every $\epsilon > 0$ there exists a constant C_ϵ such that for every $R \geq 1$*

$$\|\mathcal{E}f\|_{L^p([0,R]^{n+1})} \leq C_\epsilon R^\epsilon \|f\|_{L^2(\Omega)}^{2/q} \|f\|_{L^\infty(\Omega)}^{1-2/q} \tag{46}$$

for all $f \in L^q(\Omega)$. Then for every $\epsilon > 0$ there exists a constant C'_ϵ such that for all $R \geq 1$

$$\|\mathcal{E}f\|_{L^p(\mathbb{R}^n \times [0,R])} \leq C'_\epsilon R^\epsilon \|f\|_{L^2(\Omega)}^{2/q} \|f\|_{L^\infty(\Omega)}^{1-2/q} \tag{47}$$

for all $f \in L^q(\Omega)$.

Proof First observe that (46) holds for any translate of $[0, R]^{n+1}$ in place of $[0, R]^{n+1}$ as well, in particular on any cube $Q_y := R(y, 0) + [0, R]^{n+1}$, $y \in \mathbb{Z}^n$.

In order to pass to a corresponding estimate on the plate $\mathbb{R}^n \times [0, R]$), which decomposes into the cubes Q_y, it will suffice to perform an adapted frequency decomposition of f (a full wave packet decomposition is needed here):

Let $f_y := f * \check{\chi}_y$, where $\chi_y(\eta) := \chi(\eta/R - y)$, $y \in \mathbb{Z}^n$, and χ is a suitable compactly supported bump function chosen so that the χ_y, $y \in \mathbb{Z}^n$, form a partition of unity on $\mathbb{R}^n$. Then $f = \sum_y f_y$.

In order to prove (47), we may and shall assume that $0 \leq \xi_{n+1} \leq R$. Under this restriction, we will see that $\mathcal{E}f_y$ is essentially supported in Q_y. Indeed, note that by Fourier inversion

$$\mathcal{E}f_y(\xi) = c_n \iint \hat{f}(\eta) \chi(\eta/R - y) e^{-i[(\xi'-\eta) \cdot x + \xi_{n+1}\phi(x)]} \rho(x) dx d\eta.$$

The gradient in x of the full phase is given by $\xi' - \eta + \xi_{n+1}\nabla\phi(x) = \xi' - Ry + O(R)$. Hence, by a standard integration by parts argument in x, we see that for any $N \in \mathbb{N}$

$$|\mathcal{E}f_y(\xi)| \lesssim_N R^n |\xi' - Ry|^{-N} \|f_y\|_1, \quad \text{if } |\xi' - Ry| \gg R. \tag{48}$$

It is thus natural to split (and estimate, using Minkowski's inequality)

$$\|\mathcal{E}f\|_{L^p(\mathbb{R}^n \times [0,R])} = \left(\sum_y \| \sum_z \mathcal{E}f_{y+z} \|_{L^p(Q_y)}^p \right)^{1/p} \leq \sum_z \left(\sum_y \| \mathcal{E}f_y \|_{L^p(Q_{y-z})}^p \right)^{1/p}$$

into two parts: First we use (46) and $p \geq q$ to estimate

$$\sum_{|z| \lesssim 1} \left(\sum_y \| \mathcal{E}f_y \|_{L^p(Q_{y-z})}^p \right)^{1/p} \lesssim C_\epsilon R^\epsilon \left(\sum_y \| f_y \|_{L^2(\Omega)}^{2p/q} \| f_y \|_{L^\infty(\Omega)}^{p(1-2/q)} \right)^{1/p}$$

$$\leq C_\epsilon R^\epsilon \left(\sum_y \| f_y \|_{L^2(\Omega)}^2 \right)^{1/q} \| f \|_{L^\infty(\Omega)}^{1-2/q}$$

$$\lesssim C_\epsilon R^\epsilon \| f \|_{L^2(\Omega)}^{2/q} \| f \|_{L^\infty(\Omega)}^{1-2/q},$$

where have used Plancherel's theorem. The remainder can be estimated using (48):

$$\sum_{|z| \gg 1} \left(\sum_y \| \mathcal{E}f_y \|_{L^p(Q_{y-z})}^p \right)^{1/p} \lesssim R^{n+(n+1)/p} \sum_{|z| \gg 1} (R|z|)^{-N} \left(\sum_y \| f_y \|_1^p \right)^{1/p}$$

$$\lesssim R^{-N'} \left(\sum_y \| f_y \|_1^p \right)^{1/p},$$

which finishes the proof because $\| f_y \|_1 \leq \| f_y \|_2^{2/q} \| f_y \|_\infty^{1-2/q}$, so that from here we can proceed as before. □

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