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DECAY OF SOLUTIONS TO A POROUS MEDIA EQUATION WITH FRACTIONAL DIFFUSION

César J. Niche
Departamento de Matemática Aplicada, Instituto de Matemática
Universidade Federal do Rio de Janeiro
CEP 21941-909, Rio de Janeiro - RJ, Brazil

Rafael Orive-Illera
Instituto de Ciencias Matemáticas, CSIC-UAM-UC3M-UCM
Nicolás Cabrera 13-15, Campus de Cantoblanco-UAM
Madrid, 28049, Spain
and
Departamento de Matemáticas, Facultad de Ciencias
Universidad Autónoma de Madrid, Spain

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Abstract. In this article we prove results concerning the decay and first order asymptotic behaviour of solutions to a system of equations that model heat transfer in a porous medium by an incompressible flow with fractional dissipation.

1. Introduction and statement of results

Recently, Castro, Córdoba, Gancedo and Orive [6] introduced a system of equations to model heat transfer by an incompressible fluid in a porous medium with fractional dissipation. The transport velocity is given by Darcy's law

$$v = -k\big(\nabla p + gTe_N\big),$$

where $v \in \mathbb{R}^N$ is the liquid discharge (flux per unit area), p is the pressure, k is the matrix position-independent medium permeabilities in the different directions respectively divided by the viscosity, T is the liquid temperature, g is the acceleration due to gravity and the vector $\epsilon_N \in \mathbb{R}^N$ is the last canonical

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vector. While the Navier–Stokes equation and the Stokes equation are both microscopic equations, Darcy's law yields a macroscopic description of a flow in the porous medium [2]. To simplify the notation, we consider $k = g = 1$.

Then, this model leads to the system

$$\begin{aligned} \frac{\partial T}{\partial t} + v \cdot \nabla T &= -\nu \Lambda^{2\alpha} T \\ v &= -(\nabla p + T e_N) \\ div\, v &= 0 \\ T_0(x) &= T(x, 0), \end{aligned} \tag{1.1}$$

with $\nu > 0$ and the operator Λ given by $\Lambda \equiv (-\Delta)^{\frac{1}{2}}$, for $0 \leq \alpha \leq 1$. To simplify notation, we will choose $\nu = 1$. Castro *et al* [6] proved existence of strong and weak solutions in different spaces for the supercritical ($0 < \alpha < \frac{1}{2}$), critical ($\alpha = \frac{1}{2}$) and subcritical ($\frac{1}{2} < \alpha \leq 1$) cases. They also obtained some results on blowup of solutions and existence of global attractors. For other results about (1.1), see the work of Xiang and Yan [25], Xue [26], Yamazaki [27] and Yuan and Yuan [28]. For results concerning the inviscid case, i.e., $\nu = 0$ in (1.1), see D. Córdoba, Gancedo and Orive [8], [9].

Concerning the decay of weak solutions to (1.1), Castro *et al* [6] proved that for $T_0 \in L^p(\mathbb{R}^N)$

$$\|T(t)\|_{L^p} \leq \|T_0\|_{L^p} \big(1 + Ct\big)^{-\frac{N(p-2)}{4p\alpha}}, \tag{1.2}$$

with $2 \leq p < \infty$, $\alpha \in [0, 1]$ and $C = C(\alpha, p)$. Note that when $p = 2$, this establishes a maximum principle but provides no information on decay rates. A second result concerning decay can be deduced from a Corollary proved by A. Córdoba and D. Córdoba (Corollary 2.6, [7]) in dimension $N = 2$. This leads to

$$\|T(t)\|_{L^p} \leq \|T_0\|_{L^p} \big(1 + Ct\big)^{-\frac{p-1}{p\alpha}} \tag{1.3}$$

provided $T_0 \in L^1 \cap L^p$, with $1 < p < \infty$, $\alpha \in [0, 1]$ and $C = C(\alpha, \|T_0\|_{L^1})$.

The main goal of this article is to complement and extend these results for $\|T(t)\|_{L^2}$ and to obtain estimates for the decay of the difference between the solution and its linear part.

The first two results describe the behaviour of the L^2 norm when the initial data is in L^2 only: the norm decays to zero, but with no uniform rate.

Theorem 1.1. *Let $0 < \alpha \leq 1$. For any weak solution to* (1.1) *with $T_0 \in L^2(\mathbb{R}^N)$*

$$\lim_{t \to \infty} \|T(t)\|_{L^2} = 0.$$

Theorem 1.2. *Let $t^* > 0, r > 0, \epsilon > 0$ be arbitrary, $0 < \alpha \leq 1$. Then there exists $T_0 \in L^2(\mathbb{R}^N)$ with $\|T_0\|_{L^2} = r$ such that*

$$\frac{\|T(t^*)\|_{L^2}}{\|T_0\|_{L^2}} \geq 1 - \epsilon.$$

Theorem 1.2 is proved by choosing $\widetilde{T_0} \in L^2(\mathbb{R}^N)$ which is also in $L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$, for appropriate $p, q > 2$ and scaling it to $T_0 = \widetilde{T_0}^{\lambda}$ with large λ, leaving its L^2 norm invariant. For this rescaled initial data, the arbitrarily slow decay is obtained by estimating the linear and nonlinear terms in the integral equation associated to (1.1). For proving Theorem 1.1 we use the Fourier Splitting method, developed by M.E. Schonbek [18], [19], [20]. This method consists in obtaining a differential inequality on the L^2 norm of $T(t)$, which depends on the behaviour of small frequencies of $\widehat{T}(t)$, for large t.

We now extend and complement the results in (1.2) and (1.3).

Theorem 1.3. *Let $T_0 \in L^p(\mathbb{R}^N) \cap L^2(\mathbb{R}^N), 1 \leq p < 2$, and let $0 < \alpha \leq 1$. Then for any weak solution to* (1.1)

$$\|T(t)\|_{L^2} \leq C\big(1+t\big)^{-\frac{N}{4\alpha}\left(\frac{2}{p}-1\right)}, \qquad t > 0$$

where $C = C\big(\alpha, \|T_0\|_{L^p}\big)$.

The proof of this Theorem is again based on the Fourier Splitting method. The fact that the initial data is in $L^p(\mathbb{R}^N) \cap L^2(\mathbb{R}^N), 1 \leq p < 2$ provides the estimates needed to obtain uniform decay, as opposed to the non-uniform one of solutions in Theorem 1.2.

We note that Theorems 1.1 and 1.2 provide substantially more information about the decay than the estimate (1.2) with $p = 2$, which only gives a maximum principle. The hypotheses on T_0 in Theorem 1.3 are less restrictive than those needed to obtain (1.3) and moreover, Theorem 1.3 holds in dimension $N \geq 2$, as opposed to just $N = 2$.

We address now the decay of the L^p norm of T and its derivatives, for $1 \leq p \leq \infty$.

Theorem 1.4. *Let $T_0 \in L^1(\mathbb{R}^N) \cap L^{\frac{N}{2\alpha-1}}(\mathbb{R}^N)$ with $\frac{1}{2} < \alpha \leq 1$ and $N \geq 2$. Then for a solution T to* (1.1)

$$t^{\frac{N}{2\alpha}\left(1-\frac{1}{p}\right)}\|T(t)\|_{L^p} \leq C\|T_0\|_{L^1}, \qquad 1 \leq p \leq \infty. \tag{1.4}$$

Theorem 1.5. *Let $T_0 \in L^1(\mathbb{R}^N) \cap L^{\frac{N}{2\alpha-1}}(\mathbb{R}^N)$ with $\frac{1}{2} < \alpha \leq 1$ and $N \geq 2$. If $k \in \mathbb{N}^N$, then for a solution T to (1.1)*

$$t^{\frac{|k|}{2\alpha}+\frac{N}{2\alpha}\left(1-\frac{1}{p}\right)} \nabla^k T(\cdot,t) \in BC(0,\infty; L^p(\mathbb{R}^N)),$$

and, in particular,

$$t^{\frac{|k|}{2\alpha}+\frac{N}{2\alpha}\left(1-\frac{1}{p}\right)} \|\nabla^k T(t)\|_{L^p} \leq C\|T_0\|_{L^1}, \tag{1.5}$$

for $1 \leq p \leq \infty$.

The proof of Theorem 1.4 is based on an argument by Kato [12] (see also Carrillo and Ferreira [5] for the case of the dissipative quasigeostrophic equation), in which an iterative differential inequality is obtained for $\|T(t)\|_{L^p}$ and $\|T(t)\|_{L^{\frac{p}{2}}}$. We prove this Theorem by solving the inequality by induction on $p = 2^n$ and then interpolating. Theorem 1.5 is also proved by induction, the key point being an estimate obtained for $\Lambda^h \nabla^k T$, for $h < 1$, in terms of the decay for $\nabla^k T$.

Finally, we state the results concerning first order decay of solutions.

Theorem 1.6. *Let $T_0 \in L^1(\mathbb{R}^N) \cap L^{\frac{N}{2\alpha-1}}(\mathbb{R}^N)$ with $\frac{1}{2} < \alpha \leq 1$, $N \geq 2$ and $k \in \mathbb{N}^N$. Then*

(1) *For a solution T to (1.1) and the kernel*

$$K_\alpha(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}^N} e^{ix\xi} e^{-|\xi|^{2\alpha}t}\, d\xi,$$

we have

$$\|\nabla^k T(\cdot,t) - \nabla^k K_\alpha(t) * T_0\|_{L^p} \leq C t^{-\frac{|k|+1}{2\alpha}-\frac{N}{2\alpha}\left(1-\frac{1}{p}\right)}$$

for $t > 1$, except in the case $\alpha = 1$ and $N = 2$.

(2) *For $\alpha = 1$ and $N = 2$, i.e., $T_0 \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, and T a solution to (1.1)*

$$\|\nabla^k T(\cdot,t) - \nabla^k K(t) * T_0\|_{L^p} \leq C(1+t)^{-\frac{|k|+1}{2}-\left(1-\frac{1}{p}\right)} \ln(1+t)$$

for $t > 1$, where $K(x,t)$ denotes the heat kernel.

Now, let T_1 be the integral solution to the non-homogeneous heat equation

$$\frac{\partial T_1}{\partial t} + \big(H[K(t) * T_0] \cdot \nabla\big) K(t) * T_0 - \Delta T_1 = 0$$
$$T_1(x,0) = T_0(x), \quad x \in \mathbb{R}^2, \tag{1.6}$$

where the operator H *is defined in* (2.1) *and* K *is the heat kernel in* $\mathbb{R}^2$. *Then*

(3) *For* $\alpha = 1$ *and* $N = 2$, *i.e.,* $T_0 \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, *and* T *a solution to* (1.1)

$$\|\nabla^k T(\cdot,t) - \nabla^k T_1(\cdot,t)\|_{L^p} \leq Ct^{-\frac{|k|+1}{2}-\left(1-\frac{1}{p}\right)}$$

for $t > 1$ *and where* $T_1(x,t)$ *is solution of* (1.6).

This Theorem provides information on the decay of the first order term of solutions to (1.1), i.e., the decay of the difference between the full solution to (1.1) and its linear part. From (1) in the Theorem, we see that the first order term decays faster than the linear part, cf. Lemma 3.2. However, in (2), i.e., for in the case $\alpha = 1$ and $N = 2$, we obtain a decay which is slower than that of the linear part. Thus, in part (3) we introduce a natural corrector term T_1, such that the corrected first order term, i.e., the difference between the full solution and the corrector, again decays faster than the linear part. The proof Theorem 1.6 is based on estimates obtained as a result of Theorem 1.5.

This article is organized as follows. In Section 2 we provide a new proof of the existence of weak solutions to (1.1), following the ideas of Resnick [17]. We do so because the approximate solutions of Castro *et al* [6], obtained by parabolic regularization with a term of the form $\epsilon\Delta T$, are not suitable for using the Fourier Splitting method. Then, in Section 3 we collect estimates concerning the linear part of (1.1) and the Calderón-Zygmund kernel that relates v to T. We then prove Theorems 1.1 and 1.2. In Section 4 we prove Theorem 1.3 and in Section 5 we prove Theorems 1.4, 1.5 and 1.6.

2. Existence of weak solutions

In this section we prove existence of weak solutions to (1.1). We regularize the equation by mollifying the nonlinear term and we then obtain regular, approximate solutions T_m, which are shown to converge to a weak solution T through energy estimates. Castro *et al* [6] proved existence of weak solutions by regularizing with a parabolic term of the form $\epsilon\Delta T$. However, in this case the use of the Fourier Splitting Method for proving decay may lead to technical complications due to the simultaneous presence of the dissipative terms $\Lambda^{2\alpha}T_m$ and ΔT_m. As a result of this, we prefer to give a different proof of existence of solutions that will allow us to later use the Fourier Splitting Method in a clear manner.

Namely, we use a mollifier in time ψ_δ to regularize the advective term $(v\cdot\nabla)T$ in such a way that $\psi_\delta(v)$ depends on the values of v, hence of T, at past times $t-s$, with $t-2\delta < t-s < t-\delta$. Then, on strips $S_k = \mathbb{R}^N \times (k\delta, (k+1)\delta)$, with $\delta = \frac{\tau}{m}$ and $k = 0, \cdots, m-1$, the equation becomes linear. We glue solutions to these linear equations on the strips S_k to obtain an approximate solution T_m in $\mathbb{R}^N \times [0,\tau]$, which, through energy estimates, we prove to be convergent to a weak solution T.

We introduce now the definition of weak solutions.

Definition 2.1. *A function*

$$T = T(x,t) \in L^\infty(0,\tau; L^2(\mathbb{R}^N)) \cap L^2(0,\tau; H^\alpha(\mathbb{R}^N))$$

is a weak solution to (1.1) if for all $\phi \in C_c^\infty(\mathbb{R}^N \times [0,\tau))$

$$\langle T(t), \phi(t)\rangle - \langle T_0, \phi(0)\rangle + \int_0^\tau \big(- \langle T, \partial_t \phi\rangle + \langle T, \Lambda^{2\alpha}\phi\rangle - \langle Tv, \nabla\phi\rangle\big)\, ds = 0.$$

We now find an explicit expression for the kernel that allows us to obtain v in terms of T. The incompressibility condition and Darcy's Law lead to

$$-\Delta p = \frac{\partial T}{\partial x_N}$$

which, after taking the inverse of the Laplacian implies that

$$v = H[T] = -\Big(- \frac{\partial}{\partial x_N}\nabla\Delta^{-1} + e_N\Big)T.$$

Integrating by parts, we obtain

$$H[T](x,t) = -\frac{N-1}{N} T(x,t)\, e_N + PV \int_{\mathbb{R}^N} H_N(x-y) T(y,t)\, dy, \quad x \in \mathbb{R}^N, \tag{2.1}$$

where

$$H_N(x) = -\frac{1}{S_{N-1}} \Big(\frac{N\, x_1 x_N}{|x|^{N+2}}, \frac{N\, x_2 x_N}{|x|^{N+2}}, \cdots, \frac{(N-1)x_N^2 - \sum_{j\neq N} x_j^2}{|x|^{N+2}} \Big),$$

for any $N \geq 2$ and $S_{N-1} = 2\pi^{N/2}/\Gamma(N/2)$ is the surface area of the unit ball in $\mathbb{R}^N$. Then, the Calderón-Zygmund Theorem leads to the estimate

$$\|v\|_{L^p} \leq C\|T\|_{L^p}, \qquad 1 < p < \infty. \tag{2.2}$$

We now describe the retarded mollifier used to regularize (1.1). Let $\psi \in C_c^\infty(\mathbb{R}_+)$ such that

$$\psi \geq 0, \quad \operatorname{supp} \psi \subseteq [1,2] \quad \text{and} \quad \int_0^\infty \psi(s) ds = 1,$$

and set, for $t > 0$,

$$\psi_\delta[f](t) = \int_0^\infty \psi(\tau) \widetilde{f}(t - \delta\tau) d\tau$$

where for a given function $f = f(x,t)$, with $x \in \mathbb{R}^N$ and $t \in [0,\tau]$ we have

$$\widetilde{f}(x,t) = \left\{ \begin{array}{cc} f(x,t) & x \in \mathbb{R}^n, t \in [0,\tau] \\ 0 & \text{elsewhere.} \end{array} \right.$$

Note that $\psi_\delta[f](t)$ depends on the values of $\widetilde{f}(s)$, with $t - 2\delta < t - s < t - \delta$.

For any $\tau > 0$, given $\delta = \frac{\tau}{m}$ we obtain from (1.1) a sequence of regularized equations

$$\frac{\partial T_m}{\partial t} + \big(\psi_\delta[v_m] \cdot \nabla\big) T_m + \Lambda^{2\alpha} T_m = 0, \qquad T_m(x,0) = T_0(x), \tag{2.3}$$

where $v_m = H[T_m]$, for H the operator defined in (2.1). Note also that as the mollifier acts on the time variable only, the incompressibility condition leads to $\nabla \cdot \psi_\delta[v_m] = 0$. On every strip $S_k = \mathbb{R}^N \times [k\delta, (k+1)\delta]$, we can solve the linear equation (2.3) and then glue them together to obtain an approximate solution T_m.

In the following Proposition we address the existence of solutions to a linear problem.

Proposition 2.2. *Let $w \in L^\infty(0,\tau; L^2(\mathbb{R}^N))$ with $\nabla \cdot w = 0$ and $T_0 \in L^2(\mathbb{R}^N)$. Then there exists a unique function T such that*

$$T \in C(0,\tau; L^2(\mathbb{R}^N)) \cap L^2(0,\tau; H^\alpha(\mathbb{R}^N))$$

and is a weak solution to

$$\frac{\partial T}{\partial t} + \big(w \cdot \nabla\big) T + \Lambda^{2\alpha} T = 0, \qquad T(x,0) = T_0(x). \tag{2.4}$$

A detailed proof of this Proposition can be found in Niche and Planas [15].

If $T \in L^\infty(0,\tau; L^2(\mathbb{R}^N))$, then $\psi_\delta[T] \in L^\infty(0,\tau; L^2(\mathbb{R}^N))$, hence by (2.2) we have $\psi_\delta[v] \in L^\infty(0,\tau; L^2(\mathbb{R}^N))$. So, by Proposition 2.2 we have solutions on each strip S_k, which glued together provide solutions T_m to (2.3).

Theorem 2.3. *Let $T_0 \in L^2(\mathbb{R}^N)$. Then, for any $0 < \alpha \leq 1$, there exists a weak solution to* (1.1).

Proof. Let $\{T_m\}_{m\in\mathbb{N}}$ be a sequence of approximate solutions, as described in the last paragraph. Multiplying by T_m and integrating in space and then in time we obtain

$$\|T_m(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{2\alpha} T_m(s)\|_{L^2}^2 \, ds \leq \|T_0\|_{L^2}^2$$

which implies that we can extract a subsequence, which we again call T_m, that converges weakly-$*$ in $L^\infty(0,\tau;L^2(\mathbb{R}^N))$ and weakly in $L^2(0,\tau;H^\alpha(\mathbb{R}^N))$.

First, let $N=2$. Let $\phi \in C_c^\infty(\mathbb{R}^2 \times [0,\tau))$. Then, as $2\alpha < 2+\epsilon$, we have

$$\begin{aligned} |\langle \Lambda^{2\alpha} T_m, \phi\rangle| &= |\langle T_m, \Lambda^{2\alpha}\phi\rangle| \leq \|T_m\|_{L^2}\|\Lambda^{2\alpha}\phi\|_{L^2} \\ &\leq \|T_m\|_{L^2}\|\phi\|_{H^{2\alpha}} \leq \|T_m\|_{L^2}\|\phi\|_{H^{2+\epsilon}}, \end{aligned}$$

so in any bounded open ball $\Omega \subset \mathit{supp}\,\phi$, we have

$$\|\Lambda^{2\alpha} T_m\|_{H^{-(2+\epsilon)}(\Omega)} \leq \|T_m\|_{L^2} \leq C$$

independently of m. Similarly we have that

$$\begin{aligned} \langle \psi_\delta[v_m] T_m, \nabla\phi\rangle &\leq \|\psi_\delta[v_m]\|_{L^2}\|T_m\|_{L^2}\|\nabla\phi\|_{L^\infty} \\ &\leq \|v_m\|_{L^2}\|T_m\|_{L^2}\|\nabla\phi\|_{L^\infty} \leq C\|\nabla\phi\|_{L^\infty} \leq C\|\nabla\phi\|_{H^{1+\epsilon}}, \end{aligned}$$

where we used the embedding $H^{1+\epsilon}(\Omega) \subset L^\infty(\Omega)$ (which is valid only when $N=1,2$) , we conclude from (2.3) that $\frac{\partial T_m}{\partial t}$ is in $L^2(0,\tau;H^{-(2+\epsilon)}(\Omega))$, as both $\Lambda^{2\alpha}T_m$ and $\big(\psi_\delta[v_m]\cdot\nabla\big)T_m$ are. By the Aubin-Lions Compactness Lemma (see Th. 2.1, Chapter 3 in Temam [23]), there exists a subsequence T_m that converges strongly in $L^2(0,T;L^2(\Omega))$. Passing to the limit in the weak formulation of (2.3) we obtain a weak solution T.

Now, let $N \geq 3$. As before, consider $\phi \in C_c^\infty(\mathbb{R}^2 \times [0,\tau))$ and let $r \geq \frac{N}{2} \geq \alpha$. Then

$$|\langle \Lambda^{2\alpha} T_m, \phi\rangle| \leq \|\Lambda^\alpha T_m\|_{L^2}\|\Lambda^\alpha\phi\|_{L^2} \leq \|\Lambda^\alpha T_m\|_{L^2}\|\phi\|_{H^\alpha} \leq \|\Lambda^\alpha T_m\|_{L^2}\|\phi\|_{H^r}$$

which leads to $\|\Lambda^{2\alpha}T_m\|_{H^{-r}(\Omega)} \leq \|\Lambda^\alpha T_m\|_{L^2}$ where $\Omega \subset \mathit{supp}\,\phi$ is a bounded open ball. Then $\Lambda^{2\alpha}T_m \in L^2\big(0,\tau;H^{-r}\big)$. Now we have

$$\langle\big(\psi_\delta[v_m]\cdot\nabla\big)T_m, \phi\rangle = \langle \psi_\delta[v_m]T_m, \nabla\phi\rangle \leq \|\psi_\delta[v_m]\|_{L^2}\|\nabla\phi\|_{L^{\frac{N}{\alpha}}}\|T_m\|_{L^{\frac{2N}{N-2\alpha}}}.$$

As $N > 2\alpha$,

$$\|T_m\|_{L^{\frac{2N}{N-2\alpha}}} \leq \|T_m\|_{H^\alpha},$$

where we have used a fractional Sobolev Embedding (cf. Theorem 6.7 in Di Nezza, Palatucci and Valdinoci [14]). Moreover, our choice of r implies, again due to a Sobolev Embedding, that H^{r-1} is continuosly embedded in $L^{\frac{N}{\alpha}}$, so

$$\|\nabla\phi\|_{L^{\frac{N}{\alpha}}} \leq \|\nabla\phi\|_{H^{r-1}} \leq \|\phi\|_{H^r}.$$

Then

$$\langle\big(\psi_\delta[v_m]\cdot\nabla\big)T_m,\phi\rangle \leq \|\psi_\delta[v_m]\|_{L^2}\|\nabla\phi\|_{H^\alpha}\|T_m\|_{H^r} \leq \|T_m\|_{L^2}\|\phi\|_{H^r}\|T_m\|_{H^\alpha}$$

which implies that $\big(\psi_\delta[v_m]\cdot\nabla\big)T_m \in L^2\big(0,\tau;H^{-r}\big)$. As before, by using the Aubin-Lions Compactness Theorem we obtain a subesquence T_m that converges strongly in $L^2\big(0,\tau;L^2(\Omega)\big)$. This allows us to pass to the limit in (2.3), thus obtaining a weak solution T. □

Remark 2.4. The solution just obtained need not be unique. However, it is unique for $\frac{1}{2}<\alpha\leq 1$ amongst those in

$$T\in C(0,\tau;L^2(\mathbb{R}^N))\cap L^2(0,\tau;H^\alpha(\mathbb{R}^N))\cap L^p(0,\tau;L^q(\mathbb{R}^N))$$

where $q>\frac{N}{2\alpha-1}$ and $p=\frac{\alpha}{\alpha-\frac{N}{2q}-\frac{1}{2}}$, see Castro *et al* [6].

3. Non-uniform decay

In this section we address the long time behaviour of solutions when the initial data T_0 is just in L^2. We first list results and estimates we need in order to prove that the L^2 norm tends to zero, but with no uniform rate.

The following Lemma provides decay rates for the kernel K_α.

Lemma 3.1. (M. E. Schonbek and T. Schonbek [21]) *Let β,γ be multi-indices such that $|\gamma|<|\beta|+2\alpha\max(j,1)$, $j=0,1,2,\cdots$, $1\leq p\leq\infty$. Then*

$$\|x^\gamma\partial_t^j\nabla^\beta K_\alpha(t)\|_{L^p}=Ct^{\frac{|\gamma|-|\beta|}{2\alpha}-j-\frac{N}{2\alpha}\left(1-\frac{1}{p}\right)}$$

for some constant C depending only on $\alpha,\beta,\gamma,j,p,N$.

In order to obtain the asymptotic behaviour of the solution T, we need some estimates on the solution to the corresponding linear equation.

Lemma 3.2. (Carrillo and Ferreira [5]) *Let β be a multi-index in in $\mathbb{N}^N$. Then, given $T_0\in L^r(\mathbb{R}^N)$ and $1\leq r\leq p\leq\infty$,*

$$\|\nabla^\beta K_\alpha(t)T_0\|_{L^p}\leq Ct^{-\frac{|\beta|}{2\alpha}-\frac{N}{2\alpha}\left(\frac{1}{r}-\frac{1}{p}\right)}\|T_0\|_{L^r} \tag{3.1}$$

for some constant C depending only on α,β,p,N.

3.1. **Decay to zero of the L^2 norm of solutions.** In order to study the decay of solutions to (1.1) we use the Fourier Splitting method, developed by M.E. Schonbek to prove decay of energy for solutions to parabolic conservation laws [18] and to Navier-Stokes equations [19], [20]. The idea of the method is to obtain a differential inequality for the value of the L^2 norm at large times, in terms of the behaviour of the small frequencies of solutions.

We proceed formally, assuming the solutions are regular. After multiplying (1.1) by T, integrating in space and applying the Fourier transform, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^N} |\widehat{T}|^2\, d\xi = -2\int_{\mathbb{R}^N} |\xi|^{2\alpha}|\widehat{T}|^2\, d\xi. \tag{3.2}$$

We then choose

$$B(t) = \left\{\xi \in \mathbb{R}^N : |\xi|^{2\alpha} \leq \frac{r'(t)}{2\,r(t)}\right\}$$

where $r(t)$ is a positive, increasing function, with $r(0) = 1$ and $r(t) \geq 1$. Substituting in (3.2) we get

$$\begin{aligned}\int_{\mathbb{R}^N} |\xi|^{2\alpha}|\widehat{T}|^2\, d\xi &= \int_{B(t)} |\xi|^{2\alpha}|\widehat{T}|^2\, d\xi + \int_{B(t)^c} |\xi|^{2\alpha}|\widehat{T}|^2\, d\xi \\ &\geq \int_{B(t)} |\xi|^{2\alpha}|\widehat{T}|^2\, d\xi + \frac{r'(t)}{2\,r(t)}\int_{B(t)^c} |\widehat{T}|^2\, d\xi\end{aligned}$$

which after multiplying by $r(t)$ and integrating leads to the main inequality

$$\frac{d}{dt}\Big(r(t)\int_{\mathbb{R}^N} |\widehat{T}(t)|^2\, d\xi\Big) \leq r'(t)\Big(\int_{B(t)} |\widehat{T}(t)|^2\, d\xi\Big). \tag{3.3}$$

Proof of Theorem 1.1. We first prove the Theorem for $N \geq 3, 0 < \alpha \leq 1$ and $N = 2, 0 < \alpha < 1$. Assume the solution T is regular. We note that

$$\begin{aligned}|\widehat{v\cdot\nabla T}(s)| &\leq |\widehat{\nabla(Tv)}(s)| = |\xi||\widehat{Tv}| \leq |\xi|\|\widehat{Tv}\|_{L^\infty} \\ &\leq |\xi|\|Tv\|_{L^1} \leq |\xi|\|v\|_{L^2}\|T\|_{L^2} \leq C|\xi|\|T\|^2_{L^2},\end{aligned} \tag{3.4}$$

where we used (2.2). From an integral solution to (1.1)

$$\widehat{T}(\xi,t) = e^{-|\xi|^{2\alpha}t}\,\widehat{T_0}(\xi) + \int_0^t e^{-|\xi|^{2\alpha}(t-s)}\,\widehat{v\cdot\nabla T}(s)\, ds \tag{3.5}$$

after using (3.4) and $\|T(t)\|^2_{L^2(\mathbb{R}^N)} \leq C$ we obtain

$$\int_{B(t)} |\widehat{T}(\xi,t)|^2 d\xi \leq C\int_{B(t)} \Big(|e^{-|\xi|^{2\alpha}t}\widehat{T_0}(\xi)|^2 + \Big(\int_0^t |\widehat{v\cdot\nabla T}(s)|\, ds\Big)^2\Big)\, d\xi$$

$$\leq C\int_{B(t)}|e^{-|\xi|^{2\alpha}t}\widehat{T_0}(\xi)|^2\,d\xi + C\int_{B(t)}|\xi|^2\Big(\int_0^t \|T(s)\|_{L^2}^2\,ds\Big)^2\,d\xi$$
$$\leq C\|K_\alpha(t)T_0\|_{L^2}^2 + C\Big(\frac{r'(t)}{2\,r(t)}\Big)^{\frac{N+2}{2\alpha}}\Big(\int_0^t \|T(s)\|_{L^2}^2\,ds\Big)^2$$
$$\leq C\|K_\alpha(t)T_0\|_{L^2}^2 + Ct^2\Big(\frac{r'(t)}{2\,r(t)}\Big)^{\frac{N+2}{2\alpha}}, \tag{3.6}$$

where $K_\alpha(t)T_0$ is the solution to the fractional heat equation with initial data T_0. Taking $r(t) = (t+1)^N$ and substituting (3.6) in (3.3), we obtain

$$\frac{d}{dt}\big((t+1)^N\|T(t)\|_{L^2}^2\big) \leq C(t+1)^{N-1}\big(\|K_\alpha(t)T_0\|_{L^2}^2 + (t+1)^{2-\frac{N+2}{2\alpha}}\big)$$

which leads to

$$\|T(t)\|_{L^2}^2 \quad\leq\quad \|T_0\|_{L^2}^2(t+1)^{-N} + C(t+1)^{-N}\int_0^t (s+1)^{N-1}\|K_\alpha(s)T_0\|_{L^2}^2\,ds$$
$$+(t+1)^{2-\frac{N+2}{2\alpha}}.$$

As

$$\lim_{t\to\infty}\|K_\alpha(t)T_0\|_{L^2}^2 = 0$$

and we have $N \geq 2$ and $\alpha < 1$, we obtain the result for smooth solutions. This same argument applies to the regularized solutions T_m in (2.3) and we then obtain the same estimate for T_m. This estimate is independent of m, so semicontinuity of the norm implies it holds a.e. in t for weak solutions T, then for all time because T is in $C_w(\mathbb{R}_+, L^2)$. For full details concerning this argument, see pages 267–269 in Lemarié-Rieusset [13] and the appendix in Wiegner [24].

We address now the case $N = 2$ and $\alpha = 1$. We use the same ideas, but with a variation due to Kajikiya and Miyakawa [11]. As

$$\widehat{(v\cdot\nabla)T} \quad\leq\quad \|\widehat{(v\cdot\nabla)T}\|_{L^\infty} \leq \|(v\cdot\nabla)T\|_{L^1}$$
$$\leq\quad \|v\|_{L^2}\|\nabla T\|_{L^2} \leq C\|T\|_{L^2}\|\nabla T\|_{L^2} \leq C\|\nabla T\|_{L^2},$$

then

$$\int_0^t \widehat{(v\cdot\nabla)T}^{\;2}\,ds \leq C\int_0^t \|\nabla T\|_{L^2}^2\,ds \leq C$$

as the solution T is in $L^\infty((0,T), L^2) \cap L^2((0,T), H^1)$, as $\alpha = 1$. Then

$$\int_{B(t)}|\widehat{T}(\xi,t)|^2\,d\xi \leq C\|K_\alpha(t)T_0)\|_{L^2} + C\frac{r'(t)}{r(t)}.$$

Taking $r(t) = (t+1)^\mu$, with $0 < \mu < 1$, we have

$$\frac{d}{dt}\big((t+1)^\mu \|T(t)\|_{L^2}^2\big) \le C\mu(t+1)^{\mu-1}\big(\|K_\alpha(t)T_0\|_{L^2}^2 + \mu(t+1)^{-1}\big)$$

so

$$\|T(t)\|_{L^2}^2 \le C(t+1)^{-\mu} + C(t+1)^{-\mu}\int_0^t (s+1)^{\mu-1}\|K_\alpha(s)T_0\|_{L^2}^2 ds + C\mu^2 \le C\mu^2.$$

So

$$\limsup_{t\to\infty} \|T(t)\|_{L^2} \le C\mu^2$$

for all $0 < \mu < 1$, which proves the result for smooth solutions. An argument similar to the one before extends the result to weak solutions. □

3.2. **Non-uniform decay. Proof of Theorem 1.2:** We follow the idea of M.E. Schonbek [21] and Niche and M.E. Schonbek [16] for the Navier-Stokes and dissipative quasi-geostrophic equations respectively. Given a smooth T_0 with $\|T_0\|_{L^2} = r$, we will construct a scaled T_0^λ with the same norm such that for arbitrary t^*, the linear part of the integral solution

$$T^\lambda(x,t^*) = K_\alpha(t^*) * T_0^\lambda(x) - \int_0^{t^*} K_\alpha(t^*-s) * \big(v^\lambda\cdot\nabla\big)T(s)\,ds \tag{3.7}$$

has $L^2(\mathbb{R}^N)$ norm arbitrarily close to that of T_0^λ and such that the integral term has arbitrarily small $L^2(\mathbb{R}^N)$ norm. In order to obtain this second condition estimate, we will have to choose T_0 in $L^p(\mathbb{R}^N)\cap L^q(\mathbb{R}^N)$, for some $p,q>2$ to be determined later.

Given $T_0 \in L^2(\mathbb{R}^N)$, let $T_0^\lambda(x) = \lambda^{\frac{N}{2}} T_0(\lambda x)$. Clearly, $\|T_0\|_{L^2} = \|T_0^\lambda\|_{L^2}$. Now, if $\Theta(x,t)$ is the solution to

$$\Theta_t + \Lambda^{2\alpha}\Theta = 0, \quad \Theta_0(x) = T_0(x), \tag{3.8}$$

then $\Theta^\lambda(x,t) = \lambda^{\frac{N}{2}}\Theta(\lambda x, \lambda^{2\alpha}t)$ is the solution to (3.8) with initial data $\Theta_0^\lambda(x) = T_0^\lambda(x) = \lambda^{\frac{N}{2}} T_0(\lambda x)$. But now

$$\begin{aligned}\|\Theta^\lambda(t)\|_{L^2}^2 &= \int_{\mathbb{R}^N} |\Theta^\lambda(x,t)|^2\,dx = \lambda^N \int_{\mathbb{R}^N} |\Theta(\lambda x, \lambda^{2\alpha}t)|^2\,dx \\ &= \int_{\mathbb{R}^N} |\Theta(y,\lambda^{2\alpha}t)|^2\,dy = \int_{\mathbb{R}^N} e^{-2|\xi|^{2\alpha}\lambda^{2\alpha}t}|\widehat{T}_0(\xi)|^2\,d\xi.\end{aligned}$$

Then for the given arbitrary $t^* > 0$

$$\lim_{\lambda\to 0}\frac{\|\widehat{\Theta^\lambda}(t^*)\|_{L^2}^2}{\|\widehat{\Theta_0}\|_{L^2}^2} = \lim_{\lambda\to 0}\frac{\int_{\mathbb{R}^N} e^{-2|\xi|^{2\alpha}\lambda^{2\alpha}t^*}|\widehat{T}_0(\xi)|^2\,d\xi}{\int_{\mathbb{R}^N}|\widehat{T_0}(\xi)|^2\,d\xi} = 1. \tag{3.9}$$

So for a small enough $\lambda > 0$, we can make this ratio as close to one as we want, regardless of $t^* > 0$.

We now address the integral term in (3.7). If T_0 in $L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ with $\frac{1}{2} = \frac{1}{p} + \frac{1}{q}$, for $p, q \neq \infty$ we have that

$$\begin{aligned}
\|K_\alpha(t^* - s) * (v^\lambda \cdot \nabla) T^\lambda(s)\|_{L^2} &= \|\nabla K_\alpha(t^* - s) * (v^\lambda T^\lambda)(s)\|_{L^2} \\
&\leq \|\nabla K_\alpha(t^* - s)\|_{L^1} \|(v^\lambda T^\lambda)(s)\|_{L^2} \\
&\leq C(t^* - s)^{-\frac{1}{2\alpha}} \|v^\lambda(s)\|_{L^p} \|T^\lambda(s)\|_{L^q} \\
&\leq C(t^* - s)^{-\frac{1}{2\alpha}} \|T^\lambda(s)\|_{L^p} \|T^\lambda(s)\|_{L^q} \\
&\leq C(t^* - s)^{-\frac{1}{2\alpha}} (s+1)^{-\frac{N}{4\alpha}} \|T_0^\lambda(s)\|_{L^p} \|T_0^\lambda(s)\|_{L^q} \\
&\leq C(t^* - s)^{-\frac{1}{2\alpha}} (s+1)^{-\frac{N}{4\alpha}} \lambda^{\frac{N}{2}} \|T_0(s)\|_{L^p} \|T_0(s)\|_{L^q},
\end{aligned}$$

where we have used Lemma 3.1, (2.2), the Maximum Principle (1.2) and the scaling

$$\|T_0^\lambda\|_{L^p} = \lambda^{N\left(\frac{1}{2} - \frac{1}{p}\right)} \|T_0\|_{L^p}, \quad 1 \leq p < \infty.$$

Then

$$\int_0^{t^*} \|K_\alpha(t^* - s) * \big(v^\lambda \cdot \nabla\big) T(s)\|_{L^2}\, ds \leq C\big(t^*\big)^{1 - \frac{1}{2\alpha}} \lambda^{\frac{N}{2}} \|T_0(s)\|_{L^p} \|T_0(s)\|_{L^q}. \tag{3.10}$$

So for any $\epsilon > 0, t^* > 0$, we can find $\lambda > 0$ such that by (3.9)

$$\frac{\|K_\alpha(T) * T_0^\lambda\|_{L^2}}{\|T_0^\lambda\|_{L^2}} \geq 1 - \frac{\epsilon}{2}$$

and by (3.10)

$$\frac{\int_0^T \|K_\alpha(T - s) * (v^\lambda \cdot \nabla) T^\lambda(s)\|_{L^2}\, ds}{\|T_0^\lambda\|_{L^2}} \leq \frac{\epsilon}{2}.$$

Then

$$\frac{\|T^\lambda(T)\|_{L^2}}{\|T_0^\lambda\|_{L^2}} \geq 1 - \epsilon.$$

which proves the Theorem. □

4. UNIFORM DECAY

In this section we prove uniform decay of solutions, provided the initial data is in $L^p(\mathbb{R}^N) \cap L^2(\mathbb{R}^N), 1 \leq p < 2$. We again use the Fourier Splitting

Method, as in Section 3.1. Due to technical reasons, we address separately the cases $N = 2$ and $N \geq 3$.

We will need the following Lemma.

Lemma 4.1. (Niche and M. E. Schonbek [16]) *Let $h \in L^p(\mathbb{R}^N)$, $1 \leq p < 2$ and let $S(t) = \{\xi \in \mathbb{R}^N : |\xi| \leq g(t)^{-\frac{1}{2\alpha}}\}$, for a continuous function $g : \mathbb{R}^+ \to \mathbb{R}^+$. Then*

$$\int_{S(t)} |\hat{h}|^2 d\xi \leq C g(t)^{-\frac{N}{2\alpha}(\frac{2}{p}-1)}.$$

4.1. **Decay with $N = 2$.** From the integral solution (3.5)

$$\widehat{T}(\xi,t) = e^{-|\xi|^{2\alpha}t}\, \widehat{T_0}(\xi) + \int_0^t e^{-|\xi|^{2\alpha}(t-s)}\, \widehat{v \cdot \nabla T}(s)\, ds$$

and using (3.4), we obtain the estimate

$$|\widehat{T(\xi,t)}|^2 \leq C\big(|\widehat{T}_0(\xi)|^2 + t\int_0^t |\xi|^2 \|\widehat{T}(s)\|_{L^2}^4\, ds\big).$$

Integrating in $B(t)$ and using Lemma 4.1 we get

$$\begin{aligned}
\int_{B(t)} |\widehat{T}(t)|^2\, d\xi &\leq \int_{B(t)} |\widehat{T}_0(\xi)|^2\, d\xi + C\int_{B(t)} t\big(\int_0^t |\xi|^2 \|\widehat{T}(\tau)\|_{L^2}^4\, d\tau\big)\, d\xi \\
&\leq C\big(\frac{r'(t)}{r(t)}\big)^{\frac{1}{\alpha}\left(\frac{2}{p}-1\right)} \\
&+ Ct\int_0^{\left(\frac{r'(t)}{2r(t)}\right)^{\frac{1}{2\alpha}}} r^3\big(\int_0^t \|\widehat{T}(\tau)\|_{L^2}^4\, d\tau\big)\, dr
\end{aligned}$$

where we passed to polar coordinates in the second term. Using this in the main inequality (3.3) leads to

$$\begin{aligned}
\frac{d}{dt}\big(r(t)\int_{\mathbb{R}^2} |\widehat{T}(t)|^2\, d\xi\big) &\leq Cr'(t)\big(\frac{r'(t)}{r(t)}\big)^{\frac{1}{\alpha}\left(\frac{2}{p}-1\right)} \\
&+ Ctr'(t)\big(\frac{r'(t)}{r(t)}\big)^{\frac{2}{\alpha}} \int_0^t \|\widehat{T}(\tau)\|_{L^2}^4\, d\tau. \qquad (4.1)
\end{aligned}$$

To prove decay in $\mathbb{R}^2$, we use a bootstrap argument introduced by Zhang [29] for the $2D$ Navier-Stokes equations. We first obtain a preliminary decay.

Lemma 4.2. *Let $T(t)$ be a regular solution to (1.1). Then,*

$$\|T(t)\|_{L^2}^2 \leq C(\alpha)[\ln(e+t)]^{-\left(1+\frac{1}{\alpha}\right)}, \quad t \geq 0, \quad 0 < \alpha \leq 1.$$

Proof. First consider $0<\alpha<1$. Taking $r(t)=\big[\ln(e+t)\big]^{1+\frac{1}{\alpha}}$ in (4.1), we obtain

$$\frac{d}{dt}\big([\ln(e+t)]^{1+\frac{1}{\alpha}}\|\widehat{T}(t)\|_{L^2}^2\big)\le C\frac{[\ln(e+t)]^{\frac{2}{\alpha}(1-\frac{1}{p})}}{(e+t)^{1+\frac{1}{\alpha}(\frac{2}{p}-1)}}+C\frac{t^2}{[\ln(e+t)]^{\frac{1}{\alpha}}(e+t)^{1+\frac{2}{\alpha}}}.$$

The estimate follows after integration. Indeed, as $p<2$

$$\int_0^t\frac{\big[\ln(e+s)\big]^{\frac{2}{\alpha}\left(1-\frac{1}{p}\right)}}{(e+s)^{1+\frac{1}{\alpha}\left(\frac{2}{p}-1\right)}}ds=\int_1^{\ln(e+t)}s^{\frac{2}{\alpha}\left(1-\frac{1}{p}\right)}e^{-\frac{s}{\alpha}\left(\frac{2}{p}-1\right)}ds\le C,$$

and as $\alpha<1$

$$\int_0^t\frac{s^2ds}{\big[\ln(e+s)\big]^{\frac{1}{\alpha}}(e+s)^{1+\frac{2}{\alpha}}}\le\int_1^{\ln(e+t)}s^{-\frac{1}{\alpha}}e^{\left(\frac{2}{\alpha}-2\right)s}\,ds\le C.$$

Now let $\alpha=1$. Taking $r(t)=[\ln(e+t)]^3$ in (4.1) we have

$$\frac{d}{dt}\big([\ln(e+t)]^3\|\widehat{T}(t)\|_{L^2}^2\big)\le C\frac{[\ln(e+t)]^{\frac{2}{p}+1}}{(e+t)^{\frac{2}{p}}}+C(e+t)^{-1}. \tag{4.2}$$

As

$$\int_0^t\frac{[\ln(e+s)]^{\frac{2}{p}+1}}{(e+)^{\frac{2}{p}}}\,ds=\int_1^{\ln(e+t)}s^{\frac{2}{p}+1}e^{\left(1-\frac{2}{p}\right)s}\,ds\le C$$

and

$$\int_0^t\frac{ds}{e+s}\le\ln(e+t),$$

then the estimate follows after integrating (4.2). □

Proof of Theorem 1.3 for $N=2$. We treat the cases $0<\alpha<1$ and $\alpha=1$ separately.

We first consider $0<\alpha<1$ and choose $r(t)=(1+t)^{\frac{1}{\alpha}-\delta}$ in (4.1), for $0<\delta<\frac{1}{\alpha}$. From now on, constants are $C_\delta=C(\alpha,\delta,p)$ and remain finite when δ goes to zero, for all possible values of α and p. Only those constants that may blowup are explicitly written in closed form.

From (4.1) and Lemma 4.1, for a smooth solution $T(t)$ we get

$$\begin{aligned}\frac{d}{dt}\big((1+t)^{\frac{1}{\alpha}-\delta}\|T(t)\|_{L^2}^2\big)\;&\le\;C_\delta(1+t)^{-\frac{2}{\alpha}\left(\frac{1}{p}-1\right)-\delta-1}\\&+\;C_\delta(1+t)^{-\left(\frac{1}{\alpha}+\delta\right)}\int_0^t\frac{\|T(s)\|_{L^2}^2}{[\ln(e+s)]^{1+\frac{1}{\alpha}}}ds\end{aligned}\tag{4.3}$$

where we have used the preliminary decay from Lemma 4.2. Integrating between 0 and t we obtain

$$\begin{aligned}\|T(t)\|_{L^2}^2 &\leq C_\delta(1+t)^{-\frac{1}{\alpha}+\delta} \\ &+ \frac{C_\delta}{\frac{2}{\alpha}\big(1-\frac{1}{p}\big)-\delta}\big((1+t)^{-\frac{1}{\alpha}\left(\frac{2}{p}-1\right)}-(1+t)^{-\frac{1}{\alpha}+\delta}\big) \\ &+ C_\delta(1+t)^{-\frac{1}{\alpha}+\delta}\int_0^t \frac{\|T(s)\|_{L^2}^2}{[\ln(e+s)]^{1+\frac{1}{\alpha}}}ds.\end{aligned}$$

If $p \neq 1$, after letting δ go to zero we have

$$\|T(t)\|_{L^2}^2 \leq C(1+t)^{-\frac{1}{\alpha}} + C(1+t)^{-\frac{1}{\alpha}\left(\frac{2}{p}-1\right)} + C\int_0^t \frac{\|T(s)\|_{L^2}^2}{[\ln(e+s)]^{1+\frac{1}{\alpha}}}ds.$$

As

$$\int_0^t [\ln(e+s)]^{-\left(1+\frac{1}{\alpha}\right)}ds < \infty,$$

using Gronwall's inequality we obtain

$$\|T(t)\|_{L^2}^2 \leq C(1+t)^{-\frac{1}{\alpha}} + C(1+t)^{-\frac{1}{\alpha}\left(\frac{2}{p}-1\right)} \leq C(1+t)^{-\frac{1}{\alpha}\left(\frac{2}{p}-1\right)}.$$

For $p = 1$, from (4.3) we have

$$\begin{aligned}\|T(t)\|_{L^2}^2 &\leq C_\delta(1+t)^{-\frac{1}{\alpha}+\delta} + C_\delta(1+t)^{-\frac{1}{\alpha}}\big(\frac{(1+t)^\delta-1}{\delta}\big) \\ &+ C_\delta\int_0^t \frac{\|T(s)\|_{L^2}^2}{[\ln(e+s)]^{1+\frac{1}{\alpha}}}ds.\end{aligned}$$

Letting δ go to zero we obtain

$$\|T(t)\|_{L^2}^2 \leq C(1+t)^{-\frac{1}{\alpha}} + C\int_0^t \frac{\|T(s)\|_{L^2}^2}{[\ln(e+s)]^{1+\frac{1}{\alpha}}}ds$$

which after Gronwall's inequality leads to

$$\|T(t)\|_{L^2}^2 \leq C(1+t)^{-\frac{1}{\alpha}}.$$

We now consider $\alpha = 1$. Using $r(t) = (t+1)^2$ in (4.1) we have

$$\frac{d}{dt}\big((1+t)^2\|T(t)\|_{L^2}^2\big) \leq C + C(1+t)^{-\left(\frac{2}{p}-2\right)} + C\int_0^t \frac{\|T(s)\|_{L^2}^2}{[\ln(e+s)]^{1+\frac{1}{\alpha}}}ds.$$

Integrating between 0 and t and dividing by $(1+t)$ we obtain

$$(1+t)\|T(t)\|^2_{L^2} \leq CC(1+t)^{-\left(\frac{2}{p}-2\right)} + C\int_0^t \frac{(s+1)\|T(s)\|^2_{L^2}}{(s+1)[\ln(e+s)]^{1+\frac{1}{\alpha}}}ds$$

which has the form

$$\psi(t) \leq C + \int_0^t [\psi(\tau)b(\tau) + a'(\tau)]\,d\tau$$

for

$$\psi(t) = (1+t)\|T(t)\|^2_{L^2}, \quad a(t) = C(1+t)^{-\left(\frac{2}{p}-2\right)}, \quad b(t) = C\frac{[\ln(e+t)]^{-2}}{(1+t)}.$$

As

$$\int_0^t b(s)\,ds < \infty$$

by a version of Gronwall's inequality (see Corollary 1.6, Baĭnov and Simeonov [1])

$$\psi(t) \leq C\exp\big(\int_0^t b(s)\,ds\big) + \int_0^t a'(s)\exp\big(\int_\tau^t b(\tau)d\tau\big)ds$$

we obtain

$$\|T(t)\|^2_{L^2} \leq C(1+t)^{-1} + C(1+t)^{-(\frac{2}{p}-1)} \leq C(1+t)^{-\left(\frac{2}{p}-1\right)}$$

which proves the decay. Using the same argument at the end of the proof of Theorem 1.1, we extend the estimate to weak solutions. □

4.2. **Decay with $N \geq 3$.** As before, we need preliminary estimates and decays.

Lemma 4.3. *Let $T(t)$ be a regular solution to (1.1). Then,*

$$|\widehat{T}(\xi,t)| \leq C\big(|\widehat{T_0}(\xi)| + |\xi|^{1-2\alpha}\big), \quad \xi \in B(t).$$

Proof. From (3.4) and the Maximum Principle (1.2), we obtain

$$|\widehat{v\cdot\nabla T}(t)| \leq C|\xi|\|T\|^2_{L^2} \leq C|\xi|.$$

So the integral equation (3.5) becomes

$$\begin{aligned} |\widehat{T}(\xi,t)| &\leq |\widehat{T_0}(\xi)| + C\int_0^t e^{-|\xi|^{2\alpha}(t-s)}|\xi|\,ds \\ &= |\widehat{T_0}(\xi)| + C|\xi|^{1-2\alpha}\big(1-e^{-|\xi|^{2\alpha}t}\big) \leq C\big(|\widehat{T_0}(\xi)| + |\xi|^{1-2\alpha}\big) \end{aligned}$$

which proves the Lemma. □

We now obtain a preliminary decay.

Lemma 4.4. *For a regular solution $T(t)$ to (1.1), we have*

$$\|T(t)\|_{L^2}^2 \leq C(t+1)^{-\min\left\{\frac{N+2}{2\alpha}-2,\frac{N}{2\alpha}\left(\frac{2}{p}-1\right)\right\}}, \quad t \geq 0.$$

Proof. Considering $B(t)$ defined in (3.1) and Lemma 4.3, we have

$$\int_{B(t)} |\widehat{T}(\xi,t)|^2\, d\xi \quad \leq \quad C\int_{B(t)} |\widehat{T}_0(\xi)|^2\, d\xi + C\int_{B(t)} |\xi|^{2-4\alpha}\, d\xi,$$

and by Lemma 4.1, we obtain

$$\int_{B(t)} |\widehat{T}(\xi,t)|^2\, d\xi \quad \leq \quad C\big(\frac{r'(t)}{r(t)}\big)^{-\frac{N}{2\alpha}\left(\frac{2}{p}-1\right)} + C\big(\frac{r'(t)}{r(t)}\big)^{-\left(\frac{N+2}{2\alpha}-2\right)}.$$

Using this and $r(t) = (t+1)^N$ in (3.3), we obtain

$$\|T(t)\|_{L^2}^2 \leq C(t+1)^{-\frac{N}{2\alpha}\left(\frac{2}{p}-1\right)} + C(t+1)^{-\left(\frac{N+2}{2\alpha}-2\right)}$$

which proves the Lemma. □

Lemma 4.5. *Let $B(t)$ defined in (3.1) with $r(t) = (1+t)^N$. For $T(t)$ a regular solution to (1.1), we have*

$$|\widehat{T}(\xi,t)| \leq |\widehat{T}_0(\xi)| + C, \quad \xi \in B(t).$$

Proof. By using (3.4) in the integral equation (3.5), together with the decay estimate from Lemma 4.4 we obtain

$$\begin{aligned} |\widehat{T}(\xi,t)| &\leq |\widehat{T}_0(\xi)| + C|\xi|\int_0^t \|T(s)\|_{L^2}^2\, ds \\ &\leq |\widehat{T}_0(\xi)| + C|\xi|\int_0^t (s+1)^{-\min\left\{\frac{N+2}{2\alpha}-2,\frac{N}{2\alpha}\left(\frac{2}{p}-1\right)\right\}}\, ds \\ &\leq |\widehat{T}_0(\xi)| + C \end{aligned}$$

as $N \geq 3$. This proves the Lemma. □

Proof of Theorem 1.3 for $N \geq 3$. Let $T(t)$ be a regular solution. Using Lemma 4.5 in (3.3), Lemma 4.1 and $r(t) = (t+1)^N$ we obtain

$$\frac{d}{dt}\big((t+1)^N\|T(t)\|_{L^2}^2\big) \leq C(t+1)^{N-1}\big((t+1)^{-\frac{N}{2\alpha}\left(\frac{2}{p}-1\right)} + (t+1)^{\frac{-N}{2\alpha}}\big).$$

Integrating, we get

$$\|T(t)\|_{L^2}^2 \leq C(t+1)^{-N} + C(t+1)^{-\frac{N}{2\alpha}\left(\frac{2}{p}-1\right)} \leq C(t+1)^{-\frac{N}{2\alpha}\left(\frac{2}{p}-1\right)}$$

which proves the Theorem. As before, the argument at the end of the proof of Theorem 1.1, allows us to extend the estimate to weak solutions. □

5. Asymptotic behaviour

In this section we study decay of norms and some aspects of asymptotic behaviour of solutions to (1.1) when the initial data T_0 is in $L^1(\mathbb{R}^N) \cap L^{\frac{N}{2\alpha-1}}(\mathbb{R}^N)$. This choice is motivated by the fact that given a solution T, the scaled function $T_\lambda(x,t) = \lambda^{2\alpha} T(\lambda x, \lambda^{2\alpha-1} t)$ is also a solution and its norm is invariant precisely in $L^{\frac{N}{2\alpha-1}}(\mathbb{R}^N)$, i.e.,

$$\|T_\lambda(t)\|_{L^{\frac{N}{2\alpha-1}}} = \|T(\lambda^{2\alpha-1}t)\|_{L^{\frac{N}{2\alpha-1}}}, \quad \lambda > 0.$$

Estimates similar to the ones we address here have been proved, for initial data in Lebesgue scale-invariant spaces, by Kato [12] and Carpio [3], [4] for the Navier-Stokes and vorticity equations and by Carrillo and Ferreira [5] for the dissipative quasi-geostrophic equations.

We introduce now the definitions needed to prove existence of solutions to (1.1) with T_0 in $L^1(\mathbb{R}^N) \cap L^{\frac{N}{2\alpha-1}}(\mathbb{R}^N)$. We closely follow the Appendix in Carrillo and Ferreira [5].

We consider the space of functions

$$E_{q,T} = \{h : h \in BC(0,T; L^{\frac{N}{2\alpha-1}}(\mathbb{R}^N)), t^\nu h \in BC(0,T; L^q(\mathbb{R}^N))\}$$

where $\nu = 2 - \frac{1}{\alpha} - \frac{N}{\alpha q}$, endowed with the norm

$$\|h\|_{E_{q,T}} = \sup_{0<t<T} t^\nu \|h(t)\|_{L^q} + \sup_{0<t<T} \|h(t)\|_{L^{\frac{N}{2\alpha-1}}}.$$

A mild solution to (1.1) is a function in $E_{q,T}$ such that

$$T(x,t) = K_\alpha(t) * T_0(x) - \int_0^t \nabla K_\alpha(t-s) * \big(TH[T]\big)(s)\, ds \tag{5.1}$$

where $H[T]$ is as defined in (2.1).

Theorem 5.1. *Let $T_0 \in L^1(\mathbb{R}^N) \cap L^{\frac{N}{2\alpha-1}}(\mathbb{R}^N)$. Then, there exits a unique mild solution to (1.1) with $1/2 < \alpha \le 1$ such that*

$$T \in BC(0,\infty; L^1(\mathbb{R}^N) \cap L^{\frac{N}{2\alpha-1}}(\mathbb{R}^N)).$$

As the operator H is Calderón-Zygmund, then

$$\|H[T]\|_{L^p} \le C\|T\|_{L^p}, \qquad 1 < p < \infty$$

so the proof of the analogous existence result for the dissipative quasi-geostrophic equation by Carrillo and Ferreira [5] carries over with minimal changes. We refer the reader to [5] for the details.

Proof of Theorem 1.4. Using the inequality

$$\int_{\mathbb{R}^N} |T|^{p-2} T \Lambda^{2\alpha} T dx \geq \frac{2}{p} \int_{\mathbb{R}^N} \left(\Lambda^\alpha |T|^{\frac{p}{2}}\right)^2 dx, \quad p \geq 2, \quad \alpha \in [0,1],$$

proved by A. Córdoba and D. Córdoba [7] and Ju [10], we obtain from (1.1)

$$\frac{d}{dt}\|T(t)\|_{L^p}^p \leq -2 \int |\Lambda^\alpha T^{\frac{p}{2}}|^2 dx. \tag{5.2}$$

The fractional Gagliardo-Nirenberg inequality

$$\|u\|_{L^2} \leq C \|u\|_{L^1}^{\frac{2\alpha}{N+2\alpha}} \|\Lambda^\alpha u\|_{L^2}^{\frac{N}{N+2\alpha}}$$

leads, after taking $u = T^{p/2}$, to

$$\|\Lambda^\alpha T^{\frac{p}{2}}\|_{L^2}^2 \geq C\left(\|T\|_{L^p}^p\right)^{\frac{(N+2\alpha)}{N}} \left(\|T\|_{L^{\frac{p}{2}}}^{\frac{p}{2}}\right)^{-\frac{4\alpha}{N}}. \tag{5.3}$$

Making $y_p = \|T\|_p^p$, for $p \geq 2$, and using (5.3) in (5.2) we have

$$\frac{d}{dt} y_p \leq -2C y_{\frac{p}{2}}^{-\frac{4\alpha}{N}} y_p^{\frac{N+2\alpha}{N}}. \tag{5.4}$$

Let $p_n = 2^n$, $n \geq 1$. As $y_{\frac{p_1}{2}} = \|T(t)\|_{L^1} \leq \|T_0\|_{L^1}$, solving (5.4) for $n = 1$ we have

$$t^{\frac{N}{2\alpha}} y_{p_1}(t) \leq \|T_0\|_{L^1}^2 \left(\frac{N}{4C\alpha}\right)^{\frac{N}{2\alpha}} = M_{p_1}.$$

By induction, we get

$$\frac{d}{dt}(y_{p_n}) y_p^{-\frac{N+2\alpha}{N}} \leq -2C M_{p_{n-1}}^{-\frac{4\alpha}{N}} t^{2\left(p_{n-1}-1\right)}$$

which implies

$$t^{\frac{N}{2\alpha}(p_n-1)} y_{p_n}(t) \leq M_{p_{n-1}}^2 \left(\frac{N}{4C\alpha}\right)^{\frac{N}{2\alpha}} (p_n - 1)^{\frac{N}{2\alpha}}.$$

So

$$M_{p_n} = \prod_{k=1}^n M_{p_k} \leq \|T_0\|_1^{p_n} \left(\frac{N}{4C\alpha}\right)^{\frac{Nn}{2\alpha}} 2^{\frac{n(n+1)N}{4\alpha}}$$

which implies

$$\lim_{n\to\infty} M_{p_n}^{\frac{1}{p_n}} < \infty.$$

Having obtained the result for $p = 2^n$, we conclude the proof using interpolation for the remaining norms. $\square$

Remark 5.2. This Theorem can be generalized for any initial data $T_0 \in L^p(\mathbb{R}^N) \cap L^{\frac{N}{2\alpha-1}}(\mathbb{R}^N)$ with $1 \le p \le \frac{N}{2\alpha-1}$. Similarly, if the initial data is just $T_0 \in L^{\frac{N}{2\alpha-1}}(\mathbb{R}^N)$, we get

$$t^{1-\frac{1}{2\alpha}-\frac{N}{2p\alpha}}\|T(t)\|_{L^p} \le C\|T_0\|_{L^{\frac{N}{2\alpha-1}}} \tag{5.5}$$

for $\frac{N}{2\alpha-1} \le p \le \infty$.

Proof of Theorem 1.5. The proof is by induction. We assume the result for $\nabla^k T$ and prove a preliminary decay for $\Lambda^h\nabla^k T$ for an appropiately chosen, small enough $h < 1$. Then, as for some integer m we have $hm + j = 1$, with $j < h$, we apply this result m times with Λ^h and then once using Λ^j. The decay for $\nabla^{k+1}T$ then follows.

The case $|k| = 0$ is (1.4). We now prove the result for $\nabla^{k+1}T$. Taking derivatives in (5.1) and using the decay estimates for $\nabla^k T$ and Lemma 3.1

$$\begin{aligned}
\|\Lambda^h\nabla^T(t)\|_{L^p} &\le Ct^{-\frac{|k|+h}{2\alpha}-\frac{N}{2\alpha}\left(1-\frac{1}{p}\right)}\|T_0\|_{L^1} \\
&+ \int_0^{t/2} \Lambda^h\nabla^k\nabla K_\alpha(t-s)*(TH[T])(s) \quad {}_{L^p}ds \\
&+ \int_{t/2}^{t} \Lambda^h\nabla K_\alpha(t-s)*\nabla^k(TH[T])(s) \quad {}_{L^p}ds \\
&\le Ct^{-\frac{|k|+h}{2\alpha}}t^{-\frac{N}{2\alpha}\left(1-\frac{1}{p}\right)}\|T_0\|_{L^1} + I_1 + I_2.
\end{aligned}$$

We note that the operator H commutes with derivatives. First, we estimate I_1 for $1 < p < \infty$. Using the Kernel estimates of Lemma 3.2 and Hölder's inequality, we obtain

$$\begin{aligned}
I_1 &\le C\int_0^{t/2}(t-s)^{-\frac{1+|k|+h}{2\alpha}-\frac{N}{2\alpha}\left(\frac{1}{r}-\frac{1}{p}\right)} \quad TH[T](s) \quad {}_{L^r}ds \\
&\le C\int_0^{t/2}(t-s)^{-\frac{1+|k|+h}{2\alpha}-\frac{N}{2\alpha}\left(\frac{1}{r}-\frac{1}{p}\right)}\|T(s)\|_{L^a} \quad H[T](s) \quad {}_{L^{a'}}ds,
\end{aligned}$$

where$\frac{1}{r} = \frac{1}{a} + \frac{1}{a'}$. Using the estimates of (1.4) and (5.5) with $a \ge \frac{N}{2\alpha-1}$, we obtain

$$I_1 \le C\int_0^{t/2}(t-s)^{-\frac{1+|k|+h}{2\alpha}-\frac{N}{2\alpha}\left(\frac{1}{r}-\frac{1}{p}\right)}s^{-\frac{N}{2\alpha}\left(\frac{2\alpha-1}{N}-\frac{1}{a}\right)}s^{-\frac{N}{2\alpha}\left(1-\frac{1}{a'}\right)}ds$$

$$\leq \quad Ct^{-\frac{1+|k|+h}{2\alpha}-\frac{N}{2\alpha}\left(\frac{1}{r}-\frac{1}{p}\right)}\int_0^{t/2} s^{-1+\frac{1}{2\alpha}-\frac{N}{2\alpha}\left(1-\frac{1}{r}\right)}ds$$

Then, if $r < \frac{N}{N-1}$, we conclude that

$$I_1 \leq Ct^{-\frac{|k|+h}{2\alpha}-\frac{N}{2\alpha}\left(1-\frac{1}{p}\right)}. \tag{5.6}$$

The conditions on r and a are simultaneously satisfied taking $1 < r < \frac{N}{N-1}$.

Now Lemma 3.2 and Hölder's inequality lead to

$$\begin{aligned} I_2 \quad &\leq \quad C\int_{t/2}^{t}(t-s)^{-\frac{1+h}{2\alpha}-\frac{N}{2q\alpha}}\ \|\nabla^k(TH[T])(s)\|_{L^{\frac{qp}{q+p}}}ds \\ &\leq \quad C\int_{t/2}^{t}(t-s)^{-\frac{1+h}{2\alpha}-\frac{N}{2q\alpha}}\Big(\sum_{0\leq i\leq k}\|\nabla^{k-i}T(s)\|_{L^q}\ \|\nabla^i H[T](s)\|_{L^p}\Big)ds. \end{aligned}$$

Using the estimates (1.5) with order less or equal than $|k|$ for $1 < p < \infty$ and (1.5) for $\frac{N}{2\alpha-1} \leq q < \infty$, we get

$$I_2 \quad \leq \quad Ct^{-\frac{|k|}{2\alpha}}t^{-\frac{N(p-1)}{2p\alpha}}t^{-1+\frac{1}{2\alpha}+\frac{N}{2q\alpha}}\int_{t/2}^{t}(t-s)^{-\frac{1+h}{2\alpha}-\frac{N}{2q\alpha}}ds.$$

This integral is bounded if $h < 2\alpha - 1$ because there exists $q \geq 1$ such that the exponent in the integrand is as large as -1. Then, taking $h = \frac{2\alpha-1}{2}$, we obtain

$$I_2 \leq Ct^{-\frac{|k|+h}{2\alpha}}t^{-\frac{N(p-1)}{2p\alpha}}.$$

This, together with (5.6), proves the decay for $\Lambda^h\nabla^k T$.

Now we proceed as in the comments at the beginning of the proof, by applying this result m times with Λ^h and then once using Λ^j, where $hm+j = 1$, with $j < h$ and m an integer. The analysis of the estimates on the iterations is always the same, so we show the method for the second iteration only. We have

$$\begin{aligned} \|\Lambda^{2h}\nabla^k T(t)\|_{L^p} \quad &\leq \quad Ct^{-\frac{|k|+2h}{2\alpha}-\frac{N}{2\alpha}\left(1-\frac{1}{p}\right)}\|T_0\|_{L^1} \\ &+ \quad \int_0^{t/2}\ \|\Lambda^{2h}\nabla^k\nabla K_\alpha(t-s)*(TH[T])(s)\|_{L^p}ds \\ &+ \quad \int_{t/2}^{t}\ \|\Lambda^{h}\nabla K_\alpha(t-s)*\Lambda^h\nabla^k(TH[T])(s)\|_{L^p}ds. \end{aligned}$$

The first integral is estimated as I_1. In the second integral, we use the following general estimate for the operator Λ^s on products of functions (see

[22]), for $s>0$

$$\|\Lambda^s(fg)\|_{L^r} \le C(\|f\|_{L^{q'}}\|\Lambda^s g\|_{L^q}\| + \|g\|_{L^{q'}}\|\Lambda^s f\|_{L^q})$$

with $1<r<q'\le\infty$ and $\frac{1}{r}=\frac{1}{q}+\frac{1}{q'}$. We obtain the right estimate for this second term, working as in the case of I_2 in the estimate for the decay of $\Lambda^h\nabla^k T$.

Now , we address the cases $p=1$ and $p=\infty$. For $p=\infty$, we use the previously proven case $p=\frac{2N}{2\alpha-1}$. Indeed, using the Gagliardo-Niremberg estimates we conclude that

$$\|\nabla^k T\|_{L^\infty} \le C\|\nabla^k T\|^{\frac{3-2\alpha}{2}}_{L^{\frac{2N}{2\alpha-1}}}\|\nabla\nabla^k T\|^{\frac{2\alpha-1}{2}}_{L^{\frac{2N}{2\alpha-1}}},$$

and using (1.5) for k and $k+1$ with $p=\frac{2N}{2\alpha-1}$, we obtain the result. Finally, we prove the case $p=1$. Again, we use the integral (5.1) and decompose it in I_1 and I_2 as before. We get

$$\begin{aligned}\|\nabla^k T(t)\|_{L^1} &\le Ct^{-\frac{|k|}{2\alpha}}\|T_0\|_{L^1} + \int_0^{t/2}(t-s)^{-\frac{1+|k|}{2\alpha}}\ (TH[T])(s)\ _{L^1}ds\\ &+ \int_{t/2}^t (t-s)^{-\frac{1}{2\alpha}}\ \nabla^k(TH[T])(s)\ _1 ds\\ &\le Ct^{-\frac{|k|}{2\alpha}}\|T_0\|_{L^1} + I_1 + I_2.\end{aligned}$$

By using Hölder's inequality in the first integral we obtain

$$I_1 \le Ct^{-\frac{1+|k|}{2\alpha}}\int_0^{t/2}\ T(s)\ _{L^a}\|H[T](s)\|_{L^{a'}}ds,$$

and by (1.4) and (5.5) taking $a\ge\frac{N}{2\alpha-1}$, we get

$$I_1 \le Ct^{-\frac{1+|k|}{2\alpha}}\int_0^{t/2}s^{-1+\frac{1}{2\alpha}}ds \le Ct^{-\frac{|k|}{2\alpha}}.$$

To finish the proof

$$I_2 \le C\sum_{0\le i\le k}\int_{t/2}^t (t-s)^{-\frac{1}{2\alpha}}\|\nabla^{k-i}T(s)\|_{L^a}\ \nabla^i H[T](s)\ _{L^{a'}}ds,$$

and by (1.5) and (5.7), taking again $a\ge\frac{N}{2\alpha-1}$, we conclude that

$$I_2 \le Ct^{-1+\frac{1}{2\alpha}-\frac{|k|}{2\alpha}}\int_{t/2}^t (t-s)^{-1+\frac{1}{2\alpha}}ds \le Ct^{-\frac{|k|}{2\alpha}}. \quad \square$$

Remark 5.3. With an analogous argument, but using $T_0 \in L^{\frac{N}{2\alpha-1}}$, we obtain

$$t^{\frac{|k|}{2\alpha}} t^{1-\frac{1}{2\alpha}-\frac{N}{2p\alpha}} \|T(t)\|_{L^p} \leq C \|T_0\|_{L^{\frac{N}{2\alpha-1}}}, \qquad \frac{N}{2\alpha-1} \leq p \leq \infty. \tag{5.7}$$

Proof of Theorem 1.6, (1). First, we observe that, by estimate (1.5), we have for $t \geq 1$

$$\|\nabla^k T(t)\|_{L^p} \leq C(1+t)^{-\frac{k}{2\alpha}-\frac{N}{2\alpha}\left(1-\frac{1}{p}\right)}, \tag{5.8}$$

where $1 \leq p \leq \infty$. As a consequence of the estimate (3.1), we have for $t \geq 1$

$$\|\nabla^k K_\alpha * T_0(t)\|_{L^p} \leq C(1+t)^{-\frac{k}{2\alpha}-\frac{N}{2\alpha}\left(\frac{1}{r}-\frac{1}{p}\right)} \|T_0\|_{L^r}, \tag{5.9}$$

where $1 \leq r \leq p \leq \infty$. Now, we split the bilinear term into two parts to obtain

$$\begin{aligned} \|\nabla^k T(\cdot,t) - \nabla^k K_\alpha(t) * T_0\|_{L^p} &\leq \int_0^{t/2} \nabla^k \nabla K_\alpha(t-s) * (TH[T])(s) \,{}_{L^p} ds \\ &+ \int_{t/2}^{t} \nabla K_\alpha(t-s) * \nabla^k (TH[T])(s) \,{}_{L^p} ds \\ &\leq I_1 + I_2. \end{aligned}$$

First, we estimate I_2 as in the proof of Theorem 1.5. Note that by (5.9), we have

$$I_2 \leq C \sum_{0 \leq i \leq k} \int_{t/2}^{t} (1+t-s)^{-\frac{1}{2\alpha}-\frac{N}{2\alpha q}} \|\nabla^{k-i} T(s)\|_{L^a} \;\nabla^i H[T](s) \,{}_{L^r} ds,$$

with $\frac{1}{a} + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, and $0 \leq \frac{1}{q} \leq 1 - \frac{1}{p}$. Using (5.8), we obtain

$$I_2 \leq C(1+t)^{-\frac{|k|}{2\alpha}-\frac{N}{2\alpha}\left(2-\frac{1}{p}-\frac{1}{q}\right)} \int_{t/2}^{t} (1+t-s)^{-\frac{1}{2\alpha}-\frac{N}{2\alpha q}} ds,$$

and, taking $q > \frac{N}{2\alpha-1}$ and $q \geq \frac{p}{p-1}$,

$$I_2 \leq C(1+t)^{-\frac{|k|+1}{2\alpha}-\frac{N}{2\alpha}\left(1-\frac{1}{p}\right)} (1+t)^{1-\frac{N}{2\alpha}}. \tag{5.10}$$

Now, we estimate I_1 for $1 < p < \infty$. Using (5.9) and Hölder's inequality, we get

$$I_1 \quad \leq \quad C \int_0^{t/2} (1+t-s)^{-\frac{1+|k|}{2\alpha}-\frac{N}{2\alpha}\left(1-\frac{1}{p}\right)} \|T(s)\|_{L^a} \;H[T](s) \,{}_{L^{a'}} ds,$$

where $1 = \frac{1}{a} + \frac{1}{a'}$. Using the estimate (5.8), we obtain

$$I_1 \quad \leq \quad C(1+t)^{-\frac{1+|k|}{2\alpha}-\frac{N}{2\alpha}\left(1-\frac{1}{p}\right)} \int_0^{t/2} (1+s)^{-\frac{N}{2\alpha}} ds. \tag{5.11}$$

This last integral is bounded if $N > 2$ or $\alpha < 1$, and with (5.10) we conclude the proof for these cases. □

Proof of Theorem 1.6, (2). Using the same steps as in the proof of Theorem 1.6 (1), we have by (5.10) for $\alpha = 1$ and $N = 2$

$$I_1 \leq Ct^{-\frac{|k|+1}{2}-1+\frac{1}{p}}.$$

Now, using the estimate (5.11) for $\alpha = 1$ and $N = 2$, we conclude. □

In order to obtain the optimal decay in the region $(0, t/2)$ as in $(t/2, t)$, we consider a corrector of $K(t) * T_0$. Let be T_1 the solution of the non-homogeneous heat equation

$$\frac{\partial T_1}{\partial t} + \big(H[K(t) * T_0] \cdot \nabla\big) K(t) * T_0 - \Delta T_1 = 0$$
$$T_1(x, 0) = T_0(x), \quad x \in \mathbb{R}^2$$

where the operator H is defined in (2.1) and K is the kernel of the heat equation in $\mathbb{R}^2$. Thus, in the integral sense:

$$T_1(t) = K(t) * T_0 - \int_0^t \nabla K(t-s) * (K(s) * T_0 H[K(s) * T_0]) ds. \tag{5.12}$$

Proof of Theorem 1.6, (3). Using the integral formulas (5.1) and (5.12), we have

$$\begin{aligned} \|\nabla^k T(\cdot,t) - \nabla^k T_1(\cdot,t)\|_{L^p} \leq & \int_{t/2}^t \big\| \nabla K(t-s) * \nabla^k (TH[T])(s) \big\|_{L^p} ds \\ & + \int_{t/2}^t \big\| \nabla K(t-s) * \nabla^k (K(s) * T_0 H[K(s) * T_0])(s) \big\|_{L^p} ds \\ & + \int_0^{t/2} \big\| \nabla^k \nabla K(t-s) * [TH[T](s) - K(s) * T_0 H(K(s) * T_0)] \big\|_{L^p} ds \\ & \leq I_2 + J_2 + J_1. \end{aligned}$$

We note that by (5.10) with $\alpha = 1$ and $N = 2$, we obtain the decay estimate of the term I_2. Now, we estimate J_2 as I_2. Note that by (5.9), we have

$$J_2 \leq C \sum_{0 \leq i \leq k} \int_{t/2}^t (1+t-s)^{-\frac{1}{2}-\frac{1}{q}} \|\nabla^{k-i} K * T_0(s)\|_{L^a} \big\| \nabla^i H[K * T_0](s) \big\|_{L^r} ds,$$

with $\frac{1}{a}+\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$, and $0\le\frac{1}{q}\le 1-\frac{1}{p}$. Using (5.9) again, we obtain

$$J_2 \le C(1+t)^{-\frac{|k|}{2}-2+\frac{1}{p}+\frac{1}{q}}\int_{t/2}^{t}(1+t-s)^{-\frac{1}{2}-\frac{1}{q}}ds,$$

and, taking $q>2$ and $q\ge\frac{p}{p-1}$, we prove

$$J_2 \le C(1+t)^{-\frac{|k|+1}{2}-1+\frac{1}{p}}.$$

Finally, we estimate J_1. Using (5.9), we get

$$J_1 \quad\le\quad C(1+t)^{-\frac{1+|k|}{2}-1+\frac{1}{p}}\int_0^{t/2}\|TH[T](s)-K(s)*T_0H(K(s)*T_0)\|_{L^1}ds$$

We proved that the integral is uniformly bounded. We note that

$$\begin{aligned}&\|TH[T](s)-K(s)*T_0H(K(s)*T_0)\|_{L^1}\\ &\quad\le\|(T-K(s)*T_0)H[T](s)\|_{L^1}+\|K(s)*T_0H(T(s)-K(s)*T_0)\|_{L^1}\\ &\qquad\qquad\le C\|T(s)-K(s)*T_0\|_{L^a}(\|T(s)\|_{L^{a'}}+\|K(s)*T_0\|_{L^{a'}}),\end{aligned}$$

where we use the Hölder's inequality with $1=\frac{1}{a}+\frac{1}{a'}$. Now, we use the estimates(5.8) and (5.9), and Theorem 1.6 (2), to obtain

$$J_1\le C(1+t)^{-\frac{1+|k|}{2}-1+\frac{1}{p}}\int_0^{t/2}(1+s)^{-\frac{3}{2}}\ln(1+s)ds\le C(1+t)^{-\frac{1+|k|}{2}-1+\frac{1}{p}},$$

and we conclude the proof. □

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