Noncommutative analysis techniques in the geometry of L_p spaces and Calderón-Zygmund theory

Antonio Ismael Cano Mármol

Tesis Doctoral dirigida por
José Manuel Conde Alonso y Javier Parcet Hernández

Programa de Doctorado en Matemáticas
Departamento de Matemáticas
Universidad Autónoma de Madrid

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Répétons que le rythme profond, vrai et vivant de la musique, le “feeling” disent les Américains, ne saurait être donné uniquement par le métronome. Cette vie que procure la pulsation demeure toutefois tributaire de l’exactitude du rythme et du tempo.

Jean-Marie Londeix

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Contents

Resumen

Los contenidos de este trabajo se engloban dentro del área del análisis armónico no conmutativo. Una característica central de este campo de investigación es la sustitución de funciones sobre espacios de medida por operadores en espacios de Hilbert. Más concretamente, sea (Ω, μ) un espacio de medida semifinito y consideremos los espacios de Lebesgue $L_p(\Omega, \mu)$ para $0 < p \leq \infty$. Entonces, $L_2(\Omega, \mu)$ es un espacio de Hilbert complejo con el producto interior dado por la integral, mientras que el espacio de funciones medibles esencialmente acotadas $L_\infty(\Omega, \mu)$ puede interpretarse como una subálgebra de operadores acotados sobre $L_2(\Omega, \mu)$. En otras palabras, cualquier $f \in L_\infty(\Omega, \mu)$ induce una aplicación lineal acotada

$$T_f : \begin{array}{ccc} L_2(\Omega, \mu) & \longrightarrow & L_2(\Omega, \mu) \\ g & \longmapsto & fg \end{array}$$

con norma $\|T_f\| = \|f\|_\infty$ y la correspondencia $f \mapsto T_f$ es biyectiva. Sea $\mathcal{H} = L_2(\Omega, \mu)$ y sea $B(\mathcal{H})$ el álgebra de operadores lineales acotados sobre $\mathcal{H}$. Entonces la familia de operadores T_f es un *álgebra de von Neumann*, es decir, una C*-subálgebra de $B(\mathcal{H})$ que contiene a la identidad y es cerrada con respecto a la topología débil de operadores de $B(\mathcal{H})$. Cuando un álgebra de von Neumann $\mathcal{M}$ está equipada con una *traza* τ, un funcional lineal que juega el papel de "integral no conmutativa", decimos que el par $(\mathcal{M}, \tau)$ es un *espacio de medida no conmutativo*. Además, esto conduce a la definición, via cálculo funcional espectral y un argumento de completación, de los *espacios L_p no conmutativos* $L_p(\mathcal{M}, \tau)$ equipados respectivamente con las normas

$$\|x\|_p = \tau(|x|^p)^{1/p}.$$

El ejemplo $\mathcal{M} = L_\infty(\Omega, \mu)$ puede ser provisto de la traza dada por la integral, es decir,

$$\tau(f) = \int_\Omega f \, d\mu,$$

de forma que los espacios L_p clásicos son espacios L_p no conmutativos. A lo largo de esta tesis aparecerán varios ejemplos de álgebras de von Neumann como los contextos donde se han estudiado los dos problemas principales de este trabajo.

La primera parte de esta disertación está constituida por los capítulos 1 a 3. El punto en común de los resultados incluidos en estos capítulos es la aplicabilidad de la teoría de

funciones en el cubo de Hamming $\{-1,1\}^n$ a la geometría de espacios de Banach y la teoría de inclusiones de espacios de Banach.

Una pregunta fundamental en análisis funcional es conocer cuándo un espacio dado es isomorfo a un subespacio vectorial de otro. En el caso de los espacios $L_p(0,1)$ el panorama es bien conocido. $L_2(0,1)$ es isomorfo a un subespacio de $L_p(0,1)$ para todo p en el rango de Banach, pero no existe un embedding lineal de $L_q(0,1)$ en $L_p(0,1)$ cuando $q < \min\{2,p\}$ o $q > \max\{2,p\}$. Banach [1] conjecturó una respuesta positiva para el caso $\min\{2,p\} < q < \max\{2,p\}$. Kadec lo demostró para $p < q < 2$ en [47], mientras que Paley lo refutó para $2 < q < p$ en [65].

Nuestro trabajo está inspirado por un resultado de Naor [60] sobre la imposibilidad de un embedding en la categoría de espacios métricos del espacio de Lebesgue $L_q(0,1)$ en $L_p(0,1)$ siempre que q y p pertenezcan al rango refutado por Paley, $2 < q < p$. La no existencia de una aplicación de espacios métricos $L_q(0,1) \hookrightarrow L_p(0,1)$ es conocida desde los años setenta [50] por reducción a la teoría lineal, que aprovecha la diferenciabilidad —de aplicaciones Lipschitz— para reducir el enunciado métrico a uno lineal. Sin embargo, el enfoque propuesto por Naor y Schechtman [63] proporciona nuevos resultados que no pueden ser obtenidos a través de la teoría lineal. Consúltese la Introducción en ese trabajo para una descripción más detallada del contexto y referencias sobre la historia del problema y conexiones con otras áreas.

Nuestro interés en el trabajo de Naor y Schechtman se ve reforzado por el hecho de que este depende fuertemente del análisis armónico en el cubo de Hamming. Sea Ω_n el hipercubo $n-$dimensional $\{-1,1\} \times \{-1,1\} \times \ldots \times \{-1,1\}$, equipado con la medida de contar normalizada. Si $[n] = \{1,\ldots,n\}$, toda función admite una expansión de Fourier-Walsh [64], en otras palabras, satisface la identidad

$$f(\varepsilon) = \sum_{\mathsf{A}\subseteq[n]} \widehat{f}(\mathsf{A})\, W_{\mathsf{A}}(\varepsilon), \quad \text{donde} \quad W_{\mathsf{A}}(\varepsilon) = \prod_{j\in A} \varepsilon_j.$$

Dada una función f de media cero, Naor demostró en [60] la *desigualdad* X_p *para caos de Rademacher*: para todo $p \geq 2$ y $k \in [n]$

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subset[n]\\ |\mathsf{S}|=k}} \Big\| \sum_{\mathsf{A}\subset \mathsf{S}} \widehat{f}(\mathsf{A}) W_{\mathsf{A}} \Big\|^p_{L_p(\Omega_n)} \lesssim_p \frac{k}{n} \sum_{j=1}^n \|\partial_j f\|^p_{L_p(\Omega_n)} + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|^p_{L_p(\Omega_n)}, \tag{1}$$

donde $\partial_j f(\varepsilon) = f(\varepsilon) - f(\varepsilon - 2\varepsilon_j e_j)$. Esta desigualdad tiene aplicaciones extremadamente novedosas en geometría métrica. Más precisamente, implica la forma cuantitativamente óptima de la *desigualdad métrica* X_p para $L_p(0,1)$ [63]. En consecuencia, esto da un criterio puramente métrico de estimar una cota inferior para la distorsión L_p de un espacio métrico X. Su naturaleza métrica es extremadamente útil a la hora de resolver problemas no lineales acerca de la imposibilidad de embeber L_q en L_p para $2 < q < p$. Esto incluye, más allá de las posibilidades de la teoría lineal de inclusiones de espacios L_p, la distorsión óptima L_p de parrillas (no lineales) de ℓ_q^n o el exponente crítico L_p de un *snowflake* de L_q.

En conclusión, la desigualdad de Naor (1) y las consecuentes desigualdades X_p métricas con exponente de escalado óptimo son una contribución clave para el programa de Ribe, un esfuerzo en identificar qué propiedades de la teoría local de espacios de Banach dependen de realidad de consideraciones puramente métricas y no de la estructura lineal del espacio. Este objetivo de investigación fue iniciado tras [71] y explícitamente formulado en [3]. Consultar [59] para un visión de conjunto de este tema.

En el capítulo 1, presentamos una generalización cuántica de la desigualdad (1) que depende fuertemente de técnicas de análisis armónico no conmutativo. Aquí, las álgebras de von Neumann de grupo son el marco de referencia adecuado para nuestro objetivo. Dado un grupo discreto G, se puede asociar un operador acotado $\lambda(g) \in B(\ell_2(\mathrm{G}))$ a cada elemento $g \in G$. El *álgebra de von Neumann de grupo* $\mathcal{L}(\mathrm{G})$ se define como la clausura en la topología débil* de sumas finitas de la forma

$$f = \sum_{g \in \mathrm{G}} \widehat{f}(g) \lambda(g).$$

Cuando G es conmutativo, $\lambda(g)$ juega el papel de un caracter

$$\chi_g : \widehat{\mathrm{G}} \to \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\},$$

obteniéndose la expresión familiar $\mathcal{L}(\mathrm{G}) \simeq L_\infty(\widehat{\mathrm{G}})$, donde $\widehat{\mathrm{G}}$ es el dual de Pontryagin de G (ver [19, 74]). Por ejemplo, el espacio de funciones acotadas sobre el cubo de Hamming Ω_n puede ser identificado con $\mathcal{L}(\mathbb{Z}_2^n)$. Encontrar una versión adecuada de la desigualdad (2) en $\mathcal{L}(\mathrm{G})$ supone varias dificultades. Por ejemplo, un grupo discreto no suele estar provisto de una estructura diferencial canónica, pero esto se puede resolver con una representación apropiada G en un espacio de Hilbert, que ya posee una estructura de este tipo [40]. Más concretamente, un *cociclo ortogonal a la izquierda* $(\mathcal{H}, \alpha, \beta)$ en G está dado por una acción ortogonal $\alpha : \mathrm{G} \curvearrowright \mathcal{H}$ sobre algún espacio de Hilbert real $\mathcal{H}$ y una aplicación $\beta : \mathrm{G} \to \mathcal{H}$ satisfaciendo la relación $\alpha_g(\beta(h)) = \beta(gh) - \beta(g)$. Estas y otras dificultes son estudiadas a lo largo del capítulo, y la resolución de las mismas dan lugar a una generalización de (1). Entre los ejemplos resultantes, destacamos el productor directo de grupos: el espacio de Hilbert del cociclo asociado al producto será el producto de los espacios de Hilbert asociados a cada componente. También, se han obtenido varias aplicaciones para productos libres de grupos, así como una versión particular en el toro $\mathbb{T}^n$ y otros ejemplos de productos finitos de grupos abelianos cíclicos y grupos de Coxeter.

El capítulo 2 contiene algunas generalizaciones de la *desigualdad métrica* X_p. El resultado original de Naor y Schechtman [63] fue dado para funciones en $\Omega_n \times \mathbb{Z}_{8m}^n$ con valores en $L_p(0,1)$, por lo que nuestra primera contribución es estudiar cómo reemplazar este par por otros grupos (H, G), donde H es un grupo discreto abeliano y G puede ser no-abeliano. El argumento se sigue para desigualdades para caos en H y algunas relaciones de compatibilidad entre H y G. Aquí, el análisis armónico aparece otra vez como la forma de codificar una de estas condiciones, y proporciona una noción de traslación en el álgebra de von Neumann de grupo $\mathcal{L}(\mathrm{G})$ a través de multiplicadores semiconmutativos.

A lo largo de la segunda parte del capítulo 2, obtenemos que todo espacio L_p no conmutativo satisface la desigualdad X_p métrica para todo $p > 2$, dando lugar a resultados de no-embedabilidad para estos espacios, análogos a los clásicos.

El capítulo 3 complementa a los resultados obtenidos en los capítulos 1 y 2. En primer lugar, se introduce una versión de la desigualdad (1), proporcionando un enfoque alternativo a la desigualdad X_p métrica para espacios L_p no conmutativos que ya fue presentado en el capítulo 2. Por otro lado, estudiamos una formulación en términos de sistemas spin de la *dimension free Pisier's inequality* [33], que puede ser codificada a través de la teoría de espacios de operadores. Esta desigualdad fue originalmente enunciada para funciones $f : \Omega_n \to \mathbb{X}$ para todo espacio de Banach $\mathbb{X}$, y supone la solución de un problema longevo en geometría métrica de espacios de Banach: el tipo de Rademacher coincide con el tipo de Enflo. A pesar de que nuestra generalización no ha dado lugar a ninguna aplicación todavía, nos sugiere que sería interesante estudiar si una teoría análoga podría ser desarrollada en el contexto de los sistemas spin y los espacios de operadores.

La segunda parte de esta tesis está relacionada con la extensión de la teoría de Calderón-Zygmund al contexto de las funciones con valores en matrices. En el contexto clásico, dado un kernel $K(x, y)$ definido en $\mathbb{R}^n \times \mathbb{R}^n \setminus \{x = y\}$, la integral singular asociada a K es un operador T dado por la expresión

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy$$

bajo ciertas hipótesis. Si además T está acotado en $L_2(\mathbb{R}^n)$ y K satisface ciertas condiciones de suavidad, entonces decimos que T es un *operador de Calderón-Zygmund*.

Nuestra contribución en el capítulo 4 está relacionado con la extensión de esta teoría al contexto de funciones con valores en operadores. Dada un álgebra de von Neumann $\mathcal{M}$ equipada con una traza τ, denotemos por $\mathcal{A}$ a la clausura en la topología débil de operadores de las funciones esencialmente acotadas $f : \mathbb{R}^n \to \mathcal{M}$, que se pueden identificar con el producto tensorial $L_\infty(\mathbb{R}^n)\overline{\otimes}\mathcal{M}$ equipado con la traza

$$\varphi(f) = \int_{\mathbb{R}^n} \tau(f(x)) dx.$$

La teoría de operadores de Calderón-Zygmund puede ser trasladada a este contexto. Siempre que $1 < p < \infty$, la acotación de $L_p(\mathcal{A})$ en $L_p(\mathcal{A})$ se puede reducir al caso de funciones con valores en espacios de Banach que satisfacen la *unconditional martingale property* (UMD), como demostró Figiel [18]. Mi investigación en esta dirección ha tenido lugar en el estudio de un análogo semiconmutativo del espacio de Hardy $H_1(\mathbb{R}^n)$ y la acotación de operadores de Calderón-Zygmund de ese espacio en $L_1(\mathcal{A})$.

La forma semiconmutativa del espacio de Hardy fue examinada en profundidad en la tesis doctoral de Mei [54] a través de algunas caracterizaciones basadas en la integral de Lusin y la $g-$función de Littlewood-Paley, un enfoque análogo a [17] en el contexto clásico.

Debido a fenómenos no conmutativos, el espacio de Hardy semiconmutativo $H_1(\mathcal{A})$ debe ser considerado como la suma de un espacio columna y un espacio fila, es decir,

$$H_1(\mathcal{A}) = H_1^c(\mathcal{A}) + H_1^r(\mathcal{A}).$$

El trabajo en este capítulo se concentra en estudiar en detalle una descomposición atómica para $H_1^c(\mathcal{A})$ a través de lo que llamamos *c-átomos*. Una función $a \in L_1(\mathcal{A})$ es un *c-átomo* si admite una descomposición $a = bh$ para cierto h en la bola unidad de $L_2(\mathcal{M})$ y cierta función b en $L_2(\mathcal{A})$ satisfaciendo

$$\int_{\mathbb{R}^n} b = 0, \quad \mathrm{supp}_{\mathbb{R}^n}(b) \subseteq B, \quad \Big(\int \|b(x)\|^2_{L_2(\mathcal{M})} \, dx \Big)^{1/2} \leq \frac{1}{\sqrt{|B|}}$$

para alguna bola B. Entonces, $H_1^c(\mathcal{A})$ es el espacio de Banach

$$\Big\{ \sum_i \lambda_i a_i \ : \ (\lambda_i)_i \in \ell_1, \ (a_i)_i \ c - \text{atoms} \Big\}$$

equipado con la norma

$$\|f\|_{H_1^c} = \inf \Big\{ \sum_i |\lambda_i| \ : \ f = \sum_i \lambda_i a_i, \ (\lambda_i)_i \in \ell_1, \ (a_i)_i \ c - \text{atoms} \Big\}.$$

El capítulo está organizado como sigue. Primero, se presenta una descripción en detalle de los espacios BMO columna/fila utilizando espacios L_p no conmutativos con valores en espacios de Hilbert. Luego, se justifica la dualidad $H_1^c(\mathcal{A})^* = \mathsf{BMO}_r(\mathbb{R}, \mathcal{M})$ de forma que se obtiene

$$H_1(\mathcal{A})^* = \mathsf{BMO}(\mathbb{R}, \mathcal{M}) = \mathsf{BMO}_r(\mathbb{R}, \mathcal{M}) \cap \mathsf{BMO}_c(\mathbb{R}, \mathcal{M}).$$

Para lograr esto, nos basamos en la construcción del espacio de Hardy H_1 con valores en $L_2(\mathcal{M})$ (ver Apéndice A) y en otras propiedades del espacio $\mathsf{BMO}(\mathbb{R}^n)$ que pueden ser traducidas al contexto semiconmutativo (ver Sección 4.3).

La descomposición atómica de $H_1^c(\mathcal{A})$ es la clave que nos permite dar una nueva prueba de la acotación de operadores de Calderón-Zygmund de $H_1^c(\mathcal{A})$ en $L_1(\mathcal{A})$ con kernel escalar $K(x,y)$. Nuestra estrategia está inspirada por el enfoque seguido por Meyer y Coifman [57]. Primero, el kernel K asociado al operador T es aproximado explícitamente por una sucesión de kernels uniformemente acotados $(K_m)_{m=1}^\infty$. Entonces, si T_m es el operador de Calderón-Zygmund asociado a K_m, se tiene que

$$\|T_m(a)\|_{L_1(\mathcal{M})} \leq C$$

para todo $m \geq 1$, cualquier $c-$átomo a y una constante universal C. Cabe recalcar que T_m está bien definido sobre $c-$átomos, independientemente de la descomposición $a = bh$ para cada $c-$átomo a. Puesto que K_m está acotado, la extensión a todo el espacio de Hardy $H_1^c(\mathcal{A})$ se sigue trivialmente, y por aproximación, se sigue que T está acotado de

$H_1^c(\mathcal{A})$ en $L_1(\mathcal{A})$. Este argumento se extiende a un nuevo resultado para operadores de Calderón-Zygmund asociados a kernels K con valores en un álgebra de von Neumann $\mathcal{M}$.

La mayoría de resultados de esta tesis están contenidos en los siguientes artículos de investigación. En concreto, los capítulos 1 y 2 se corresponden con los dos primeros papers, mientras que el tercer artículo ha dado lugar al capítulo 4:

- A.I. Cano-Mármol, J.M. Conde-Alonso, and J. Parcet. Trigonometric chaos and X_p inequalities I: Balanced Fourier truncations over discrete groups. Submitted.
- A.I. Cano-Mármol, J.M. Conde-Alonso, and J. Parcet. Trigonometric chaos and X_p inequalities II: X_p inequalities with sharp scaling parameter. Submitted.
- A.I. Cano-Mármol, É. Ricard. Calderón-Zygmund theory with noncommuting kernels via H_1. Preprint.

A pesar de que no incluiremos un capítulo introduciendo los conceptos propios del análisis armónico no conmutativo necesarios para entender los resultados de esta tesis, estas herramientas serán introducidas conforme sea preciso. De esto modo, la longitud de este trabajo quedará dentro de unos límites razonables, y será tan autocontenido como sea posible.

Abstract

The contents of this dissertation can be encompassed within the area of noncommutative harmonic analysis. A central feature of this research field is the substitution of functions defined over measure spaces by operators acting on Hilbert spaces. More clearly, let (Ω, μ) be a semifinite measure space and consider the associated Lebesgue spaces $L_p(\Omega, \mu)$ for $0 < p \leq \infty$. Then, $L_2(\Omega, \mu)$ is a complex Hilbert space with inner product given by the integral, while the space of essentially bounded measurable functions $L_\infty(\Omega, \mu)$ can be interpreted as a subalgebra of bounded operators on $L_2(\Omega, \mu)$. In other words, any $f \in L_\infty(\Omega, \mu)$ induces a bounded linear map

$$T_f : \begin{array}{ccc} L_2(\Omega, \mu) & \longrightarrow & L_2(\Omega, \mu) \\ g & \longmapsto & fg \end{array}$$

with norm $\|T_f\| = \|f\|_\infty$ and the correspondence $f \mapsto T_f$ is a bijection. Let $\mathcal{H} = L_2(\Omega, \mu)$ and let $B(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. Then the family of operators T_f is a von Neumann algebra, that is, a C*-subalgebra of $B(\mathcal{H})$ which contains the identity and is closed with respect to the weak operator topology of $B(\mathcal{H})$. When a von Neumann algebra $\mathcal{M}$ is equipped with a *trace* τ, a linear functional which plays the role of the "noncommutative integral", the pair $(\mathcal{M}, \tau)$ is a *noncommutative measure space*. Moreover, this leads to the definition, via spectral functional calculus and a completion argument, of the *noncommutative L_p spaces* $L_p(\mathcal{M}, \tau)$ equipped with the norms

$$\|x\|_p = \tau(|x|^p)^{1/p}.$$

The example $\mathcal{M} = L_\infty(\Omega, \mu)$ can be provided with the trace given by the integral, that is,

$$\tau(f) = \int f \, d\mu,$$

so that classical L_p spaces are examples of noncommutative L_p spaces. Several examples of von Neumann algebras will appear through this thesis as the frameworks where the two main problems of this work have been studied.

The first part of this dissertation is constituted by chapters 1 to 3. The meeting point of the results included there is the applicability of the theory of functions on the Hamming cube $\{-1, 1\}^n$ to the geometry of Banach spaces and Banach space embedding theory.

In functional analysis, it is a fundamental question to know when a given space is isomorphic to a linear subspace of another. For the important example of $L_p(0,1)$ and $L_q(0,1)$ the landscape is well known. $L_2(0,1)$ is isomorphic to a subspace of $L_p(0,1)$ for all p in the Banach range, but there is no linear embedding from $L_q(0,1)$ to $L_p(0,1)$ if either $q < \min\{2,p\}$ or $q > \max\{2,p\}$. Banach conjectured a positive answer for $\min\{2,p\} < q < \max\{2,p\}$. Kadec proved it for $p < q < 2$ in [47], while Paley disproved it for $2 < q < p$ in [65].

Our work finds its inspiration in a result by Naor [60] about the nonembeddability as a metric space of the Lebesgue space $L_q(0,1)$ into $L_p(0,1)$ whenever q and p belong to the range disproved by Paley, $2 < q < p$. The non-existence of such a map between metric spaces $L_q(0,1) \hookrightarrow L_p(0,1)$ has been known since the seventies [50] by reduction to the linear theory, since it goes through differentiability —of Lipschitz maps— to reduce the metric statement to a linear one. However, the approach proposed by Naor and Schechtman [63] provides new results that cannot be attained through the linear theory. One can consult the Introduction in that work for a more detailed context and references regarding the history of the problem and its connections with other areas.

Our interest in the work by Naor and Schechtman is reinforced by the fact that it strongly relies on harmonic analysis on the Hamming cube. Let Ω_n be the n-hypercube $\{-1,1\} \times \{-1,1\} \times \cdots \times \{-1,1\}$ equipped with its normalized counting measure. If $[n] := \{1,2,\ldots,n\}$, every function $f : \Omega_n \to \mathbb{C}$ admits a Fourier-Walsh expansion [64], in other words, it satisfies the identity

$$f(\varepsilon) = \sum_{\mathsf{A} \subseteq [n]} \widehat{f}(\mathsf{A}) W_{\mathsf{A}}(\varepsilon), \quad \text{where} \quad W_{\mathsf{A}}(\varepsilon) = \prod_{j \in A} \varepsilon_j.$$

Given a mean-zero f, Naor proved in [60] the X_p *inequality for Rademacher chaos*: for each $p \geq 2$ and $k \in [n]$

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subset [n] \\ |\mathsf{S}| = k}} \Big\| \sum_{\mathsf{A} \subset \mathsf{S}} \widehat{f}(\mathsf{A}) W_{\mathsf{A}} \Big\|_{L_p(\Omega_n)}^p \lesssim_p \frac{k}{n} \sum_{j=1}^n \|\partial_j f\|_{L_p(\Omega_n)}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_{L_p(\Omega_n)}^p, \tag{2}$$

where $\partial_j f(\varepsilon) = f(\varepsilon) - f(\varepsilon - 2\varepsilon_j e_j)$. This inequality has groundbreaking applications in metric geometry. More precisely, it implies the quantitatively optimal form of the so-called *metric* X_p *inequality* for $L_p(0,1)$ [63]. In turn, this gives a purely metric criterion to estimate the L_p-distortion of a metric space X from below. Its metric nature is extremely useful in solving nonlinear problems around the nonembedability of L_q into L_p for $2 < q < p$. This includes, beyond the scope of linear L_p-embedding theory, the optimal L_p-distortion of (nonlinear) grids in ℓ_q^n or the critical L_p snowflake exponent of L_q. In conclusion, Naor's differential inequality (2) and subsequent X_p inequalities with sharp scaling parameter are a key contribution to the Ribe program, an effort to identify which properties from the local theory of Banach spaces ultimately rely on purely metric considerations and not on the whole strength of the linear structure of the space. This research goal was initiated after [71] and explicitly formulated in [3]. See [59] for an overview on this topic.

In chapter 1, we present a quantum generalization of the inequality (2) which strongly relies on noncommutative Fourier analysis. Here, group von Neumann algebras are the right framework for our purpose. Given a discrete group G, one can associate a bounded operator $\lambda(g) \in B(\ell_2(\mathrm{G}))$ to each $g \in \mathrm{G}$. Then, the *group von Neumann algebra* $\mathcal{L}(\mathrm{G})$ is defined as the w*-closure of finite sums of the form

$$f = \sum_{g \in \mathrm{G}} \widehat{f}(g) \lambda(g).$$

Whenever G is commutative, $\lambda(g)$ plays the role of a character

$$\chi_g : \widehat{\mathrm{G}} \to \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$$

and one gets the familiar expression $\mathcal{L}(\mathrm{G}) \simeq L_\infty(\widehat{\mathrm{G}})$, where $\widehat{\mathrm{G}}$ is the Pontryagin dual of G (see [19, 74]). For instance, the space of bounded functions on the Hamming cube Ω_n can be identified with $\mathcal{L}(\mathbb{Z}_2^n)$. Finding a suitable version of the inequality (2) on $\mathcal{L}(\mathrm{G})$ for an arbitrary group G encounters several difficulties. For instance, general discrete groups fail to admit canonical differential structures, but this can be solved with an appropriate representation of G into a Hilbert space, which already carries such a structure [40]. Indeed, an orthogonal *left cocycle* $(\mathcal{H}, \alpha, \beta)$ on G is given by an orthogonal action $\alpha : \mathrm{G} \curvearrowright \mathcal{H}$ into some $\mathbb{R}-$Hilbert space $\mathcal{H}$ and a map $\beta : \mathrm{G} \to \mathcal{H}$ satisfying the relation $\alpha_g(\beta(h)) = \beta(gh) - \beta(g)$. These and other difficulties are studied along this chapter, and their solutions give rise to a wide generalization of (2). Among the resulting examples, we highlight the direct product of groups: the cocycle Hilbert space of the product is the product of the Hilbert space corresponding to each component. Also, several applications for free products of groups have been obtained, as well as a particular version on the torus $\mathbb{T}^n$ and other examples on finite products of abelian cyclic groups and Coxeter groups.

Chapter 2 contains several generalizations of the *metric* X_p *inequality*. The original result by Naor and Schechtman [63] was given for functions on $\Omega_n \times \mathbb{Z}_{8m}^n$ with values in $L_p(0,1)$, so our first contribution is studying how to replace this pair by other pairs of groups (H, G), where H is a discrete abelian group but G is allowed to be nonabelian. The argument follows from X_p inequalities for chaos in H and some compatibility relations between H and G. At this point, harmonic analysis appears again as the way to encode one of this conditions, while providing a notion of translations in the group von Neumann algebra $\mathcal{L}(\mathrm{G})$ via semicommutative multipliers.

Along the second part of chapter 2, we obtain that any noncommutative L_p space satisfies the metric X_p inequality for each $p > 2$, yielding nonembeddability results for these spaces analogous to the classical ones.

Chapter 3 complements the results obtained in chapter 1 and 2. First, a version of inequality (2) for spin systems is introduced, providing an alternative approach for the metric X_p inequality for $L_p(\mathcal{M})$ which was presented in chapter 2. Second, we study a formulation in terms of spin systems of the dimension-free Pisier's inequality [33], which can be encoded through the theory of operator spaces. This inequality was originally stated for functions

$f : \Omega_n \to \mathbb{X}$ for any Banach space $\mathbb{X}$, and it supposes the solution to a long-standing problem in the metric geometry of Banach spaces: Rademacher type and Enflo type coincide. Although our generalization has not yielded any application yet, it suggests that it would be interesting to study whether an analogous theory could be developed in the context of spin systems and operator spaces.

The second part of this thesis is related to the extension of Calderón-Zygmund theory to the context of matrix-valued functions. In the classical setting, given a kernel $K(x, y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{x = y\}$, the singular integral associated to K is an operator T admitting the expression

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy$$

under some suitable hypotheses. If, in addition, T is bounded on $L_2(\mathbb{R}^n)$ and K satisfies certain smoothness conditions, then T is classified as a *Calderón-Zygmund operator*.

Our contribution in chapter 4 is concerned with the extension of this theory to the context of operator-valued functions. Given a von Neumann algebra $\mathcal{M}$ equipped with a trace τ, denote by $\mathcal{A}$ the weak operator closure of essentially bounded functions $f : \mathbb{R}^n \to \mathcal{M}$, which can be identified as the tensor product $L_\infty(\mathbb{R}^n)\overline{\otimes}\mathcal{M}$ equipped with the trace

$$\varphi(f) = \int_{\mathbb{R}^n} \tau(f(x)) dx.$$

The theory of Calderón-Zygmund operators can also be translated to this framework. Whenever $1 < p < \infty$, the boundedness from $L_p(\mathcal{A})$ to $L_p(\mathcal{A})$ can be reduced to the case of functions with values on Banach spaces satisfying the unconditional martingale property (UMD), as showed by Figiel [18]. My research along this direction has taken place in the study of the semicommutative analogue of the Hardy space $H_1(\mathbb{R}^n)$ and the boundedness of Calderón-Zygmund operators from that space into $L_1(\mathcal{A})$.

The semicommutative form of the Hardy space was deeply examined in Mei's Ph.D. thesis [54] via characterizations relying on Lusin's integral and the Littlewood-Paley $g-$function, an approach analogous to the one [17] for the classical setting. Due to well-understood noncommutative phenomena, the semicommutative Hardy space $H_1(\mathcal{A})$ must be considered as the sum of certain column and row spaces, that is,

$$H_1(\mathcal{A}) = H_1^c(\mathcal{A}) + H_1^r(\mathcal{A}).$$

The work of this chapter focuses on studying in detail an atomic decomposition for $H_1^c(\mathcal{A})$ via what we call *c-atoms*. A function $a \in L_1(\mathcal{A})$ is a *c-atom* whenever it admits a decomposition $a = bh$ for some h in the unit ball of $L_2(\mathcal{M})$ and some function b in $L_2(\mathcal{A})$ satisfying

$$\int_{\mathbb{R}^n} b = 0, \quad \mathrm{supp}_{\mathbb{R}^n}(b) \subseteq B, \quad \Big(\int \|b(x)\|_{L_2(\mathcal{M})}^2 \, dx \Big)^{1/2} \leq \frac{1}{\sqrt{|B|}}$$

for some ball B. Then, $H_1^c(\mathcal{A})$ is the Banach space

$$\Big\{ \sum_i \lambda_i a_i \ : \ (\lambda_i)_i \in \ell_1, \ (a_i)_i \text{ c}-\text{atoms} \Big\}$$

equipped with the norm

$$\|f\|_{H_1^c} = \inf \Big\{ \sum_i |\lambda_i| \ : \ f = \sum_i \lambda_i a_i, \ (\lambda_i)_i \in \ell_1, \ (a_i)_i \text{ c}-\text{atoms} \Big\}.$$

The chapter is organized as follows: first, an in-depth description of the column/row BMO spaces is introduced using Hilbert-valued noncommutative L_p spaces. Then, the duality identity $H_1^c(\mathcal{A})^* = \mathsf{BMO}_r(\mathbb{R}, \mathcal{M})$ is fully justified so there holds

$$H_1(\mathcal{A})^* = \mathsf{BMO}(\mathbb{R}, \mathcal{M}) = \mathsf{BMO}_r(\mathbb{R}, \mathcal{M}) \cap \mathsf{BMO}_c(\mathbb{R}, \mathcal{M}).$$

In order to do that, we rely on the construction of the $L_2(\mathcal{M})-$valued H_1 space (see Appendix A), and some properties of the $\mathsf{BMO}(\mathbb{R}^n)$ space which can be translated to the semicommutative setting (see Section 4.3).

The atomic decomposition of $H_1^c(\mathcal{A})$ is the key point which allows us to give a new proof for the boundedness of Calderón-Zygmund operators with scalar-valued kernel from $H_1^c(\mathcal{A})$ into $L_1(\mathcal{A})$. Our strategy is inspired by the approach followed by Meyer and Coifman [57]. First, the kernel K associated to the operator T is explicitly approximated by a sequence of uniformly bounded kernels $(K_m)_{m=1}^\infty$. Then, if T_m is a Calderón-Zygmund operator associated to K_m, there holds

$$\|T_m(a)\|_{L_1(\mathcal{M})} \leq C$$

for any $m \geq 1$, any $c-$atom a and some universal constant C. Recall that T_m is well-defined on atoms, regardless the decomposition $a = bh$ for the $c-$atom a. Since K_m is bounded, the extension to the whole Hardy space $H_1^c(\mathcal{A})$ follows trivially, and by approximation, it follows that T is bounded from $H_1^c(\mathcal{A})$ to $L_1(\mathcal{A})$. This argument extends to a result, which is new, for Calderón-Zygmund operators associated to kernels K with values in the von Neumann algebra $\mathcal{M}$.

Most of the results contained in this thesis are contained in the following papers. In particular, chapters 1 and 2 correspond to the first two papers, while the third publication has given rise to chapter 4:

- A.I. Cano-Mármol, J.M. Conde-Alonso, and J. Parcet. Trigonometric chaos and X_p inequalities I: Balanced Fourier truncations over discrete groups. Submitted.
- A.I. Cano-Mármol, J.M. Conde-Alonso, and J. Parcet. Trigonometric chaos and X_p inequalities II: X_p inequalities with sharp scaling parameter. Submitted.
- A.I. Cano-Mármol, É. Ricard. Calderón-Zygmund theory with noncommuting kernels via H_1. Preprint.

Although we have chosen not to include a chapter introducing the concepts from noncommutative harmonic analysis required to understand the results in this thesis, these tools will be presented as they become necessary. In this manner, the extension of this work remains within reasonable length limits, and it is still as self-contained as possible.

Chapter 1

Trigonometric chaos and X_p inequalities

Let Ω_n be the n-hypercube $\{-1,1\} \times \{-1,1\} \times \cdots \times \{-1,1\}$ equipped with its normalized counting measure. If $[n] := \{1,2,\ldots,n\}$, every function $f : \Omega_n \to \mathbb{C}$ admits a Fourier expansion in terms of *Walsh characters*

$$W_{\mathsf{A}}(\varepsilon) = \prod_{j \in A} \varepsilon_j \text{ for any } \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \Omega_n$$

which can be understood as A-products of Rademacher coordinates ε_j for any $\mathsf{A} \subseteq [n]$ [64]. More clearly, there holds

$$f(\varepsilon) = \sum_{\mathsf{A} \subseteq [n]} \widehat{f}(\mathsf{A}) W_{\mathsf{A}}(\varepsilon)$$

for some complex coefficients $\widehat{f}(A)$. Given a mean-zero f, Naor proved in [60] the following inequality for each $p \geq 2$ and $k \in [n]$

$$(\mathrm{N}_p) \qquad \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subseteq [n] \\ |\mathsf{S}| = k}} \Big\| \sum_{\mathsf{A} \subseteq \mathsf{S}} \widehat{f}(\mathsf{A}) W_{\mathsf{A}} \Big\|_{L_p(\Omega_n)}^p \lesssim_p \frac{k}{n} \sum_{j=1}^n \|\partial_j f\|_{L_p(\Omega_n)}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_{L_p(\Omega_n)}^p.$$

The above S-truncations of the Walsh expansion of f are *conditional expectations* denoted by $\mathsf{E}_{[n]\setminus\mathsf{S}} f$, while $\partial_j f$ stands for the j-th *directional (discrete) derivative* of f, given by $\varepsilon \mapsto f(\varepsilon) - f(\varepsilon - 2\varepsilon_j e_j)$.

Naor's inequality (N_p) for functions with a linear Fourier-Walsh expansion becomes a form of Rosenthal inequality for symmetrically exchangeable random variables [34, 22, 35, 73]. More precisely, let Π_k be the space of sets $\mathsf{S} \subseteq [n]$ with $|\mathsf{S}| = k$ equipped with its normalized counting measure and define $\Sigma_{n,k} = \Omega_n \otimes \Pi_k$. Then, if $\widehat{f}(\mathsf{A}) = 0$ when $|\mathsf{A}| \neq 1$, the left hand side of (N_p) becomes

$$\Big\| \sum_{j=1}^n \widehat{f}(\{j\}) \sigma_j \Big\|_{L_p(\Sigma_{n,k})}^p \quad \text{with} \quad \sigma_j(\varepsilon, \mathsf{S}) = \varepsilon_j \otimes \delta_{j \in \mathsf{S}},$$

and the linear model for Naor's inequality follows from [34]

$$\Big\| \sum_{j=1}^n \widehat{f}(\{j\}) \sigma_j \Big\|_{L_p(\Sigma_{n,k})} \asymp_p \Big(\frac{k}{n} \sum_{j=1}^n |\widehat{f}(\{j\})|^p \Big)^{\frac{1}{p}} + \Big(\frac{k}{n} \sum_{j=1}^n |\widehat{f}(\{j\})|^2 \Big)^{\frac{1}{2}}.$$

Its general form (N_p) can be regarded as an extension for Rademacher chaos. Our primary goal in this chapter is to produce similar inequalities when we replace the hypercube by other (nonnecessarily abelian) discrete groups. Fourier series with frequencies on a given discrete group G must be written in terms of its left regular representation $\lambda : \mathrm{G} \to \mathcal{B}(\ell_2(\mathrm{G}))$. The unitaries $\lambda(g)$ replace Walsh characters and we work with operators of the form

$$f = \sum_{g \in \mathrm{G}} \widehat{f}(g) \lambda(g).$$

The "quantum" probability space where we place them is the group von Neumann algebra $\mathcal{L}(\mathrm{G})$. Understanding how to replace Rademacher chaos by some sort of "trigonometric chaos" has to do with identifying elementary generating families. Our construction is somehow delicate and we start with a model case which originally motivated us.

Let $\mathbb{F}_n = \mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}$ be the free group with n generators $g_1, g_2, \ldots, g_n$. The unitaries $\lambda(g_j)$ are an archetype of Voiculescu's free random variables, which play the role of Rademacher variables above. The tensor products $\zeta_j(\mathsf{S}) = \lambda(g_j) \otimes \delta_{j \in \mathsf{S}}$ in $\Sigma'_{n,k} = \mathcal{L}(\mathbb{F}_n) \otimes \Pi_k$ satisfy the inequality

$$\Big\| \sum_{j=1}^n \widehat{f}(g_j) \zeta_j \Big\|_{L_p(\Sigma'_{n,k})} \asymp_p \Big(\frac{k}{n} \sum_{j=1}^n |\widehat{f}(g_j)|^p \Big)^{\frac{1}{p}} + \Big(\frac{k}{n} \sum_{j=1}^n |\widehat{f}(g_j)|^2 \Big)^{\frac{1}{2}}.$$

The desired free form of Naor's inequality looks as follows

$$(\mathrm{FN}_p) \qquad \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subseteq [n] \\ |\mathsf{S}|=k}} \Big\| \sum_{w \in \mathbb{F}_\mathsf{S}} \widehat{f}(w) \lambda(w) \Big\|_p^p \lesssim_p \frac{k}{n} \sum_{j=1}^n \big\| \partial_j f \big\|_p^p + \Big(\frac{k}{n} \Big)^{\frac{p}{2}} \|f\|_p^p.$$

Here $\mathbb{F}_\mathsf{S}$ denotes the free subgroup with generators in S and

$$\partial_j f = 2\pi i \Big(\sum_{w \geq g_j} \widehat{f}(w) \lambda(w) + \sum_{w \geq g_j^{-1}} \widehat{f}(w) \lambda(w) \Big),$$

where $w \geq g_j$ is used to pick those words starting with the letter g_j when written in reduced form. Let us briefly comment on the two inequalities above. The first one follows from the noncommutative Burkholder/Rosenthal inequality [45, 46]. On the other hand, the second inequality reduces to the first one when f lives in the linear span of $\lambda(g_j)$'s as a consequence of the free Khintchine inequality [28]. It is therefore an extension of the linear model for free chaos. A look at Naor's original inequality shows that both group elements and collections of generators (respectively denoted by A and S there) become subsets of $[n]$. This curious coincidence in the hypercube must be decoupled for other discrete groups and our inner sum in the left hand side is taken over those words w with letters living in free coordinates located in S. On the other hand, our choice for $\partial_j f$ comes from [40] and will be properly justified in due time. It is worth mentioning that some nonlinear extensions of the free Rosenthal inequality where investigated in [44] for free chaos, but none of them include a free form of Naor's inequality along the lines suggested above.

The above reasoning settles a free model for Naor's inequality and illustrates how trigonometric chaos fits in for free groups. Answering these questions amounts to considering Fourier truncations and somehow related differential operators over discrete groups. Other than lattices of Lie groups, discrete groups fail to admit canonical differential structures. This difficulty was successfully solved in [39, 40] with affine representations. More precisely, an orthogonal cocycle of G is a pair (α, β) given by an orthogonal action $\alpha : \mathrm{G} \curvearrowright \mathcal{H}$ into some $\mathbb{R}$-Hilbert space together with a map $\beta : \mathrm{G} \to \mathcal{H}$ satisfying the cocycle law

$$\alpha_g(\beta(h)) = \beta(gh) - \beta(g).$$

The latter ensures that $g \mapsto \alpha_g(\cdot) + \beta(g)$ is an affine representation of G, so that the cocycle map β establishes a good Hilbert space lift of G and one can expect to import the differential structure of $\mathcal{H}$. Naively, we "identify" the unitary $\lambda(g)$ with the Euclidean character $\exp(2\pi i \langle \beta(g), \cdot \rangle)$ and define $\mathcal{H}$-directional derivatives on $\mathcal{L}(\mathrm{G})$ as follows for any $u \in \mathcal{H}$

$$\partial_u(\lambda(g)) = 2\pi i \langle \beta(g), u \rangle \lambda(g) \quad \text{and} \quad \Delta(\lambda(g)) = -4\pi^2 \|\beta(g)\|^2 \lambda(g).$$

This strategy has been extremely useful to establish L_p-boundedness criteria for Fourier multipliers on group von Neumann algebras. We now introduce the right setup for the problem. Given a discrete group G equipped with an orthogonal cocycle (α, β) and a positive integer n, we say that

$$\mathcal{A} = \Big\{ \mathrm{B}_{\mathsf{S}} \subseteq \mathrm{G} : \mathsf{S} \subseteq [n] \Big\}$$

is an *admissible family of Fourier truncations* when we have:

- $\Big\| \sum_{g \in \mathrm{B}_{\mathsf{S}}} \widehat{f}(g) \lambda(g) \Big\|_p \le_{\mathrm{cb}} C_p \Big\| \sum_{g \in \mathrm{G}} \widehat{f}(g) \lambda(g) \Big\|_p$ for $p \ge 2$.
- Pairwise β-orthogonality:

$$\mathcal{H} = \bigoplus_{j=1}^n \mathcal{H}_j \quad \text{with} \quad \beta(\mathrm{B}_{\mathsf{S}}), \beta(\mathrm{B}_{\mathsf{S}}^{-1}) \subseteq \bigoplus_{j \in \mathsf{S}} \mathcal{H}_j = \mathcal{H}_{\mathsf{S}}.$$

Given an orthonormal basis $(u_{j\ell})_\ell$ of $\mathcal{H}_j$, define the *j-th gradients*

$$\mathrm{D}_j f = \sum_{\ell\geq 1} \partial_{u_{j\ell}} f \otimes e_{\ell,1} \quad \text{so that} \quad |\mathrm{D}_j f| = \Big(\sum_{\ell \geq 1} |\partial_{u_{j\ell}} f|^2 \Big)^{\frac{1}{2}}.$$

Theorem 1.0.1. *Let* G *be a discrete group equipped with an orthogonal cocycle* (α,β) *whose associated laplacian* Δ *has a positive spectral gap* $\sigma > 0$*. Let us consider an admissible family of Fourier truncations* $\mathcal{A} = \{\mathrm{B}_\mathsf{S} : \mathsf{S} \subseteq [n]\}$*. Then, given* $p \geq 2$ *and* $k \in [n]$*, the following inequality holds for any mean-zero* f

$$\frac{1}{\binom{n}{k}} \sum_{|\mathsf{S}|=k} \Big\| \sum_{g \in \mathrm{B}_\mathsf{S}} \widehat{f}(g)\lambda(g) \Big\|_p^p \lesssim_{p,\sigma} \frac{k}{n} \sum_{j=1}^n \Big[\big\| |\mathrm{D}_j(f)| \big\|_p^p + \big\| |\mathrm{D}_j(f^*)| \big\|_p^p \Big] + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_p^p.$$

Naor's inequality follows as a particular case of Theorem 1.0.1 by taking $\mathrm{G} = \widehat{\Omega}_n = \mathbb{Z}_2^n$ equipped with the cocycle into the n-dimensional space $\mathcal{H} = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ determined by the inclusion map β and

$$\alpha_A = (-1)^{\delta_{1\in A}} \mathrm{Id} \times \ldots \times (-1)^{\delta_{n \in A}} \mathrm{Id},$$

as well as the truncations $\mathrm{B}_\mathsf{S} = \{\mathsf{A} \subseteq \mathsf{S}\} = \beta^{-1}(\mathcal{H}_\mathsf{S})$. Recall that $\mathrm{D}_j = \partial_j$ in this case since $\dim \mathcal{H}_j = 1$. Moreover, $|\partial_j(f)| = |\partial_j(f^*)|$ in the abelian framework of the hypercube. Two generalizations of Naor's inequality for large classes of discrete groups easily follow from Theorem 1.0.1:

i) **Direct products.** If $\mathrm{G} = \mathrm{G}_1 \times \mathrm{G}_2 \times \cdots \times \mathrm{G}_n$ is a direct product of discrete groups equipped with orthogonal cocycles (α_j, β_j), consider the product cocycle (α,β) and set B_S be the subgroup of G generated by group elements whose nontrivial entries lie in S. Then, the Fourier truncations become (completely contractive) conditional expectations and we get an admissible family of Fourier truncations. The gradients D_j correspond to the different factors and cocycles in the direct product above.

ii) **Equivariant decompositions.** If G is a discrete group equipped with an orthogonal cocycle (α,β), any direct sum decomposition of the Hilbert space $\mathcal{H}$ into α-equivariant subspaces gives rise to an admissible family of Fourier truncations. More precisely, assume

$$\mathcal{H} = \bigoplus_{j=1}^n \mathcal{H}_j \quad \text{and} \quad \alpha_g(\mathcal{H}_j) \subseteq \mathcal{H}_j \text{ for every } (g,j) \in \mathrm{G} \times [n].$$

Then, the family of sets

$$\mathrm{B}_\mathsf{S} = \beta^{-1}\Big(\bigoplus_{j \in \mathsf{S}} \mathcal{H}_j \Big)$$

are subgroups of G. In particular, the associated Fourier truncations are conditional expectations (henceforth L_p-contractions) and the B_{S}'s satisfy pairwise β-orthogonality. This more general construction does not impose a direct product structure on the discrete group G.

Let $\mathcal{A}$ be an admissible family of Fourier truncations on G as defined above. Let us say that a group element $g \in \mathrm{G}$ is an $\mathcal{A}$-generator when $\beta(g) \in \mathcal{H}_j$ for some $1 \le j \le n$. Theorem 1.0.1 may be regarded as a nonlinear form of an inequality for linear combinations of $\mathcal{A}$-generators

$$f = \sum_{j=1}^{n} \sum_{\beta(g)\in\mathcal{H}_j} \widehat{f}(g)\lambda(g) = \sum_{j=1}^{n} A_j(f).$$

This inequality controls balanced averages of S-truncations $\sum_{j\in\mathsf{S}} A_j(f)$ in terms of f and the j-th gradients of $A_j(f)$. This linear model seems to be new for general discrete groups/cocycles and Theorem 1.0.1 gives a nonlinear generalization in terms of trigonometric chaos over $\mathcal{A}$-generators.

Theorem 1.0.1 does not recover the conjectured free form of Naor's inequality (FN_p). Indeed, the free inequality relies on the standard cocycle of $\mathbb{F}_n$ associated with the word length, which yields $\mathcal{H} \simeq \ell_2(\mathbb{F}_n \setminus \{e\})$ and infinitely many free derivatives of the form

$$\partial_u f = \sum_{w \ge u} \widehat{f}(w)\lambda(w) \quad \text{for any} \quad u \in \mathbb{F}_n \setminus \{e\}.$$

However, we only need to use n free directional derivatives

$$\partial_j = \partial_{g_j} + \partial_{g_j^{-1}} \quad \text{with} \quad 1 \le j \le n$$

and these are not coupled into a family of gradients, as we do in Theorem 1.0.1. The key point to achieve this is the fact that free derivatives associated to free generators include all free derivatives in the sense that

$$u \neq e \ \Rightarrow\ u \ge g_j \text{ or } u \ge g_j^{-1} \text{ for some } 1 \le j \le n \ \Rightarrow\ \partial_u \circ \partial_j = \partial_j \circ \partial_u = \partial_u.$$

In general, assume that $\mathcal{A} = \{\mathrm{B}_{\mathsf{S}} : \mathsf{S} \subseteq [n]\}$ is an admissible family of Fourier truncations in G with respect to (α, β). We will say that $\mathcal{J} = \{\partial_j : 1 \le j \le n\}$ is a *distinguished family of derivatives* when $\partial_u \circ \partial_j = \partial_u$ for any $u \in \mathcal{H}_j$ with $1 \le j \le n$. Throughout the chapter, we shall consistently use u for vectors in $\mathcal{H}$ and $j \in [n]$, so that no confusion should arise when using ∂_u and ∂_j. The following result refines Theorem 1.0.1 when we can find such a family.

Theorem 1.0.2. *Let* G *be a discrete group equipped with an orthogonal cocycle* (α, β) *and an admissible family of Fourier truncations* $\mathcal{A} = \{\mathrm{B}_{\mathsf{S}} : \mathsf{S} \subseteq [n]\}$*. Assume that* $\mathcal{J} = \{\partial_j : 1 \le j \le n\}$ *is a distinguished family of derivatives. Then, given* $p \ge 2$ *and* $k \in [n]$*, the following inequality holds for any mean-zero* f

$$\frac{1}{\binom{n}{k}} \sum_{|\mathsf{S}|=k} \Big\| \sum_{g\in\mathrm{B}_{\mathsf{S}}} \widehat{f}(g)\lambda(g) \Big\|_p^p \lesssim_p \frac{k}{n} \sum_{j=1}^{n} \Big(\big\|\partial_j(f)\big\|_p^p + \big\|\partial_j(f^*)\big\|_p^p \Big) + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_p^p.$$

When the distinguished family of derivatives ∂_j is a proper subset of the cocycle derivatives ∂_u, it turns out that Theorem 1.0.2 gives a stronger inequality (compared to that of Theorem 1.0.1) at the cost of additional assumptions, which fortunately hold in several important cases considered below. Note as well that the spectral gap assumption is unnecessary under the presence of distinguished derivatives. Here are our main applications of Theorem 1.0.2:

i) **Free chaos.** Our discussion on free derivatives illustrates how to apply Theorem 1.0.2 to obtain an inequality which gets very close to (FN_p). The extra term $\partial_j(f^*)$ is anyway removable due to a special property of free groups, for which word-length derivatives become free forms of directional Hilbert transforms [55]. This "good pathology" leads us to an even stronger inequality than the free analog of Naor's inequality (FN_p). This could be useful in other directions of free harmonic analysis. We shall also explore the free products $\mathbb{Z}_{2m} * \mathbb{Z}_{2m} * \cdots * \mathbb{Z}_{2m}$.

ii) **Continuous and discrete tori.** We also analyze $\mathbb{T}^n = \widehat{\mathbb{Z}}^n$ and $\mathbb{Z}_m^n = \widehat{\mathbb{Z}}_m^n$ equipped with different geometries. Theorem 1.0.2 is applicable for the Cayley graph metric and the resulting inequality improves the one coming from the Euclidean metric. These forms of Naor's inequality can be regarded as refinements of the classical Poincaré inequality.

iii) **Infinite Coxeter groups.** Any group presented by

$$\mathrm{G} = \Big\langle g_1, g_2, \ldots, g_n \ \Big| \ (g_j g_k)^{s_{jk}} = e \Big\rangle$$

with $s_{jj} = 1$ and $s_{jk} \geq 2$ for $j \neq k$ is called a Coxeter group. Bożejko proved in [5] that the word length is conditionally negative for any infinite Coxeter group. The Cayley graph of these groups is more involved and we will not construct here a natural ONB for the cocycle, we invite the reader to do it and to derive inequalities in the lines of Theorems 1.0.1 and 1.0.2.

Our proof of Theorems 1.0.1 and 1.0.2 streamlines Naor's original argument. The key point in this general setting is to identify the right notions, such as admissible families of Fourier truncations or distinguished families of derivatives. Once this is done, the proof heavily relies on dimension-free estimates for noncommutative Riesz transforms [40] in the same way Naor's inequality did in terms of Lust-Piquard results [49]. Another crucial point in our argument is the Banach X_p nature of noncommutative L_p-spaces. Generalizing previous work of Naor and Schechtman [63, Theorem 7.1], we shall establish it with a much simpler argument based on Junge/Xu's noncommutative Burkholder and Rosenthal inequalities [45, 46]. Of course, one could expect that Theorems 1.0.1 and 1.0.2 may lead to noncommutative X_p-type inequalities, very much like in [60]. We have obtained some inequalities in this direction (see chapter 2 or [9]). Our hope was to deduce nontrivial bounds for L_p-distortions of Schatten q-classes or other noncommutative L_q-spaces. Unfortunately, our efforts so far have not been fruitful in this direction.

1.1 Trigonometric chaos

Before introducing the concepts related to trigonometric chaos, we include a summary about noncommutative analysis which will be useful along the whole text of this thesis. Unless otherwise stated, $\mathcal{H}$ will denote a complex Hilbert space, and $B(\mathcal{H})$ will be the C*-algebra of bounded linear operators $\mathcal{H}$ with the usual adjoint as involution. A *von Neumann algebra* $\mathcal{M}$ on $\mathcal{H}$ is a C*-subalgebra of $B(\mathcal{H})$ which contains the identity operator Id on $\mathcal{H}$ and is closed with respect to the weak operator topology. Let $\mathcal{M}_+$ denote the positive cone of $\mathcal{M}$, that is,

$$\mathcal{M}_+ = \{x \in \mathcal{M} \ : \ \langle xh, h\rangle_{\mathcal{H}} \geq 0 \text{ for any } h \in \mathcal{H}\}.$$

We will consider von Neumann algebras $\mathcal{M}$ which admit a *normal semifinite faithful (n.s.f.) trace* τ, that is, a positive linear functional for which there holds

$$\tau(x^*x) = \tau(xx^*) \text{ for every } x \in \mathcal{M}$$

and is

- *normal*, that is, $\sup_i \tau(x_i) = \tau(\sup_i x_i)$ for any bounded increasing net $(x_i)_i \subseteq \mathcal{M}_+$,
- *faithful*, that is, $\tau(x) = 0$ for some $x \in \mathcal{M}_+$ implies $x = 0$,
- *semifinite*, that is, for any nonzero $x \in \mathcal{M}_+$ there is a nonzero $y \in \mathcal{M}_+$ such that $y \leq x$ and $\tau(y) < \infty$.

Moreover, τ will be said to be *finite* whenever $\tau(\mathrm{Id}) < \infty$.

By spectral functional calculus, one can define the *modulus* of $x \in \mathcal{M}$ as

$$|x| = (x^*x)^{1/2} \in \mathcal{M}$$

and set the *polar decomposition* $x = u|x|$ for some partial isometry $u \in \mathcal{M}$. Whenever $x \in \mathcal{M}_+$, there holds $u^*u = uu^*$ and $s(x) := u^*u = uu^*$ is the least orthogonal projection e in $\mathcal{M}$ satisfying $ex = xe = x$. For that reason, $s(x)$ is called the *support* of x. Then, we set

$$\mathcal{S}(\mathcal{M}) = \mathrm{span}\{x \in \mathcal{M}_+ \ : \ \tau(s(x)) < \infty\}$$

as the ideal of operators in $\mathcal{M}$ *supported by a τ-finite projection.* Moreover, given $0 < p < \infty$, $|x|^p \in \mathcal{S}(\mathcal{M})$ whenever $x \in \mathcal{S}(\mathcal{M})$, what leads to defining the *noncommutative L_p space* $L_p(\mathcal{M})$ as the completion of $\mathcal{S}(\mathcal{M})$ with respect to

$$\|x\|_p = \tau(|x|^p)^{1/p}.$$

In other words,

$$L_p(\mathcal{M}) = \overline{\mathcal{S}(\mathcal{M})}^{\|\cdot\|_p}.$$

Whenever $1 \leq p \leq \infty$, $L_p(\mathcal{M})$ is a Banach space, and most of the properties holding for classical (commutative) L_p spaces remain valid in this framework: Hölder inequality, duality $L_{p'}(\mathcal{M}) = L_p(\mathcal{M})^*$ for $1 \leq p < \infty$, interpolation techniques, etc.

As a first example, whenever $\mathcal{M} = L_\infty(\Omega, \mu)$ for some semifinite measure space (Ω, μ), the trace τ is given $\tau(f) = \int f \, d\mu$ and $L_p(\mathcal{M})$ coincides with $L_p(\Omega, \mu)$. On the other hand, whenever $\mathcal{M} = B(\mathcal{H})$ and $\tau = \mathrm{Tr}$ is the usual trace of matrices, we recover the *Schatten classes* $S_p(\mathcal{H}) = L_p(B(\mathcal{H}), \mathrm{Tr})$. For more context on noncommutative L_p spaces, see [70, 81, 23].

Another crucial notion in noncommutative harmonic analysis is the concept of operator spaces [69, 15]. An *operator space* E is a closed linear subspace of $B(\mathcal{H})$. The morphisms in this category must keep track of the information given by the inclusion into $B(\mathcal{H})$: these are called completely bounded maps. Let $\mathbb{M}_n(E)$ denote the space of matrices $n \times n$ with entries in E. Given another operator space $F \subseteq B(\mathcal{K})$, a map $u : E \to F$ is *completely bounded* if and only if the matrix amplifications

$$\begin{array}{rccc} \mathrm{Id} \otimes u : & \mathbb{M}_n(E) \subseteq B(\mathcal{H}^n) & \longrightarrow & \mathbb{M}_n(F) \subseteq B(\mathcal{K}^n) \\ & (x_{ij})_{i,j=1}^n & \longmapsto & (u(x_{ij}))_{i,j=1}^n \end{array}$$

are uniformly bounded for $n \geq 1$. Then we define

$$\|u\|_{cb} = \sup_{n \geq 1} \|\mathrm{Id}_{\mathbb{M}_n(\mathbb{C})} \otimes u : \mathbb{M}_n(E) \to \mathbb{M}_n(F)\|.$$

Moreover, as in Banach space theory, we can consider several tensor products. The *minimal tensor product* $E \otimes_{\min} F$ is defined as the completion of the algebraic tensor $E \otimes F$ with respect to the norm of $B(\mathcal{H} \otimes_2 \mathcal{K})$. This tensor product plays the analogous role of the injective tensor product in Banach space theory. On the other hand, an operator space version of the *projective tensor product* $E \widehat{\otimes} F$ can be introduced. We refer to [69] for its definition. These constructions lead to consider *vector-valued noncommutative L_p spaces* $L_p(\mathcal{M}; E)$ [68]. We redirect the reader to Section 3.2 for a brief introduction on this topic, and to chapter 4 for a disgression in the theory whenever E is a Hilbert space equipped with the column/row operator structure. Nevertheless, let illustrate this concept through the case $(\mathcal{M}, \tau) = (B(\ell_2), \mathrm{Tr})$. Set $S_\infty[E] = B(\ell_2) \otimes_{\min} E$ and $S_1[E] = S_1(\ell_2) \widehat{\otimes} E$. Then, the Schatten class with entries in E is defined, via complex interpolation of operator spaces, as

$$S_p[E] = (S_\infty[E], S_1[E])_{1/p}.$$

Analogously, $S_p^n[E]$ can be defined for any $n \geq 1$, yielding a characterization for the notion of completely bounded maps: a map $u : E \to F$ is completely bounded if and only if

$$\sup_{n \geq 1} \|\mathrm{Id}_{S_p^n} \otimes u : S_p^n[E] \to S_p^n[F]\| < \infty.$$

We will particularly interested in the case $E = L_p(\mathcal{N})$ for some von Neumann algebra $\mathcal{N}$, equipped with the operator space structure given by complex interpolation $(\mathcal{M}, L_1(\mathcal{M}))_{1/p} \simeq$

$L_p(\mathcal{M})$. The reason for this interest is that, whenever a inequality that relates the $L_p(\mathcal{N})$-norm of several operators holds and the constant does not depend on $\mathcal{N}$, then the inequality automatically holds for the norm in $S_p^n[L_p(\mathcal{N})] \simeq L_p(B(\ell_2^n)\overline{\otimes}\mathcal{N})$, and therefore, for the *cb* norm.

1.1.1 Harmonic analysis on discrete groups

Let G be a discrete group. The left regular representation of G on $\ell_2(\mathrm{G})$ is the unitary representation determined by

$$[\lambda(g)\varphi](h) = \varphi(g^{-1}h), \quad g, h \in \mathrm{G}, \; \varphi \in \ell_2(\mathrm{G}).$$

The *group von Neumann algebra of* G is denoted by $\mathcal{L}(\mathrm{G})$. It is the weak operator closure of the linear span of $\{\lambda(g)\}_{g\in\mathrm{G}}$ in $\mathcal{B}(\ell_2(\mathrm{G}))$. Its canonical trace τ is linearly determined by $\tau(\lambda(g)) = \langle\lambda(g)\delta_e, \delta_e\rangle_{\ell_2(\mathrm{G})} = \delta_{g=e}$. Every element $f \in \mathcal{L}(\mathrm{G})$ admits a Fourier series

$$f = \sum_{g\in\mathrm{G}} \widehat{f}(g)\lambda(g) \quad \text{where} \quad \widehat{f}(g) = \tau(\lambda(g)^* f).$$

This shows that $\tau(f) = \widehat{f}(e)$. For $1 \le p < \infty$, we denote by $L_p(\mathcal{L}(\mathrm{G}))$ the associated noncommutative L_p space. We emphasize here that in case G is abelian, its Pontryagin dual $\widehat{\mathrm{G}}$ [19, 74] is a compact abelian group and we have

$$L_p(\mathcal{L}(\mathrm{G})) \simeq L_p(\widehat{\mathrm{G}}).$$

Therefore, in that case $L_p(\mathcal{L}(G))$ is a classical (commutative) L_p space. In all instances below, we will consider all of our L_p spaces as noncommutative ones so that we can give a unified treatment to all the examples.

An *orthogonal cocycle* for G is a triple $(\mathcal{H}, \alpha, \beta)$ given by a real Hilbert space $\mathcal{H}$, an orthogonal action $\alpha : \mathrm{G} \to \mathcal{O}(\mathcal{H})$, and a map $\beta : \mathrm{G} \to \mathcal{H}$ satisfying the cocycle law

$$\alpha_g(\beta(h)) = \beta(gh) - \beta(g).$$

Orthogonal cocycles are in one-to-one correspondence with length functions. We say that a map $\psi : \mathrm{G} \to \mathbb{R}_+$ is a *length function* if it vanishes at the identity e, it is symmetric $\psi(g) = \psi(g^{-1})$, and it is conditionally negative

$$\sum_{g\in\mathrm{G}} a_g = 0 \quad \Rightarrow \quad \sum_{g,h\in\mathrm{G}} \overline{a}_g a_h \psi(g^{-1}h) \le 0$$

for any finitely supported family $\{a_g\}_{g\in\mathrm{G}}$. Given a cocycle $(\mathcal{H}, \alpha, \beta)$, the function $\psi(g) = \|\beta(g)\|_{\mathcal{H}}^2$ is a length function. On the other hand, any length function ψ determines a Gromov form $\langle\cdot,\cdot\rangle_\psi$, a semidefinite positive form on the group algebra $\mathbb{R}[\mathrm{G}]$ given by

$$\langle\delta_g, \delta_h\rangle_\psi = \frac{\psi(g) + \psi(h) - \psi(g^{-1}h)}{2}.$$

Then, the Hilbert completion $\mathcal{H}$ of $(\mathbb{R}[G]/\mathrm{Ker}(\langle\cdot,\cdot\rangle_\psi), \langle\cdot,\cdot\rangle_\psi)$, together with the map $\beta : g \mapsto \delta_g + \mathrm{Ker}(\langle\cdot,\cdot\rangle_\psi)$, and the orthogonal action $\alpha_g(\delta_h) = \delta_{gh} - \delta_h + \mathrm{Ker}(\langle\cdot,\cdot\rangle_\psi)$ form a cocycle. Moreover, Schoenberg's theorem [78] claims that $\psi : \mathrm{G} \to \mathbb{R}_+$ is a length function if and only if the maps $S_t : \lambda(g) \mapsto e^{-t\psi(g)}\lambda(g)$ form a Markov semigroup $(S_t)_{t\geq 0}$ on $\mathcal{L}(\mathrm{G})$, see [39, 40]. In this case $(S_t)_{t\geq 0}$ admits an infinitesimal generator

$$-\Delta := \lim_{t\to 0^+} \frac{S_t - \mathrm{Id}_{\mathcal{L}(\mathrm{G})}}{t} \quad \text{so that} \quad S_t = \exp(-t\Delta).$$

As is standard, we shall call the generator Δ the *ψ-Laplacian* on G. Since we have $\Delta(\lambda(g)) = \psi(g)\lambda(g)$ for $g \in \mathrm{G}$, it turns out that Δ is an unbounded Fourier multiplier whose fractional powers can be defined by

$$\Delta^\gamma f := \sum_{g\in \mathrm{G}} \psi(g)^\gamma f(g)\lambda(g).$$

Let $(\mathcal{H}, \alpha, \beta)$ be the orthogonal cocycle naturally associated to the length function $\psi : \mathrm{G} \to \mathbb{R}_+$ as explained above. Given an orthonormal basis $\{e_\ell\}_{\ell\geq 1}$ of $\mathcal{H}$, we consider the corresponding *directional derivatives* as follows

$$\partial_{e_\ell}\lambda(g) := 2\pi i \langle \beta(g), e_\ell\rangle_\psi \lambda(g) \quad \text{so that} \quad -4\pi^2\Delta = \sum_{\ell\geq 1} \partial_{e_\ell}^2.$$

The corresponding *Riesz transforms* associated to ψ are then defined as

$$R_\ell f = R_{e_\ell} f := \partial_{e_\ell}\Delta^{-\frac{1}{2}} f = 2\pi i \sum_{g\in \mathrm{G}} \frac{\langle \beta(g), e_\ell\rangle_\psi}{\sqrt{\psi(g)}} \widehat{f}(g)\lambda(g).$$

Riesz transforms act on elements of $L_p(\mathcal{L}(\mathrm{G}))$ with null Fourier coefficients on the kernel of β. More precisely, maps R_ℓ are well-defined over the mean-zero subspaces

$$L_p^\circ(\mathcal{L}(\mathrm{G})) = \Big\{ f \in L_p(\mathcal{L}(\mathrm{G})) : \widehat{f}(g) = 0 \text{ if } \beta(g) = 0 \Big\}.$$

Dimension-free estimates for noncommutative Riesz transforms were studied in [40].

Theorem 1.1.1 (Theorem A1 – [40]). *If $2 \leq p < \infty$ and $f \in L_p^\circ(\mathcal{L}(\mathrm{G}))$*

$$\|f\|_p \asymp_p \max\left\{ \Big\| \Big(\sum_{\ell\geq 1} |R_\ell(f)|^2 \Big)^{\frac{1}{2}} \Big\|_p, \ \Big\| \Big(\sum_{\ell\geq 1} |R_\ell(f^*)|^2 \Big)^{\frac{1}{2}} \Big\|_p \right\}.$$

Finally, our *Fourier truncations* will be written in the form

$$\mathsf{E}_{[n]\setminus \mathsf{S}} f = \sum_{g\in \mathrm{B}_\mathsf{S}} \widehat{f}(g)\lambda(g) \quad \text{with} \quad \mathsf{S} \subseteq [n].$$

When B_S is a subgroup of G, $\mathsf{E}_{[n]\setminus\mathsf{S}}$ is a (L_p-contractive) conditional expectation onto $\mathcal{L}(\mathrm{B}_\mathsf{S})$.

1.1.2 Noncommutative L_p-spaces are Banach X_p spaces

Linear forms of X_p inequalities are vector-valued extensions of Rosenthal inequality for symmetrically exchangeable random variables [34]. More precisely, a Banach space $\mathbb{X}$ is said to satisfy the *Banach* X_p *inequality* if the inequality of Theorem 1.1.2 below is satisfied for vectors $\{x_j\}_{j\in[n]} \subseteq \mathbb{X}$ (and with norms taken in $\mathbb{X}$). In [63, Theorem 7.1] Naor and Schechtman proved such inequalities for Schatten p-classes. A noncommutative Burkholder martingale inequality for the conditioned square function [45] led Junge and Xu to obtain noncommutative Rosenthal inequalities for symmetric variables in [46]. The precise result that we used below is the following (see [46, Corollary 6.6]): let $\mathcal{N}$ and $\mathcal{M}$ be von Neumann algebras and $p \geq 2$. If $\{x_j\}_{j\in[n]} \subseteq L_p(\mathcal{M})$ satisfy that

$$\Big\| \sum_{j=1}^n \delta_j a_{\pi(j)} \otimes x_j \Big\|_{L_p(\mathcal{N}\overline{\otimes}\mathcal{M})} \lesssim \Big\| \sum_{j=1}^n a_j \otimes x_j \Big\|_{L_p(\mathcal{N}\overline{\otimes}\mathcal{M})},$$

holds for all random signs $\delta = (\delta_1, \ldots, \delta_n) \in \Omega_n$, all permutations π on $[n]$ and coefficients $\{a_j\}_{j\in[n]} \subseteq L_p(\mathcal{N})$ — that is, the variables are symmetric —, then

$$\Big\|\sum_{j=1}^n a_j\otimes x_j\Big\|_p \simeq \frac{1}{n^{1/p}}\sum_{j,j'=1}^n \|a_j\|_p\|x_{j'}\|_p + \frac{1}{n^{1/2}}\Big\|\Big(\sum_{j=1}^n x_j^*x_j + x_jx_j^*\Big)^{1/2}\Big\|_p \Big\|\Big(\sum_{j=1}^n a_j^*a_j\Big)^{1/2}\Big\|_p$$

We use this result below to establish the Banach X_p nature of arbitrary noncommutative L_p-spaces. Naor/Schechtman's argument can be extended to work as well for other noncommutative L_p-spaces, but our argument below is much shorter.

Theorem 1.1.2. *Let $(\mathcal{M}, \tau)$ be a von Neumann algebra equipped with a normal semifinite faithful trace. Then, if $\mathbb{E}$ denotes the expectation over independently equidistributed random signs $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ and $x_j \in L_p(\mathcal{M})$, the following inequality holds for any $p \geq 2$ and $k \in [n]$*

$$\frac{1}{\binom{n}{k}}\sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \mathbb{E}\Big\|\sum_{j\in \mathsf{S}}\varepsilon_j x_j\Big\|^p_{L_p(\mathcal{M})} \lesssim_p \frac{k}{n}\sum_{j=1}^n \|x_j\|^p_{L_p(\mathcal{M})} + \Big(\frac{k}{n}\Big)^{\frac p2}\mathbb{E}\Big\|\sum_{j=1}^n\varepsilon_j x_j\Big\|^p_{L_p(\mathcal{M})}.$$

Proof. Define random variables $\sigma_j \in \Sigma_{n,k} = \Omega_n \otimes \Pi_k$ as defined at the beginning of this chapter by $\sigma_j(\varepsilon, \mathsf{S}) = \varepsilon_j \otimes \delta_{j\in\mathsf{S}}$ for $1 \leq j \leq n$ and $\mathsf{S} \subseteq [n]$. We claim that, for any choice of signs $\delta_j = \pm 1$ and any permutation π of $[n]$, it holds

$$\mathrm{A} := \Big\| \sum_{j=1}^n \delta_j \sigma_{\pi(j)} \otimes x_j \Big\|_{L_p(\Sigma_{n,k}\overline{\otimes}\mathcal{M})} \lesssim \Big\| \sum_{j=1}^n \sigma_j \otimes x_j \Big\|_{L_p(\Sigma_{n,k}\overline{\otimes}\mathcal{M})} =: \mathrm{B}.$$

Indeed, applying noncommutative Khintchine's inequality [48] in $L_p(\mathcal{M})$ twice

$$\begin{aligned}
\mathrm{A}^p &= \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \Big\| \sum_{\pi(j)\in\mathsf{S}} \varepsilon_{\pi(j)} \otimes \delta_j x_j \Big\|^p_{L_p(\Omega_n;L_p(\mathcal{M}))} \\
&\asymp_p \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \max\Big\{ \Big\| \Big(\sum_{\pi(j)\in\mathsf{S}} x_j^* x_j \Big)^{\frac{1}{2}} \Big\|_{L_p(\mathcal{M})}, \Big\| \Big(\sum_{\pi(j)\in\mathsf{S}} x_j x_j^* \Big)^{\frac{1}{2}} \Big\|_{L_p(\mathcal{M})} \Big\} \\
&= \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \max\Big\{ \Big\| \Big(\sum_{j\in\mathsf{S}} x_j^* x_j \Big)^{\frac{1}{2}} \Big\|_{L_p(\mathcal{M})}, \Big\| \Big(\sum_{j\in\mathsf{S}} x_j x_j^* \Big)^{\frac{1}{2}} \Big\|_{L_p(\mathcal{M})} \Big\} \asymp_p \mathrm{B}^p.
\end{aligned}$$

Hence, we can apply [46, Corollary 6.6] to get

$$\mathrm{B}^p \lesssim_p \frac{1}{n} \sum_{j,j'=1}^{n} \|\sigma_j\|_p^p \|x_{j'}\|_p^p + \Big(\frac{1}{n}\Big)^{\frac{p}{2}} \Big\| \Big(\sum_{j=1}^n x_j^* x_j + x_j x_j^* \Big)^{\frac{1}{2}} \Big\|_p^p \Big\| \Big(\sum_{j=1}^n \sigma_j^2 \Big)^{\frac{1}{2}} \Big\|_p^p.$$

Now, we have

$$\begin{aligned}
\|\sigma_j\|^p_{L_p(\Sigma_{n,k})} &= \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \delta_{j\in\mathsf{S}} = \frac{k}{n}, \\
\Big\| \Big(\sum_{j=1}^n \sigma_j^2 \Big)^{\frac{1}{2}} \Big\|^p_{L_p(\Sigma_{n,k})} &= \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \Big(\sum_{j=1}^n \delta_{j\in\mathsf{S}} \Big)^{\frac{p}{2}} = k^{\frac{p}{2}}.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
\mathrm{B} &\lesssim_p \frac{k}{n} \sum_{j=1}^n \|x_j\|^p_{L_p(\mathcal{M})} + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \Big\| \Big(\sum_{j=1}^n x_j^* x_j + x_j x_j^* \Big)^{\frac{1}{2}} \Big\|^p_{L_p(\mathcal{M})} \\
&\asymp_p \frac{k}{n} \sum_{j=1}^n \|x_j\|^p_{L_p(\mathcal{M})} + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \mathbb{E} \Big\| \sum_{j=1}^n \varepsilon_j x_j \Big\|^p_{L_p(\mathcal{M})},
\end{aligned}$$

applying once again noncommutative Khintchine's inequality. This proves the result since the random variables σ_j are chosen so that B equals the left hand side in the inequality of the statement. □

Remark 1.1.3. Theorem 1.1.2 says that $L_p(\mathcal{M})$ is an Banach X_p space. The conclusion also holds in the completely bounded setting since the constants that appear in the inequality of the statement do not depend on the von Neumann algebra $\mathcal{M}$.

1.1.3 Proof of Theorem 1.0.1

According to Theorem 1.1.1

$$\begin{aligned}
\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \Big\| \sum_{g\in \mathrm{B}_\mathsf{S}} \widehat{f}(g)\lambda(g) \Big\|_p^p &= \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \big\| \mathsf{E}_{[n]\setminus \mathsf{S}} f \big\|_p^p \\
&\asymp_p \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \Big\| \Big(\sum_{\substack{j\in[n]\\ \ell\geq 1}} |R_{j\ell}(\mathsf{E}_{[n]\setminus \mathsf{S}} f)|^2 \Big)^{\frac12} \Big\|_p^p \\
&+ \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \Big\| \Big(\sum_{\substack{j\in[n]\\ \ell\geq 1}} |R_{j\ell}((\mathsf{E}_{[n]\setminus \mathsf{S}} f)^*)|^2 \Big)^{\frac12} \Big\|_p^p =: \mathrm{A} + \mathrm{B},
\end{aligned}$$

where $R_{j\ell} := R_{u_{j\ell}}$ and $\{u_{j\ell} : j \in [n],\ \ell \geq 1\}$ is the orthonormal basis of $\mathcal{H}$ considered before the statement of Theorem 1.0.1. Since $\beta(\mathrm{B}_\mathsf{S}) \subseteq \mathcal{H}_\mathsf{S}$, we observe that $\langle \beta(g), u_{j\ell}\rangle_\psi = 0$ whenever $g \in \mathrm{B}_\mathsf{S}$ and $j \notin \mathsf{S}$. Moreover, Fourier truncations commute with Riesz transforms —as both are Fourier multipliers— and we deduce

$$R_{j\ell} \circ \mathsf{E}_{[n]\setminus \mathsf{S}} = \delta_{j\in \mathsf{S}}\ \mathsf{E}_{[n]\setminus \mathsf{S}} \circ R_{j\ell}.$$

Using the complete L_p-boundedness of our Fourier truncations, we get

$$\begin{aligned}
\mathrm{A} &\lesssim_p \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \Big\| \Big(\sum_{j\in \mathsf{S}} \sum_{\ell\geq 1} |R_{u_{j\ell}} f|^2 \Big)^{\frac12} \Big\|_p^p \\
&\lesssim_p \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \mathbb{E} \Big\| \sum_{j\in \mathsf{S}} \varepsilon_j \Big[\sum_{\ell\geq 1} |R_{u_{j\ell}} f|^2 \Big]^{\frac12} \Big\|_p^p =: \mathrm{A}'.
\end{aligned}$$

The last inequality follows from the noncommutative Khintchine inequality [48] applied to independent equidistributed signs $\varepsilon_j = \pm 1$. Next, we use the Banach $\mathrm{X_p}$ nature of noncommutative L_p-spaces. More precisely, applying Theorem 1.1.2 we get

$$\mathrm{A}' \lesssim_p \frac{k}{n} \sum_{j=1}^n \Big\| \Big(\sum_{\ell\geq 1} |R_{u_{j\ell}} f|^2 \Big)^{\frac12} \Big\|_p^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \mathbb{E}_\varepsilon \Big\| \sum_{j=1}^n \varepsilon_j \Big(\sum_{\ell\geq 1} |R_{u_{j\ell}} f|^2 \Big)^{\frac12} \Big\|_p^p = \mathrm{A}_1' + \mathrm{A}_2'.$$

Since $R_{u_{j\ell}} = \partial_{u_{j\ell}} \Delta^{-\frac12} = \Delta^{-\frac12} \partial_{u_{j\ell}}$, [38, Proposition 1.1.5] yields

$$\begin{aligned}
\mathrm{A}_1' &= \frac{k}{n} \sum_{j=1}^n \Big\| \sum_{\ell\geq 1} R_{u_{j\ell}} f \otimes e_{\ell,1} \Big\|_{S_p[L_p(\mathcal{L}(\mathrm{G}))]}^p \\
&\lesssim_{p,\sigma} \frac{k}{n} \sum_{j=1}^n \Big\| \sum_{\ell\geq 1} \partial_{u_{j\ell}} f \otimes e_{\ell,1} \Big\|_{S_p[L_p(\mathcal{L}(\mathrm{G}))]}^p = \frac{k}{n} \sum_{j=1}^n \big\| |\mathrm{D}_j(f)| \big\|_p^p.
\end{aligned}$$

Moreover, noncommutative Khintchine inequality and Theorem 1.1.1 give

$$\mathrm{A}_2' \lesssim_p \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \Big\| \Big(\sum_{j=1}^{n} \sum_{\ell \geq 1} |R_{u_{j\ell}} f|^2 \Big)^{\frac{1}{2}} \Big\|_p^p \lesssim_p \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_p^p.$$

Therefore, the term A satisfies the expected estimate and it remains to justify the assertion for B. We now analyze the behavior of our Fourier truncations under adjoints. Observe that

$$(\mathsf{E}_{[n]\setminus \mathsf{S}} f)^* = \sum_{g \in \mathrm{B}_\mathsf{S}} \overline{\widehat{f}(g)} \lambda(g^{-1}) =: \mathsf{E}'_{[n]\setminus \mathsf{S}}(f^*).$$

In particular, since $\mathsf{E}'_{[n]\setminus \mathsf{S}}$ commutes with $R_{j\ell}$

$$R_{j\ell}\big((\mathsf{E}_{[n]\setminus \mathsf{S}} f)^*\big) = \mathsf{E}'_{[n]\setminus \mathsf{S}}\big(R_{j\ell}(f^*)\big) = \mathsf{E}_{[n]\setminus \mathsf{S}}\big(R_{j\ell}(f^*)^*\big)^*.$$

At this point is where we need the condition $\beta(\mathrm{B}_\mathsf{S}^{-1}) \subseteq \mathcal{H}_\mathsf{S}$, to make sure that the above terms vanish when $j \notin \mathsf{S}$ since we find the inner products $\langle \beta(g^{-1}), u_{j\ell} \rangle_\psi$ for $g \in \mathrm{B}_\mathsf{S}$. Thus, we obtain

$$\begin{aligned}
\Big\| \Big(\sum_{\substack{j \in [n] \\ \ell \geq 1}} |R_{j\ell}((\mathsf{E}_{[n]\setminus \mathsf{S}} f)^*)|^2 \Big)^{\frac{1}{2}} \Big\|_p &= \Big\| \sum_{j \in [n]} \sum_{\ell \geq 1} \mathsf{E}_{[n]\setminus \mathsf{S}}\big(R_{j\ell}(f^*)^*\big) \otimes e_{1,(j,\ell)} \Big\|_p \\
&\lesssim_p \Big\| \sum_{j \in \mathsf{S}} \sum_{\ell \geq 1} R_{j\ell}(f^*) \otimes e_{(j,\ell),1} \Big\|_p \\
&= \Big\| \Big(\sum_{j \in \mathsf{S}} \sum_{\ell \geq 1} |R_{j\ell}(f^*)|^2 \Big)^{\frac{1}{2}} \Big\|_p.
\end{aligned}$$

Therefore, we may follow the above argument for A just replacing f by f^*. $\square$

Remark 1.1.4. A careful reading of the proof of Theorems 1.0.1 and 1.0.2 shows that we may use different Hilbert space decompositions $\mathcal{H} = \oplus_j \mathcal{H}_j = \oplus_j \mathcal{K}_j$ for B_S and its inverse —with $\beta(\mathsf{B}_\mathsf{S}) \subseteq \mathcal{H}_\mathsf{S}$ and $\beta(\mathsf{B}_\mathsf{S}^{-1}) \subseteq \mathcal{K}_\mathsf{S}$— as long as we can find an orthonormal basis $\{u_\ell : \ell \geq 1\}$ of $\mathcal{H}$ satisfying that

$$\forall \ \ell \geq 1 \ \exists \ j_1, j_2 \in [n] \ \text{such that} \ u_\ell \in \mathcal{H}_{j_1} \cap \mathcal{K}_{j_2}. \tag{1.1}$$

More precisely, under these more flexible assumption we get

$$\frac{1}{\binom{n}{k}} \sum_{|\mathsf{S}|=k} \Big\| \sum_{g \in \mathrm{B}_\mathsf{S}} \widehat{f}(g) \lambda(g) \Big\|_p^p \lesssim_{p,\sigma} \frac{k}{n} \sum_{j=1}^{n} \Big[\big\| |\mathrm{D}_j(f)| \big\|_p^p + \big\| |\mathrm{D}_j^\dagger(f^*)| \big\|_p^p \Big] + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_p^p,$$

where $\mathrm{D}_j^\dagger := \sum_{u_\ell \in \mathcal{K}_j} \partial_{u_\ell}(\cdot) \otimes e_{\ell,1}$ is the gradient over the basis vectors living in $\mathcal{K}_j$.

Remark 1.1.5. The constant depending on σ in Theorem 1.0.1 grows as $\sigma^{-\frac{p}{2}}$. One can also track the dependence on p of the constant. Using free generators in place of random signs —Theorem 1.1.2 holds as well— we keep constants uniformly bounded replacing noncommutative by free Khintchine inequalities [28]. The constants in Theorem 1.1.1 are bounded by $p^{3/2}$, but it is still open whether this is optimal.

1.1.4 Proof of Theorem 1.0.2

Again Theorem 1.1.1 gives

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n] \\ |\mathsf{S}|=k}} \|\mathsf{E}_{[n]\setminus\mathsf{S}} f\|_p^p \asymp_p \mathrm{A} + \mathrm{B}$$

as in the proof of Theorem 1.0.1. Following our argument there, we use our estimate $\mathrm{A} \lesssim_p \mathrm{A}_1' + \mathrm{A}_2'$ and we bound A_2' in the same way. To estimate A_1' we use our distinguished family of derivatives and Theorem 1.1.1 to deduce

$$\mathrm{A}_1' = \frac{k}{n} \sum_{j=1}^{n} \Big\| \Big(\sum_{\ell\geq 1} |R_{u_{j\ell}} \partial_j f|^2 \Big)^{\frac{1}{2}} \Big\|_p^p \lesssim_p \frac{k}{n} \sum_{j=1}^{n} \big\| \partial_j f \big\|_p^p.$$

The estimate for B then follows by the same considerations as in Theorem 1.0.1. □

1.2 Applications to abelian groups

We now focus our attention on concrete realizations of Theorems 1.0.1 and 1.0.2 for certain commutative group algebras. In all the cases in this section, we choose $\mathsf{E}_{[n]\setminus\mathsf{S}}$ of the form

$$\mathsf{E}_{[n]\setminus\mathsf{S}} f = \sum_{g\in \mathrm{B}_\mathsf{S}} \widehat{f}(g)\lambda(g) \quad \text{for some subgroup } \mathrm{B}_\mathsf{S} \text{ of } \mathrm{G}.$$

Due to that fact, we know that they are conditional expectations, and therefore completely contractive maps. This allows us to safely apply Theorems 1.0.1 and 1.0.2 without checking that hypothesis. We will give the details for the cases $\mathrm{G} = \mathbb{Z}^n$ and $\mathrm{G} = \mathbb{Z}_{2m}^n$, yielding inequalities in $L_p(\mathbb{T}^n)$ and $L_p(\mathbb{Z}_{2m}^n)$, respectively.

1.2.1 Classical tori

Define

$$\begin{aligned} \psi_1(g) &= |g_1| + \ldots + |g_n|, \\ \psi_2(g) &= g_1^2 + g_2^2 + \ldots + g_n^2, \end{aligned}$$

with $g = (g_1, g_2, \ldots, g_n) \in \mathbb{Z}^n$. Both functions are symmetric and vanish at 0. Moreover, conditional negativity follows easily. In the case of ψ_1, it suffices to check it for each summand $|g_j|$ which are conditionally negative from subordination with respect to g_j^2.

These functions are denoted as the word and the Euclidean length respectively. We analyze balanced Fourier truncations using both geometries.

A) The Euclidean length. The length ψ_2 induces the standard cocycle $(\mathcal{H}, \alpha, \beta)$ where $\mathcal{H} = \mathbb{R}^n$ with the usual Euclidean inner product, the trivial action and the canonical inclusion $\beta = \mathrm{Id}$. We use the standard decomposition $\mathcal{H} = \bigoplus_j \mathcal{H}_j$ with $\mathcal{H}_j = \mathbb{R}e_j$ the subspace generated by the j-th element of the canonical basis. Therefore, given $\mathsf{S} \subseteq [n]$, denote by $\mathbb{Z}^{\mathsf{S}}$ the subgroup of elements with vanishing entries outside S and consider the truncations

$$\mathsf{E}_{[n]\setminus\mathsf{S}} f(x) = \sum_{g \in \mathbb{Z}^{\mathsf{S}}} \widehat{f}(g) e^{2\pi i \langle x, g\rangle} \quad \text{for any} \quad f \in L_p(\mathbb{T}^n) \simeq L_p(\mathcal{L}(\mathbb{Z}^n)),$$

where $e^{2\pi i\langle \cdot, g\rangle} \mapsto \lambda(g)$ defines a trace-preserving $*$-homomorphism. The cocycle derivatives coincide in this case with the classical ones $\partial_{e_j}\lambda(g) = 2\pi i g_j \lambda(g)$ and the infinitesimal generator Δ is the usual Laplacian (up to a universal constant) with spectral gap 1. Then, Theorem 1.0.1 yields

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n] \\ |\mathsf{S}|=k}} \Big\| \sum_{g \in \mathbb{Z}^{\mathsf{S}}} \widehat{f}(g) e^{2\pi i\langle \cdot, g\rangle} \Big\|_{L_p(\mathbb{T}^n)}^p \lesssim_p \frac{k}{n} \sum_{j=1}^n \|\partial_{e_j} f\|_{L_p(\mathbb{T}^n)}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_{L_p(\mathbb{T}^n)}^p \tag{1.2}$$

for any mean-zero $f \in L_p(\mathbb{T}^n)$. This seems to be the most natural generalization of Naor's inequality for classical tori, but it is not the most efficient. Indeed, using the same Hilbert space decomposition as above, one can consider the alternative absorbent derivatives $\partial_j \lambda(g) = \delta_{g_j \neq 0}\lambda(g)$, which satisfy $\partial_{e_j} \circ \partial_j = \partial_{e_j}$. In particular Theorem 1.0.2 yields

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n] \\ |\mathsf{S}|=k}} \Big\| \sum_{g \in \mathbb{Z}^{\mathsf{S}}} \widehat{f}(g) e^{2\pi i\langle \cdot, g\rangle} \Big\|_{L_p(\mathbb{T}^n)}^p \lesssim_p \frac{k}{n} \sum_{j=1}^n \|\partial_j f\|_{L_p(\mathbb{T}^n)}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_{L_p(\mathbb{T}^n)}^p. \tag{1.3}$$

This is a stronger inequality since

$$\|\partial_j f\|_{L_p(\mathbb{T}^n)} = \frac{1}{2\pi} \Big\| \sum_{g_j \neq 0} \frac{1}{g_j} \widehat{\partial_{e_j} f}(g) e^{2\pi i\langle \cdot, g\rangle} \Big\|_{L_p(\mathbb{T}^n)} \leq C_p \|\partial_{e_j} f\|_{L_p(\mathbb{T}^n)}.$$

Indeed, the symbol $m(g) = 1/g_j$ defines an L_p-bounded multiplier as a consequence of K. de Leeuw restriction theorem and Hörmander-Mikhlin multiplier theorem [13, 31, 58]. As we shall see (1.3) naturally appears using the word length.

Remark 1.2.1. Consider $f : \mathbb{T}^n \to \mathbb{C}$ with

$$f(x) = \sum_{g \in \mathbb{Z}^n} \widehat{f}(g) e^{2\pi i\langle x, g\rangle} \quad \text{and} \quad \widehat{f}(0) = 0.$$

Given $\mathsf{S} \subseteq [n]$, the classical Poincaré inequality gives

$$\begin{aligned}
\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\ |\mathsf{S}|=k}} \Big\| \underbrace{\sum_{g\in\mathbb{Z}^{\mathsf{S}}} \widehat{f}(k) e^{2\pi i\langle\cdot,g\rangle}}_{f_{\mathsf{S}}} \Big\|_p^p &\le \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\ |\mathsf{S}|=k}} \big\| |\nabla f_{\mathsf{S}}| \big\|_p^p \\
&\asymp \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\ |\mathsf{S}|=k}} \Big\| \sum_{j\in\mathsf{S}} \varepsilon_j \partial_{e_j} f_{\mathsf{S}} \Big\|_p^p \\
&= \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\ |\mathsf{S}|=k}} \Big\| \Big[\sum_{j\in\mathsf{S}} \varepsilon_j \partial_{e_j} f \Big]_{\mathsf{S}} \Big\|_p^p \\
&\le \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\ |\mathsf{S}|=k}} \Big\| \sum_{j\in\mathsf{S}} \varepsilon_j \partial_{e_j} f \Big\|_p^p = \Big\| \sum_{j=1}^n \sigma_j \partial_{e_j} f \Big\|_p^p
\end{aligned}$$

for $\sigma_j(\varepsilon, \mathsf{S}) = \varepsilon_j \otimes \delta_{j\in\mathsf{S}}$, as defined at the beginning of this chapter. Applying [34] gives

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\ |\mathsf{S}|=k}} \Big\| \sum_{g\in\mathbb{Z}^{\mathsf{S}}} \widehat{f}(g) e^{2\pi i\langle\cdot,g\rangle} \Big\|_p^p \lesssim_p \frac{k}{n} \sum_{j=1}^n \|\partial_{e_j} f\|_p^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \big\| |\nabla f| \big\|_p^p.$$

Inequalities (1.2) and (1.3) improve the above inequality replacing $|\nabla f|$ by f.

B) The word length. Let us now study which inequality do we get with the word length. The cocycle associated to it is infinite-dimensional, with an orthonormal basis which can be described as oriented edges in the coordinate axes of the Cayley graph of $\mathbb{Z}^n$. More precisely, the associated Gromov form on $\mathbb{R}[\mathbb{Z}^n]$ is

$$\langle \delta_g, \delta_h \rangle_{\psi_1} = \frac{1}{2}\big(\psi_1(g) + \psi_1(h) - \psi_1(h-g)\big) = \sum_{j=1}^n \min\big\{|h_j|, |g_j|\big\} \delta_{g_j\cdot h_j>0}.$$

Given $g \in \mathbb{Z}^n$ and $j \in [n]$, define

$$g_{[j]}^- = g - \operatorname{sgn}(g_j) e_j \quad \text{with} \quad \operatorname{sgn}(0) = 0.$$

Then, we may construct the following elements in $\mathbb{R}[\mathbb{Z}^n]$

$$w_{g,j} = \delta_g - \delta_{g_{[j]}^-} \quad \text{and} \quad u_j(\ell) = w_{\ell e_j, j}.$$

Lemma 1.2.2. *If $\mathcal{H}_{\psi_1} = \mathbb{R}[\mathbb{Z}^n]/\mathrm{Ker}\langle\cdot,\cdot\rangle_{\psi_1}$, the following properties hold:*

- $\langle u_j(\ell), u_j(\ell)\rangle_{\psi_1} = 1$ *for all* $(j,\ell) \in [n] \times \mathbb{Z}\setminus\{0\}$.
- $\langle u_j(\ell), u_{j'}(\ell')\rangle_{\psi_1} = 0$ *whenever* $j \neq j'$ *or* $\ell \neq \ell'$.

- $w_{g,j} = u_j(g_j)$ *since the difference belongs to* $\operatorname{Ker}\langle\cdot,\cdot\rangle_{\psi_1}$.

Proof. First, let $(j,\ell) \in [n] \times \mathbb{Z}\setminus\{0\}$. Then, there holds

$$\begin{aligned}\langle u_j(\ell), u_j(\ell)\rangle_{\psi_1} &= \langle \delta_{\ell e_j} - \delta_{(\ell-\operatorname{sgn}(\ell))e_j}, \delta_{\ell e_j} - \delta_{(\ell-\operatorname{sgn}(\ell))e_j}\rangle_{\psi_1}\\ &= |\ell| - 2(|\ell|-1) + |\ell| - 1 = 1.\end{aligned}$$

Whenever $\ell \neq \ell'$, it follows that

$$\begin{aligned}\langle u_j(\ell), u_j(\ell')\rangle_{\psi_1} &= \langle \delta_{\ell e_j} - \delta_{(\ell-\operatorname{sgn}(\ell))e_j}, \delta_{\ell' e_j} - \delta_{(\ell'-\ell')e_j}\rangle_{\psi_1}\\ &= \langle \delta_{\ell e_j}, \delta_{\ell' e_j}\rangle_{\psi_1} - \langle \delta_{\ell e_j}, \delta_{(\ell'-\ell')e_j}\rangle_{\psi_1} - \langle \delta_{(\ell-\operatorname{sgn}(\ell))e_j}, \delta_{\ell' e_j}\rangle_{\psi_1} + \langle \delta_{(\ell-\operatorname{sgn}(\ell))e_j}, \delta_{(\ell'-\ell')e_j}\rangle_{\psi_1}\\ &= \min\{|\ell|,|\ell'|\}\ \delta_{\ell\cdot\ell'>0} - \min\{|\ell|,|\ell'|-1\}\ \delta_{\ell\cdot(\ell'-\operatorname{sgn}(\ell'))>0}\\ &\quad - \min\{|\ell|-1,|\ell'|\}\ \delta_{(\ell-\operatorname{sgn}(\ell))\cdot\ell'>0} + \min\{|\ell|-1,|\ell'|-1\}\ \delta_{(\ell-\operatorname{sgn}(\ell))\cdot(\ell'-\operatorname{sgn}(\ell'))>0},\end{aligned}$$

so we can assume that $\ell\cdot\ell' > 0$ and $|\ell| < |\ell'|$, what yields that the scalar product above vanishes. Whenever $j \neq k$, it is trivial that $\langle u_j(\ell), u_k(\ell')\rangle_{\psi_1} = 0$. On the other hand, the third claim follows by induction from the identity

$$w_{g,j} = w_{g^-_{[k]},j}$$

for any $j \neq k$. Indeed, denoting $g^-_{[j,k]} = (g^-_{[j]})^-_{[k]}$, the inner product

$$\begin{aligned}\langle w_{g,j} - w_{g^-_{[k]},j}, w_{g,j} - w_{g^-_{[k]},j}\rangle_{\psi_1} &= \langle \delta_g - \delta_{g^-_{[j]}} - \delta_{g^-_{[k]}} + \delta_{g^-_{[j,k]}}, \delta_g - \delta_{g^-_{[j]}} - \delta_{g^-_{[k]}} + \delta_{g^-_{[j,k]}}\rangle_{\psi_1}\\ &= \langle\delta_g,\delta_g\rangle_{\psi_1} - \langle\delta_g,\delta_{g^-_{[j]}}\rangle_{\psi_1} - \langle\delta_g,\delta_{g^-_{[k]}}\rangle_{\psi_1} + \langle\delta_g,\delta_{g^-_{[j,k]}}\rangle_{\psi_1}\\ &\quad - \langle\delta_{g^-_{[j]}},\delta_g\rangle_{\psi_1} + \langle\delta_{g^-_{[j]}},\delta_{g^-_{[j]}}\rangle_{\psi_1} + \langle\delta_{g^-_{[j]}},\delta_{g^-_{[k]}}\rangle_{\psi_1} - \langle\delta_{g^-_{[j]}},\delta_{g^-_{[j,k]}}\rangle_{\psi_1}\\ &\quad - \langle\delta_{g^-_{[k]}},\delta_g\rangle_{\psi_1} + \langle\delta_{g^-_{[k]}},\delta_{g^-_{[j]}}\rangle_{\psi_1} + \langle\delta_{g^-_{[k]}},\delta_{g^-_{[k]}}\rangle_{\psi_1} - \langle\delta_{g^-_{[k]}},\delta_{g^-_{[j,k]}}\rangle_{\psi_1}\\ &\quad + \langle\delta_{g^-_{[j,k]}},\delta_g\rangle_{\psi_1} - \langle\delta_{g^-_{[j,k]}},\delta_{g^-_{[j]}}\rangle_{\psi_1} - \langle\delta_{g^-_{[j,k]}},\delta_{g^-_{[k]}}\rangle_{\psi_1} + \langle\delta_{g^-_{[j,k]}},\delta_{g^-_{[j,k]}}\rangle_{\psi_1}\end{aligned}$$

vanishes due to cancellations. □

Lemma 1.2.2, together with the identity

$$\begin{aligned}\delta_g &= \sum_{j=1,\ g_j\neq 0}^{n}\Bigg(\sum_{k=0}^{|g_j|-1} \delta_{(g_1,\dots,g_j-k\ \operatorname{sgn}(g_j),0,\dots,0)} - \delta_{(g_1,\dots,g_j-(k+1)\operatorname{sgn}(g_j),0,\dots,0)}\\ &= \sum_{j=1,\ g_j\neq 0}^{n}\Bigg(\sum_{k=0}^{|g_j|-1} \delta_{g_j-k\ \operatorname{sgn}(g_j)} - \delta_{g_j-(k+1)\operatorname{sgn}(g_j)},\end{aligned}$$

this implies that the set

$$\Big\{u_j(\ell) : (j,\ell) \in [n]\times\mathbb{Z}\setminus\{0\}\Big\}\Bigg($$

is an orthonormal basis for $\mathcal{H}_{\psi_1}$. The cocycle map is given by $\beta(g) = \delta_g$ and the orthogonal action α satisfies $\alpha_g(\delta_h) = \delta_{g+h} - \delta_g$. This means that for any $g \in \mathbb{Z}^n$ we have

$$\alpha_g(u_j(\ell)) = \delta_{g+\ell e_j} - \delta_{g+\ell e_j-(\mathrm{sgn}(\ell))e_j}.$$

Therefore, the subspaces $\mathcal{H}_{\psi_1,j} = \mathrm{span}\{u_j(\ell) : \ell \in \mathbb{Z} \setminus \{0\}\}$ are α-invariant for $j \in [n]$. This proves that the same conditional expectations $\mathsf{E}_{[n]\setminus \mathsf{S}}$ considered before still define an admissible family of Fourier truncations. The cocycle derivative associated to $u_j(\ell)$ acts as follows:

$$\begin{aligned}
\partial_{u_j(\ell)}\lambda(g) &= 2\pi i \, \langle u_j(\ell), \delta_g\rangle_{\psi_1} \lambda(g) \\
&= 2\pi i \Big(\min\big\{|g_j|, |\ell|\big\} \, \delta_{g_j \cdot \ell > 0} - \min\big\{|g_j|, |\ell| - 1\big\} \, \delta_{g_j\cdot(\ell-\mathrm{sgn}(\ell))>0} \Big) \lambda(g) \\
&= 2\pi i \, \delta_{\{g_j\cdot\ell>0, |g_j|\geq|\ell|\}} \lambda(g).
\end{aligned}$$

The Laplacian is

$$\Delta_{\psi_1} f = \sum_{g\in\mathbb{Z}^n} \psi_1(g) \widehat{f}(g) \lambda(g),$$

whose spectral gap is still $\sigma = \min_j \psi_1(e_j) = 1$. Theorem 1.0.1 yields

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n] \\ |\mathsf{S}|=k}} \Big\| \sum_{g\in\mathbb{Z}^{\mathsf{S}}} \widehat{f}(g) e^{2\pi i\langle\cdot,g\rangle} \Big\|_{L_p(\mathbb{T}^n)}^p \lesssim_p \frac{k}{n} \sum_{j=1}^n \big\| \, |\mathrm{D}_j f| \, \big\|_{L_p(\mathbb{T}^n)}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_{L_p(\mathbb{T}^n)}^p, \tag{1.4}$$

with

$$\big\| \, |\mathrm{D}_j(f)| \, \big\|_{L_p(\mathbb{T}^n)} = \big\| \, |\mathrm{D}_j(f^*)| \, \big\|_{L_p(\mathbb{T}^n)} = \Big\| \Big(\sum_{\ell\in\mathbb{Z}\setminus\{0\}} |\partial_{u_j(\ell)} f|^2 \Big)^{\frac{1}{2}} \Big\|_{L_p(\mathbb{T}^n)}.$$

Remark 1.2.3. Note that $|\partial_{u_j(\ell)}(f)| \neq |\partial_{u_j(\ell)}(f^*)|$. Thus, nontrivial cocycle actions lead to noncommutative phenomena even when working with abelian groups, as pointed out in [40]. In spite of that, observe that $\langle \delta_{-g}, u_j(\ell)\rangle_\psi = \langle \delta_g, u_j(-\ell)\rangle_\psi$ which implies $\| \, |\mathrm{D}_j(f)| \, \|_p = \| \, |\mathrm{D}_j(f^*)| \, \|_p$ as claimed above.

On the other hand, taking

$$\partial_j \lambda(g) := \frac{1}{2\pi i} \Big(\partial_{u_j(1)} + \partial_{u_j(-1)} \Big) = \delta_{g_j\neq 0} \lambda(g)$$

we get $\partial_{u_j(\ell)} \circ \partial_j = \partial_{u_j(\ell)}$ for any $(j,\ell) \in [n] \times \mathbb{Z}\setminus\{0\}$. Thus, Theorem 1.0.2 gives

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n] \\ |\mathsf{S}|=k}} \Big\| \sum_{g\in\mathbb{Z}^{\mathsf{S}}} \widehat{f}(g) e^{2\pi i\langle\cdot,g\rangle} \Big\|_{L_p(\mathbb{T}^n)}^p \lesssim_p \frac{k}{n} \sum_{j=1}^n \|\partial_j f\|_{L_p(\mathbb{T}^n)}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_{L_p(\mathbb{T}^n)}^p$$

for any mean-zero $f \in L_p(\mathbb{T}^n)$. This recovers inequality (1.3) and improves (1.4).

1.2.2 Discrete tori

Consider the word length $|g| = \min\{g, 2m-g\}$ in $\mathbb{Z}_{2m}$. As shown in [42], it defines a conditionally negative symmetric length. In particular the same holds for the corresponding length in the product $\mathbb{Z}_{2m}^n$

$$\psi(g) = |g_1| + |g_2| + \ldots + |g_n| \quad \text{for} \quad g = (g_1, \ldots, g_n) \in \mathbb{Z}_{2m}^n.$$

This word length has many similarities with the previous one

$$\langle \delta_g, \delta_h \rangle_\psi = \frac{1}{2}\big(\psi(g) + \psi(h) - \psi(h-g)\big) = \frac{1}{2}\sum_{j=1}^n |g_j| + |h_j| - |h_j - g_j|.$$

Given $g \in \mathbb{Z}_{2m}^n$ and $j \in [n]$, define

$$w_{g,j} = \delta_g - \delta_{g-e_j} \quad \text{and} \quad u_j(\ell) = w_{\ell e_j, j} \quad \text{for} \quad 1 \leq \ell \leq 2m.$$

If $\mathcal{H}_\psi = \mathbb{R}[\mathbb{Z}_{2m}^n]/\mathrm{Ker}\langle \cdot, \cdot \rangle_\psi$, we find that, in an analogous way as in Lemma 1.2.2,

- $\langle u_j(\ell), u_j(\ell) \rangle_\psi = 1$ for all $(j, \ell) \in [n] \times [m]$.
- $\langle u_j(\ell), u_{j'}(\ell') \rangle_\psi = 0$ whenever $j \neq j'$ or $\ell \neq \ell', \ell' + m$.
- $w_{g,j} = u_j(g_j)$ since the difference belongs to $\mathrm{Ker}\langle \cdot, \cdot \rangle_\psi$.
- $u_j(\ell) = -u_j(\ell + m)$ since the difference belongs to $\mathrm{Ker}\langle \cdot, \cdot \rangle_\psi$.
- $\langle \delta_{\ell e_j}, \delta_{\ell' e_j} \rangle_\psi = \min\big\{\ell, 2m - \ell', \max\{0, m - \ell' + \ell\}\big\}$ for $1 \leq \ell \leq \ell' \leq 2m$.

Altogether, this implies that the set

$$\Big\{u_j(\ell) : (j, \ell) \in [n] \times [m]\Big\}$$

is an orthonormal basis for $\mathcal{H}_\psi$. The cocycle map is given by $\beta(g) = \delta_g$ and the orthogonal action α satisfies $\alpha_g(\delta_h) = \delta_{g+h} - \delta_g$. This means that for any $g \in \mathbb{Z}_{2m}^n$ we have

$$\alpha_g(u_j(\ell)) = \delta_{g+\ell e_j} - \delta_{g+(\ell-1)e_j}.$$

Therefore, the subspaces $\mathcal{H}_{\psi,j} = \mathrm{span}\{u_j(\ell) : \ell \in [m]\}$ give again an α-invariant splitting of $\mathcal{H}_\psi$ with j running over $[n]$. In particular, the conditional expectations $\mathsf{E}_{[n]\setminus \mathsf{S}}$ over the subgroups $\mathbb{Z}_{2m}^{\mathsf{S}}$ define an admissible family of Fourier truncations and the cocycle derivatives are given by

$$\partial_{u_j(\ell)}\lambda(g) = 2\pi i \; \delta_{\{\ell \leq g_j < \ell + m\}}\lambda(g).$$

The associated Laplacian has spectral gap equal to 1. As before, Theorem 1.0.1 yields a statement that we omit because it can readily be improved. If we set as before $\partial_j \lambda(g) :=$

$\delta_{g_j \neq 0}\lambda(g)$ for $j \in [n]$, we immediately see that $\partial_{u_j(\ell)} \circ \partial_j = \partial_{u_j(\ell)}$. Moreover, we can rewrite it as follows

$$\partial_j \lambda(g) = \frac{1}{2\pi i}\Big(\partial_{u_j(1)}\lambda(g) + \partial_{u_j(m)}\lambda(g)\Big) - \delta_{g_j=m}\lambda(g).$$

Next, note that we may write the last term as follows

$$\delta_{g_j=m}\lambda(g) = \mathsf{E}_{\{0,m\},j}\Big(\frac{1}{2\pi i}\partial_{u_j(1)}\lambda(g)\Big) = \frac{1}{4\pi i}\mathsf{E}_{\{0,m\},j}\Big(\partial_{u_j(1)}\lambda(g) + \partial_{u_j(m)}\lambda(g)\Big),$$

where $\mathsf{E}_{\{0,m\},j}$ is the conditional expectation onto $\mathbb{Z}_{2m}^{j-1} \times \{0, m\} \times \mathbb{Z}_{2m}^{n-j}$. Then, after applying Theorem 1.0.2 one gets the following for mean-zero $f : \mathbb{Z}_{2m}^n \to \mathbb{C}$

$$\text{(1.5)} \qquad \frac{1}{\binom{n}{k}}\sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}}\Big\|\sum_{g\in\mathbb{Z}_{2m}^{\mathsf{S}}}\widehat{f}(g)e^{\frac{\pi i}{m}\langle\cdot,g\rangle}\Big\|_p^p \lesssim_p \frac{k}{n}\sum_{j=1}^n\|\partial_j f\|_p^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}}\|f\|_p^p$$

$$\lesssim \frac{k}{n}\sum_{j=1}^n\big\|(\partial_{u_j(1)} + \partial_{u_j(m)})f\big\|_p^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}}\|f\|_p^p.$$

Remark 1.2.4. Inequality (1.5) for $\mathbb{Z}_{2m}^n$ with $m = 1$ generalizes Naor's inequality (N_p) for the hypercube. Just identify $g \in \mathbb{Z}_2$ with $\exp(\pi i g) \in \{\pm 1\}$. Then note that $\partial_j^1 = 2\partial_j^2$, where ∂_j^1 is the discrete derivative used by Naor and ∂_j^2 is our choice of ∂_j in (1.5) for $m = 1$. Moreover, we can consider weighted forms of Naor's inequality by considering different measures on the same group $\mathbb{Z}_2^n$ to get different cocycle representations. We next show that this does not lead to an improvement over the result in [60]. Indeed, let $\mathrm{G} = \mathbb{Z}_2^n$ and equip $\Gamma = \widehat{\mathrm{G}} = \Omega_n$ with the measure

$$\mu = \sum_{j=1}^n \alpha_j \delta_{w_j},$$

with $\alpha_j \geq 0$ and $w_j = (1, \ldots, 1, -1, 1, \ldots, 1) = (1, 1, \ldots, 1) - 2e_j$ for $j \in [n]$. Identifying G with the power set of $[n]$ we identify $g = e_A = \sum_{j\in A} e_j$ with A. Following Example B from [40, Subsection 1.4], we consider the conditionally negative length function

$$\psi(\mathsf{A}) := \big\|1 - W_{\mathsf{A}}\big\|_{L_2(\Gamma,\mu)}^2.$$

Then ψ may be represented by the cocycle $(\mathcal{H}_\psi, \alpha, \beta)$ with

$$\mathcal{H}_\psi = L_2(\Gamma, \mu), \quad \alpha_{\mathsf{A}}(u) = W_{\mathsf{A}} \cdot u, \quad \beta(\mathsf{A}) = 1 - W_{\mathsf{A}}.$$

Then $\big\{u_j = \alpha_j^{-\frac{1}{2}}\delta_{w_j} : j \in [n]\big\}$ is an ONB and the cocycle derivatives are given by

$$\begin{aligned}\partial_{u_j} W_{\mathsf{A}} &= \frac{2\pi i}{\sqrt{\alpha_j}}\langle\beta(\mathsf{A}), \delta_{w_j}\rangle_\psi W_{\mathsf{A}}\\ &= 4\pi i\sqrt{\alpha_j}\delta_{j\in\mathsf{A}}W_{\mathsf{A}} = 2\pi i\sqrt{\alpha_j}\partial_j W_{\mathsf{A}},\end{aligned}$$

where ∂_j denotes the j-th discrete derivative. Then, Riesz transforms take the form

$$R_{u_j} f = \sum_{\mathsf{A} \subseteq [n]} \frac{\langle \beta(\mathsf{A}), \delta_{w_j} \rangle_\psi}{\sqrt{\alpha_j \psi(\mathsf{A})}} \widehat{f}(\mathsf{A}) W_{\mathsf{A}} = \sum_{\substack{\mathsf{A} \subseteq [n] \\ j \in \mathsf{A}}} \left(\frac{\sqrt{\alpha_j}}{\sqrt{\sum_{\ell \in A} \alpha_\ell}} \right) \widehat{f}(\mathsf{A}) W_{\mathsf{A}}.$$

Consider the decomposition $\mathcal{H}_{\psi,j} = \mathbb{R}\delta_{w_j}$. Note $\alpha_{\mathsf{A}}(\delta_{w_j}) = W_{\mathsf{A}} \delta_{w_j} = (-1)^{\delta_{j \in \mathsf{A}}} \delta_{w_j}$, so $\alpha_{\mathsf{A}}(\mathcal{H}_{\psi,j}) \subseteq \mathcal{H}_{\psi,j}$ and the decomposition is equivariant. Therefore, the associated conditional expectation can be chosen to be

$$\mathsf{E}_{[n] \setminus \mathsf{S}} f = \sum_{\beta(\mathsf{A}) \in \mathcal{H}_{\mathsf{S}}} \widehat{f}(\mathsf{A}) W_{\mathsf{A}} = \sum_{\mathsf{A} \subseteq \mathsf{S}} \widehat{f}(\mathsf{A}) W_{\mathsf{A}}.$$

Finally, Theorem 1.0.1 yields

$$\begin{aligned} \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subseteq [n] \\ |\mathsf{S}| = k}} \left(\sum_{\mathsf{A} \subseteq \mathsf{S}} \left(\widehat{f}(\mathsf{A}) W_{\mathsf{A}} \right)^p_{L_p(\Omega_n)} \right. \\ &\lesssim \frac{1}{\sigma^{p/2}} \frac{k}{n} \sum_{j=1}^n \left(\alpha_j^{\frac{p}{2}} \|\partial_j f\|^p_{L_p(\Omega_n)} + \left(\frac{k}{n} \right)^{\frac{p}{2}} \|f\|^p_{L_p(\Omega_n)} \right. \\ &= \frac{k}{n} \sum_{j=1}^n \left(\left(\frac{\alpha_j}{\min_{k \in [n]} \alpha_k} \right)^{\frac{p}{2}} \|\partial_j f\|^p_{L_p(\Omega_n)} + \left(\frac{k}{n} \right)^{\frac{p}{2}} \|f\|^p_{L_p(\Omega_n)}, \right. \end{aligned}$$

since $\sigma = \min_{k \in [n]} \psi(\{k\}) = 4 \min_{k \in [n]} \alpha_k$. Thus taking $\alpha_j = 1$ for all j is optimal.

Remark 1.2.5. It is natural to ask if the situation changes much when the cyclic groups under consideration have odd cardinal. The function $\psi(g) = \sum_{j=1}^n |g_j|$ with $|g_j| = \min\{g_j, 2m+1-g_j\}$ is a conditionally negative length on $\mathbb{Z}_{2m+1}^n$, and so there exists an associated cocycle induced by the Gromov form

$$\begin{aligned} \langle \delta_g, \delta_h \rangle &= \frac{1}{2} \big(\psi(g) + \psi(h) - \psi(g-h) \big) \\ &= \sum_{j=1}^n \min \Big\{ g_j, 2m+1-h_j, \max \Big\{ 0, m - h_j + g_j + \frac{1}{2} \Big\} \Big\}. \end{aligned}$$

It defines a cocycle Hilbert space $\mathcal{H}_\psi$ with dimension $2mn$. Indeed, it holds

$$w_{g,j} = u_j(g_j) \text{ and } u_j(\ell) = -u_j(\ell + m - 1) \text{ for any } j \in [n],$$

so our question can be reduced to the study of the case $n = 1$. Therefore, the matrix associated to scalar product is

$$
\begin{array}{c}
 \\ 1 \\ 2 \\ \dots \\ m-1 \\ m \\ m+1 \\ m+2 \\ \dots \\ 2m
\end{array}
\begin{pmatrix}
1 & 2 & \dots & m-1 & m & m+1 & m+2 & \dots & 2m-1 & 2m \\
1 & 1 & \dots & 1 & 1 & 1/2 & 0 & \dots & 0 & 0 \\
0 & 2 & \dots & 2 & 2 & 3/2 & 1/2 & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\
1 & 2 & \dots & m-1 & m-1 & \frac{2(m-2)+1}{2} & \frac{2(m-3)+1}{2} & \dots & 1/2 & 0 \\
1 & 2 & \dots & m-1 & m & \frac{2(m-1)+1}{2} & \frac{2(m-2)+1}{2} & \dots & 3/2 & 1/2 \\
1/2 & 3/2 & \dots & \frac{2(m-2)+1}{2} & \frac{2(m-1)+1}{2} & m & m-1 & \dots & 2 & 1 \\
0 & 1/2 & \dots & \dots & \frac{2(m-2)+1}{2} & m-1 & m-1 & \dots & 2 & 1 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\
0 & 0 & \dots & 0 & 1/2 & 1 & 1 & \dots & 1 & 1
\end{pmatrix}
$$

Let us denote this matrix by A. By applying the elementary row operations $row_j = row_j - row_{j-1}$ for $j = 2, \dots, m$ and $row_j = row_j - row_{j+1}$ for $j = m+1, \dots, 2m+1$, we get the matrix

$$
\begin{array}{c}
 \\ 1 \\ 2 \\ \dots \\ m-1 \\ m \\ m+1 \\ m+2 \\ \dots \\ 2m
\end{array}
\begin{array}{c}
\begin{array}{cccccccccc}
1 & 2 & \dots & m-1 & m & m+1 & m+2 & \dots & 2m-1 & 2m
\end{array} \\
\begin{pmatrix}
1 & 1 & \dots & 1 & 1 & 1/2 & 0 & \dots & 0 & 0 \\
0 & 1 & \dots & 1 & 1 & 1 & 1/2 & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\
0 & 0 & \dots & 1 & 1 & 1 & 1 & \dots & 1/2 & 0 \\
0 & 0 & \dots & 0 & 1 & 1 & 1 & \dots & 1 & 1/2 \\
1/2 & 1 & \dots & 1 & 1 & 1 & 0 & \dots & 0 & 0 \\
0 & 1/2 & \dots & 1 & 1 & 1 & 1 & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\
0 & 0 & \dots & 0 & 1/2 & 1 & 1 & \dots & 1 & 1
\end{pmatrix}
\end{array}
$$

By doing the operations $row_j = row_j - row_{j+m}$, and considering the linear system $Ax = 0$, we obtain $x_j = x_{j-m}$ for $j \in [m]$. Then, this reduces our problem to solving the system $By = 0$ where $y = (x_1, \dots, x_m)$ and B is the $m \times m$ matrix

$$
\begin{array}{c}
 \\ 1 \\ 2 \\ \dots \\ m-1 \\ m
\end{array}
\begin{array}{c}
\begin{array}{ccccc}
1 & 2 & \dots & m-1 & m
\end{array} \\
\begin{pmatrix}
3/2 & 1 & \dots & 1 & 1 \\
1 & 3/2 & \dots & 1 & 1 \\
\dots & \dots & \dots & \dots & \dots \\
1 & 1 & \dots & 3/2 & 1 \\
1 & 1 & \dots & 1 & 3/2
\end{pmatrix}
\end{array}
$$

Now, by doing row operations $row_j = row_j - row_m$, we obtain $x_j = x_m$ for any $j \in [m-1]$, so $(m + 1/2)x_m = 0$ and every variable x_j vanishes for any $j \in [2m]$. Therefore, A has rank $2m$, and since it is symmetric, it is diagonalizable with nonzero diagonal elements. This provides an orthogonal basis of cardinal $2m$ for $\mathbb{R}[\mathbb{Z}_{2m+1}]$. Therefore, Theorems 1.0.1 and 1.0.2 apply in this setting.

1.3 Applications to free products

We now explore applications of Theorem 1.0.2 after replacing the direct products in the previous section by free products. Given a free product $G = G_1 * G_2 * \cdots * G_n$ a general element $g \in G$ can always be written in reduced form $g = g_{i_1} g_{i_2} \cdots g_{i_s}$ where $g_{i_k} \in G_{i_k}$ and $i_1 \neq i_2 \neq \cdots \neq i_s$. We shall be working with the free group $\mathbb{F}_n = \mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}$ and with the free product $\mathbb{Z}_{2m}^{*n}$ of n copies of $\mathbb{Z}_{2m}$. In both cases we shall write $g_1, g_2, \ldots, g_n$ for the canonical generators and a generic element will be a word of the form

$$w = g_{i_1}^{\ell_1} g_{i_2}^{\ell_2} \cdots g_{i_s}^{\ell_s}$$

with $i_1 \neq i_2 \neq \cdots \neq i_s$ and ℓ_k in $\mathbb{Z}$ or $\mathbb{Z}_{2m}$ accordingly.

1.3.1 The free group

Define

$$|w| = \sum_{j=1}^{r} |\ell_j| \quad \text{for} \quad w = g_{i_1}^{\ell_1} \ldots g_{i_r}^{\ell_r}.$$

Haagerup proved in [27] that it is conditionally negative. The cocycle structure naturally induced by the word length $|\cdot|$ can be described through the Hilbert space generated by outgoing oriented edges in its Cayley graph. To be more precise, let us consider the following partial order on $\mathbb{F}_n$. Given $w_1 = g_{i_1}^{\ell_1} \ldots g_{i_r}^{\ell_r}$ and $w_2 = g_{j_1}^{t_1} \ldots g_{j_s}^{t_s}$ with $\ell_j, t_j \in \mathbb{Z} \backslash \{0\}$, we say that $w_1 \leq w_2$ when

- $r \leq s$,
- $g_{i_k}^{\ell_k} = g_{j_k}^{t_k}$ for $1 \leq k \leq r-1$,
- $g_{i_r} = g_{j_r}$, $\ell_r t_r > 0$ and $|\ell_r| \leq |t_r|$.

Any $w_1 \leq w_2$ is called an initial subchain of w_2. As we did with elements of cyclic groups equipped with their natural order structure, we can now define predecessors. If $w = g_{i_1}^{\ell_1} \ldots g_{i_r}^{\ell_r} \neq e$, we denote

$$w^- = g_{i_1}^{\ell_1} \ldots g_{i_r}^{\ell_r - \operatorname{sgn}(\ell_r)}.$$

The Gromov form takes the following expression in this case

$$\langle \delta_{w_1}, \delta_{w_2} \rangle_{|\cdot|} = \frac{1}{2}\big(|w_1| + |w_2| - |w_1^{-1} w_2|\big) = |\min\{w_1, w_2\}|,$$

where $\min\{w_1, w_2\}$ denotes the longest word which is an initial chain of both w_1 and w_2. Given $w \neq e$ in $\mathbb{F}_n$, we define $u_w = \delta_w - \delta_{w^-} \in \mathbb{R}[\mathbb{F}_n]$.

Lemma 1.3.1. *If $\mathcal{H}_{|\cdot|} = \mathbb{R}[\mathbb{F}_n] / \mathrm{Ker}\langle \cdot, \cdot \rangle_{|\cdot|}$, the following properties hold:*

- $\mathrm{Ker}(\langle \cdot, \cdot \rangle_{|\cdot|}) = \mathbb{R}\delta_e$.

- $\langle u_w, u_w \rangle_{|\cdot|} = 1$ *for* $w \in \mathbb{F}_n \setminus \{e\}$.
- $\langle u_{w_1}, u_{w_2} \rangle_{|\cdot|} = 0$ *for* $w_1 \neq w_2$ *in* $\mathbb{F}_n \setminus \{e\}$.

Proof. The first claim is trivial. On the other hand, for any $w \in \mathbb{F}_n \setminus \{0\}$,

$$\begin{aligned}\langle u_w, u_w \rangle_{|\cdot|} &= \langle \delta_w - \delta_{w^-}, \delta_w - \delta_{w^-} \rangle = \langle \delta_w, \delta_w \rangle - 2\langle \delta_w, \delta_{w^-} \rangle + \langle \delta_{w^-}, \delta_{w^-} \rangle \\ &= |w| - 2(|w| - 1) + |w| - 1 = 1.\end{aligned}$$

Moreover, if $w_1 \neq w_2$, some cases can be distinguished. Suppose $w_1 < w_2$. Then,

$$\begin{aligned}\langle u_{w_1}, u_{w_2} \rangle &= \langle \delta_{w_1}, \delta_{w_2} \rangle - \langle \delta_{w_1}, \delta_{w_2^-} \rangle - \langle \delta_{w_1^-}, \delta_{w_2} \rangle + \langle \delta_{w_1^-}, \delta_{w_2^-} \rangle \\ &= |w_1| - |w_1| - (|w_1| - 1) + |w_1| - 1 = 0.\end{aligned}$$

In any other case, w_1 and w_2 belong to two different branches of the free group. Then, denote $w(w_1, w_2)$ as the longest common subchain for w_1 and w_2. Therefore,

$$\begin{aligned}\langle u_{w_1}, u_{w_2} \rangle &= \langle \delta_{w_1}, \delta_{w_2} \rangle - \langle \delta_{w_1}, \delta_{w_2^-} \rangle - \langle \delta_{w_1^-}, \delta_{w_2} \rangle + \langle \delta_{w_1^-}, \delta_{w_2^-} \rangle \\ &= w(w_1, w_2) - w(w_1, w_2) - w(w_1, w_2) + w(w_1, w_2) = 0,\end{aligned}$$

and the statement is proved. □

Lemma 1.3.1, together with the identity

$$\delta_{w_0} = \sum_{w \leq w_0} u_w \text{ for any } w_0 \in \mathbb{F}_n \setminus \{e\},$$

this implies that

$$\Big\{ u_w : w \in \mathbb{F}_n \setminus \{e\} \Big\}$$

is an orthonormal basis of $\mathcal{H}_{|\cdot|} = \mathbb{R}[\mathbb{F}_n]/\mathbb{R}\delta_e$. The cocycle map and the cocycle action are determined as usual by $\beta(w) = \delta_w$ and $\alpha_w(\delta_{w'}) = \delta_{ww'} - \delta_w$.The cocycle derivative in the direction of u_w is

$$\partial_{u_w} \lambda(w') = 2\pi i \langle \beta(w'), u_w \rangle \lambda(w') = 2\pi i \delta_{w \leq w'} \lambda(w') \Rightarrow \partial_{u_w} f = 2\pi i \sum_{w \leq w'} \widehat{f}(w') \lambda(w').$$

Next, we decompose $\mathcal{H}_{|\cdot|}$ as

$$\mathcal{H}_{|\cdot|} = \bigoplus_{j=1}^{n} \mathcal{H}_{|\cdot|,j} \quad \text{with} \quad \mathcal{H}_{|\cdot|,j} = \text{span}\Big\{ u_w : g_j \leq w \text{ or } g_j^{-1} \leq w \Big\}.$$

This leads to consider the Fourier truncations

$$\mathsf{E}_{[n]\setminus \mathsf{S}} f := \sum_{w \in \mathbb{F}_{\mathsf{S}}} \widehat{f}(w) \lambda(w).$$

Being conditional expectations, these Fourier truncations are completely contractive and pairwise β-orthogonality holds since we trivially have $\beta(\mathbb{F}_\mathsf{S}) = \beta(\mathbb{F}_\mathsf{S}^{-1}) \subseteq \mathcal{H}_{|\cdot|,\mathsf{S}}$. Next, taking the derivatives

$$\partial_j := \frac{1}{2\pi i}\Big(\partial_{u_{g_j}} + \partial_{u_{g_j^{-1}}}\Big) \quad \text{for} \quad j \in [n],$$

we can readily check that $\partial_{u_w} \circ \partial_{j_1} = \partial_{u_w}$ whenever $u_w \in \mathcal{H}_{|\cdot|,j_1}$. In conclusion, we have checked all the hypotheses to apply Theorem 1.0.2 for our family of Fourier truncations. In this case we get

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \Big\| \sum_{w\in\mathbb{F}_\mathsf{S}} \widehat{f}(w)\lambda(w) \Big\|_p^p \lesssim_p \frac{k}{n}\sum_{j=1}^n \Big[\big\|\partial_j(f)\big\|_p^p + \big\|\partial_j(f^*)\big\|_p^p \Big] + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_p^p. \tag{1.6}$$

Inequality (1.6) is very close to the conjectured free form of Naor's inequality (FN_p) at the beginning of this chapter, with an extra adjoint term which we shall eliminate at the end of the chapter by proving an even stronger inequality.

1.3.2 The free product $\mathbb{Z}_{2m}^{*n}$

A similar analysis applies as well in this case. Given two reduced words $w_1 = g_{i_1}^{\ell_1} \dots g_{i_r}^{\ell_r}$ and $w_2 = g_{j_1}^{t_1} \dots g_{j_s}^{t_s}$ with $\ell_j, t_j \in [2m-1]$, we say that $w_1 \leq w_2$ if and only if

- $r \leq s$,
- $i_k = j_k$ for any $k \in [r]$ and $\ell_k = t_k$ for any $k \in [r-1]$,
- either $\ell_r, t_r \in [m]$ and $i_r \leq j_r$, or $i_r, j_r \in [2m-1] \setminus [m-1]$ and $i_r \geq j_r$.

The map $\psi : \mathbb{Z}_{2m}^{*n} \to \mathbb{R}_+$ given by

$$w = g_{i_1}^{\ell_1} \dots g_{i_r}^{\ell_r} \mapsto \psi(w) = \sum_{k=1}^r |g_{i_j}^{\ell_j}| = \sum_{k=1}^r \min\{\ell_k, 2m-\ell_k\}$$

is a conditionally negative length function [27], with associated Gromov form

$$\begin{aligned} \langle \delta_{w_1}, \delta_{w_2} \rangle_\psi &= \frac{1}{2}\big(\psi(w_1) + \psi(w_2) - \psi(w_1^{-1}w_2)\big) \\ &= \psi\big(\min\{w_1,w_2\}\big) + \frac{1}{2}\big(\psi(\eta_1) + \psi(\eta_2) - \psi(\eta_1^{-1}\eta_2)\big), \end{aligned} \tag{1.7}$$

where $\min\{w_1, w_2\}$ is again the longest common subchain and $w_j = \min\{w_1, w_2\}\eta_j$ for $j = 1, 2$. The second term above is always 0 in the free group $\mathbb{F}_n$, but not necessarily in this case. Given $w = g_{i_1}^{\ell_1} \dots g_{i_r}^{\ell_r} \neq e$ we define $w^- = g_{i_1}^{\ell_1} \dots g_{i_r}^{\ell_r - 1}$ and construct $u_w = \delta_w - \delta_{w^-}$ as usual. Then, we find that

- $\langle u_w, u_w \rangle_\psi = 1$ for every $w \in \mathbb{Z}_{2m}^{*n} \setminus \{e\}$.

- $\langle u_{w_1}, u_{w_2}\rangle_\psi = 0$ when $e \neq w_1^{-1}w_2 \neq g_j^m$ for $j \in [n]$.
- $\langle u_{w_1}, u_{w_2}\rangle_\psi = 0$ when $w_1^{-1}w_2 = g_j^m$ and both w_1, w_2 end with $g_j^{\pm 1}$.
- If $w = g_{i_1}^{\ell_1} \ldots g_{i_r}^{\ell_r}$, then $u_w = -u_{wg_{i_r}^m}$ in $\mathcal{H}_\psi = \mathbb{R}[\mathbb{Z}_{2m}^{*n}]/\mathrm{Ker}\langle\cdot,\cdot\rangle_\psi$.

This proves that

$$\Big\{ u_w : w = g_{i_1}^{\ell_1} \ldots g_{i_r}^{\ell_r} \in \mathbb{Z}_{2m}^{*n} \setminus \{e\} \text{ with } \ell_r \in [m] \Big\}$$

is an orthonormal basis of $\mathcal{H}_\psi = \mathbb{R}[\mathbb{Z}_{2m}^{*n}]/\mathrm{Ker}\langle\cdot,\cdot\rangle_\psi$. We set as usual $\beta(w) = \delta_w$ and $\alpha_w(\delta_{w'}) = \delta_{ww'} - \delta_w$. Among the above properties it is perhaps convenient to justify the last one. Note that $\langle u_w + u_{wg_{i_r}^m}, u_w + u_{wg_{i_r}^m}\rangle_\psi = 0$ if and only if $\langle u_w, u_{wg_{i_r}^m}\rangle_\psi = -1$ but we have

$$\begin{aligned}\langle u_w, u_{wg_{i_r}^m}\rangle_\psi &= \frac{1}{2}\Big(- \psi(g_{i_r}^m) + \psi\big((w^-)^{-1}wg_{i_r}^m\big) + \psi(g_{i_r}^{m-1}) - \psi\big((w^-)^{-1}wg_{i_r}^{m-1}\big)\Big) \\ &= \frac{1}{2}\Big(- \psi(g_{i_r}^m) + \psi(g_{i_r}^{m+1}) + \psi(g_{i_r}^{m-1}) - \psi(g_{i_r}^m)\Big) \\ &= \frac{1}{2}\Big(- m + m - 1 + m - 1 - m\Big) = -1.\end{aligned}$$

If $w = g_{i_1}^{\ell_1} \ldots g_{i_r}^{\ell_r}$ with $\ell_r \in [m]$, derivatives are given by

$$\partial_{u_w}\lambda(w') = 2\pi i\, \langle \beta(w'), u_w\rangle_\psi \lambda(w') = 2\pi i\, \delta_{w' \in W(w)}\lambda(w') \tag{1.8}$$

where $W(w)$ is the set of those words $w' = g_{j_1}^{t_1} \ldots g_{j_s}^{t_s}$ satisfying

$$r \le s, \quad i_k = j_k \text{ for } k \le r, \quad \ell_k = t_k \text{ for } k \le r-1 \quad \text{and} \quad \ell_r \le t_r \le \ell_r + m - 1. \tag{1.9}$$

Indeed, just write $\beta(w') = \delta_{w'} = u_{w'} + \delta_{w'^-} = u_{w'} + u_{w'^-} + \delta_{w'^{--}}$ and so on. The inner product with u_w will be 0 unless we find u_w in our telescopic sum above just once, in which case we get the value 1. Note that it could appear twice due to the identity $u_w = -u_{wg_{i_r}^m}$ recalled above. In that case, they get mutually cancelled and we get 0. This happens when $t_r - \ell_r \in [2m-1] \setminus [m-1]$.

It remains to consider Fourier truncations. As for the free group, our choice is the conditional expectation into the subgroup $\mathbb{Z}_{2m}^{*\mathsf{S}} = \langle g_j : j \in \mathsf{S}\rangle$. Then we consider the decomposition

$$\mathcal{H}_\psi = \bigoplus_{j=1}^n \mathcal{H}_{\psi,j} \quad \text{with} \quad \mathcal{H}_{\psi,j} = \mathrm{span}\Big\{ u_w : g_j \le w \text{ or } g_j^{-1} \le w \Big\}.$$

Our Fourier truncations form an admissible family. Define

$$\partial_j\lambda(w) = \frac{1}{2\pi i}\Big(\partial_{u_{g_j}}\lambda(w) + \partial_{u_{g_j^m}}\lambda(w)\Big) - \delta_{g_{i_1}^{\ell_1} = g_j^m}\lambda(w) \quad \text{for any} \quad w = g_{i_1}^{\ell_1} \ldots g_{i_r}^{\ell_r}.$$

In other words, $\partial_j \lambda(w) = \delta_{i_1=j}\lambda(w)$ for $w \neq e$ and $\partial_{u_w} \circ \partial_j = \partial_{u_w}$ for $u_w \in \mathcal{H}_{\psi,j}$. The construction above yields the form of Theorem 1.0.2 on the von Neumann algebra of the free product $\mathbb{Z}_{2m}^{*n}$

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n] \\ |\mathsf{S}|=k}} \Big\| \sum_{w\in\mathbb{Z}_{2m}^{*\mathsf{S}}} \widehat{f}(w)\lambda(w) \Big\|_p^p \lesssim_p \frac{k}{n}\sum_{j=1}^n \Big(\|\partial_j(f)\|_p^p + \|\partial_j(f^*)\|_p^p \Big) + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_p^p.$$

1.3.3 Free Hilbert transforms

Compared to (FN_p), the form of Theorem 1.0.2 for free groups gives the additional terms $\partial_j(f^*)$. These terms seem to be necessary in the general context of Theorem 1.0.2, but they are removable for free groups—in fact, we shall prove an even stronger inequality—due to a singular behavior of word-length derivatives for free groups. This comes from the following identity

$$\langle \delta_{w_1}, u_{w_2}\rangle_{|\cdot|} = \mathrm{sgn}\big(\langle \delta_{w_1}, u_{w_2}\rangle_{|\cdot|}\big) = \langle \beta(w_1), u_{w_2}\rangle,$$

since the above inner product can only take the values 0 or 1. This means in particular that word-length derivatives can be regarded as free forms of directional Hilbert transforms, which were recently investigated by Mei and Ricard in [55]. To be more precise, if $\mathbb{A}_{\mathsf{S}}$ is the set of free words whose first letter is in $\mathbb{F}_{\mathsf{S}}$, it turns out that distinguished derivatives satisfy

$$\partial_j = \frac{1}{2\pi i}\Big(\partial_{u_{g_j}} + \partial_{u_{g_j^{-1}}}\Big) = \text{Projection onto } \mathbb{A}_{\{j\}}.$$

The free Hilbert transforms for mean-zero f are defined as

$$H_\varepsilon(f) = \sum_{j=1}^n \varepsilon_j \partial_j(f) \quad \text{for} \quad \varepsilon_j = \pm 1.$$

Mei and Ricard proved in [55] the following crucial inequality

$$\|H_\varepsilon f\|_{L_p(\mathcal{L}(\mathbb{F}_n))} \asymp_p \|f\|_{L_p(\mathcal{L}(\mathbb{F}_n))} \quad \text{for any} \quad 1 < p < \infty. \tag{1.10}$$

Theorem 1.3.2. *If $p \geq 2$ and $k \in [n]$, every mean-zero $f \in L_p(\mathcal{L}(\mathbb{F}_n))$ satisfies*

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n] \\ |\mathsf{S}|=k}} \Big\| \sum_{w\in\mathbb{A}_{\mathsf{S}}} \widehat{f}(w)\lambda(w) \Big\|_{L_p(\mathcal{L}(\mathbb{F}_n))}^p \lesssim_p \frac{k}{n}\sum_{j=1}^n \|\partial_j(f)\|_{L_p(\mathcal{L}(\mathbb{F}_n))}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_{L_p(\mathcal{L}(\mathbb{F}_n))}^p.$$

Proof. Define

$$h = \sum_{w\in\mathbb{A}_{\mathsf{S}}} \widehat{f}(w)\lambda(w) = \sum_{j\in\mathsf{S}} \partial_j(f).$$

Applying inequality (1.10) we obtain

$$\|h\|_p \asymp_p \mathbb{E}\, \Big\| H_\varepsilon(h) \Big\|_p = \mathbb{E}\, \Big\| \sum_{j\in\mathsf{S}} \varepsilon_j \partial_j(f) \Big\|_p .$$

The result follows from Theorem 1.1.2 and another application of (1.10) for f. □

Corollary 1.3.3. *Inequality* (FN$_p$) *holds for* $p \geq 2$ *and any mean-zero* $f \in L_p(\mathcal{L}(\mathbb{F}_n))$.

Proof. It follows from Theorem 1.3.2 and the trivial inequality

$$\Big\| \sum_{w\in\mathbb{F}_\mathsf{S}} \widehat{f}(w)\lambda(w) \Big\|_p = \Big\| \sum_{w\in\mathbb{F}_\mathsf{S}} \widehat{h}(w)\lambda(w) \Big\|_p \leq \|h\|_p = \Big\| \sum_{w\in\mathbb{A}_\mathsf{S}} \widehat{f}(w)\lambda(w) \Big\|_p ,$$

where h is defined as in the proof of Theorem 1.3.2, since we note that $\mathbb{F}_\mathsf{S} \subseteq \mathbb{A}_\mathsf{S}$. □

Remark 1.3.4. It is conceivable that Theorem 1.3.2 or at least Corollary 1.3.3 could have been proved as well from a generalized form of Theorem 1.0.2 in the line of Remark 1.1.4, but we have not found an argument using such an approach.

Remark 1.3.5. Hilbert transforms can also be constructed on $\mathcal{L}(\mathbb{Z}_{2m}^{*n})$. They are L_p-bounded maps as well there, as shown in [55, Theorem 3.5]. Therefore, Theorem 1.3.2 can also be proved with this technique replacing $\mathbb{F}_n$ by $\mathbb{Z}_{2m}^{*n}$ in the statement.

Chapter 2

$\mathrm{X_p}$ inequalities and the metric geometry of Banach spaces

A key innovation in [63] is the introduction of a family of inequalities, termed X_p inequalities. Let us briefly recall the results in that paper. We view the set $\mathbb{Z}_{8m}^n$ as a probability space equipped with the normalized counting measure. A metric space $(\mathbb{X}, d)$ is said to be a *metric* X_p *space* if for each $n \in \mathbb{N}$ and $k \in [n]$, there exists $m \in \mathbb{N}$ such that every mapping $f : \mathbb{Z}_{8m}^n \to \mathbb{X}$ satisfies the following estimate for $p \geq 2$

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subseteq [n] \\ |\mathsf{S}| = k}} \mathsf{E}\big[d\big(\mathrm{M}_{4m\varepsilon_{\mathsf{S}}} f(x), f(x)\big)^p\big]$$

$$(\mathrm{MX}_p) \qquad \lesssim_p m^p \left(\frac{k}{n} \sum_{j=1}^n \mathsf{E}\big[d\big(\mathrm{M}_{e_j} f(x), f(x)\big)^p\big] + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \mathsf{E}\big[d\big(\mathrm{M}_{\varepsilon} f(x), f(x)\big)^p\big] \right)$$

Above, the dummy variable (ε, x) belongs to $\Omega_n \times \mathbb{Z}_{8m}^n$, ε_{S} denotes the S-truncated vector $\sum_{j \in \mathsf{S}} \varepsilon_j e_j$, the expectation E is taken in the product probability space, and the operators M_v are just translations given by

$$\mathrm{M}_v f(x) = f(x + v).$$

They are Fourier multipliers, which will become relevant later. Consider the Lebesgue spaces $L_p = L_p(0,1)$. The main result in [63] then asserts that L_p is a metric X_p space, while L_q is not if $2 < q < p$. This and the fact that being a metric X_p space is invariant under the action of bi-Lipschitz maps provides the quantitative nonembeddability argument of L_q into L_p via bi-Lipschitz embeddings. Given two metric spaces $(\mathbb{X}, d_{\mathbb{X}})$ and $(\mathbb{Y}, d_{\mathbb{Y}})$, we say that that $(\mathbb{X}, d_{\mathbb{X}})$ admits a *bi-Lipschitz embedding* into $(\mathbb{Y}, d_{\mathbb{Y}})$ whenever there exist $s \in (0, \infty)$ and $D \geq 1$ and a map $f : \mathbb{X} \to \mathbb{Y}$ such that there holds

$$s \ d_{\mathbb{X}}(x, y) \leq d_{\mathbb{Y}}(f(x), f(y)) \leq sD \ d_{\mathbb{X}}(x, y) \text{ for any } x, y \in \mathbb{X}.$$

Then, the *bi-Lipschitz distortion* of $\mathbb{X}$ into $\mathbb{Y}$ will be defined as the infimum over those constants D for which this happens for some s and f, and we will denote it by $c_{\mathbb{Y}}(\mathbb{X})$.

Note that the metric X_p inequality for L_p is not true independently of the value of m. It is necessary that it scales at least so that $m \gtrsim \sqrt{n/k}$, even though the bound was not found yet to be sharp in [63]. Little after it, Naor found in [60] a different proof of the metric X_p inequality which was quantitatively optimal. Namely, it turns out that the necessary bound $m \gtrsim \sqrt{n/k}$ is, indeed, sufficient. Once (N_p) has been established, the metric X_p inequality for all classical L_p spaces follows from it plus the natural injection of the multiplicative group Ω_n into $\mathbb{Z}_{2m}^n$ needed to define the multipliers $\mathrm{M}_\varepsilon(f)(x) = f(x+\varepsilon)$ for $f : \mathbb{Z}_{2m}^n \to \mathbb{C}$. This is the starting point of our work. We seek new metric X_p inequalities that can be proven using appropriate versions of (N_p). In chapter 1 (or [8]), a wide range of inequalities of this kind is established for the (noncommutative) L_p spaces associated with the group von Neumann algebra $\mathcal{L}(\mathrm{G})$ of a discrete group G. Seeking the greatest possible generality, we shall consider below different choices of abelian groups H that will play the role of $\mathbb{Z}_2^n$ in [60]. Therefore, in the evaluation $\mathrm{M}_\varepsilon f(x) := f(x+\varepsilon)$, the variable $\varepsilon \in \widehat{\mathrm{H}}$ but that may not be the case for x, which imposes a compatibility condition between the two variables. This is a crucial difficulty and the main contribution in this chapter is to clarify when a chaos-type inequality like (N_p) —as those obtained in chapter 1— can be used to deduce a new metric X_p-type inequality.

Let us now explain our abstract result, we refer to chapter 1 for notation regarding analysis on group von Neumann algebras. Let H be a discrete abelian group which can be written as a direct product:

$$\mathrm{H} = \mathrm{H}_1 \times \mathrm{H}_2 \times \cdots \times \mathrm{H}_n,$$

equipped with its counting measure. Given $\mathsf{S} \subseteq [n]$, we define the truncation of H on S as the subgroup $\mathrm{H}_{\mathsf{S}} = \times_{j \in \mathsf{S}} \mathrm{H}_j \hookrightarrow \mathrm{H}$. Truncations are transferred to the dual group $\widehat{\mathrm{H}}$, which is a probability space with Haar measure dx. Assume also that G is an arbitrary discrete group, and let $p \geq 2$. We say that (H, G) is an X_p-*representable pair* if the following conditions hold:

(RP-1) **Balanced truncations.** We have

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subseteq [n] \\ |\mathsf{S}| = k}} \big\| (\mathsf{E}_{[n]\setminus \mathsf{S}} \otimes \mathrm{Id}) h \big\|_p^p \lesssim_p \frac{k}{n} \sum_{j=1}^n \big\| (\partial_j \otimes \mathrm{Id}) h \big\|_p^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|h\|_p^p,$$

for $k \in [n]$, any mean-zero $h \in L_p(\widehat{\mathrm{H}}; L_p(\mathcal{L}(\mathrm{G})))$, and where

$$\mathsf{E}_{[n]\setminus \mathsf{S}} f(x) = \int_{\widehat{\mathrm{H}}} f(x_{\mathsf{S}} + z_{[n]\setminus \mathsf{S}})\, dz$$

is the conditional expectation onto functions which only depend on variables in $\widehat{\mathrm{H}}_{\mathsf{S}}$. The ∂_j are linear maps which play the role of directional derivatives.

(RP-2) **Compatibility conditions.** There exists an abelian group

$$\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$$

and a map $\eta : \widehat{\mathrm{H}} \to \widehat{\Gamma}$ such that the following compatibility conditions hold:

i) **Symmetric inclusion.** Given $\mathsf{S} \subseteq [n]$, the variables $\eta_{\mathsf{S}}(y) := (\eta(y))_{\mathsf{S}}$ and $-\eta_{\mathsf{S}}(y)$ (using here additive notation) have the same distribution.

ii) **Uniformly bounded translations.** There exists a group of Fourier multipliers $\{\mathrm{M}_\gamma\}_{\gamma \in \widehat{\Gamma}}$, completely and uniformly bounded on $L_p(\mathcal{L}(\mathrm{G}))$.

Theorem 2.0.1. *Let $p \geq 2$ and (H, G) be an X_p-representable pair. Let $k \in [n]$ and suppose that $m \in \mathbb{N}$ satisfies $m \geq \sqrt{n/k}$. Then, for every $f \in L_p(\mathcal{L}(\mathrm{G}))$ there holds*

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subseteq [n] \\ |\mathsf{S}| = k}} \int_{\widehat{\mathrm{H}}} \big\| \mathrm{M}_{4m\eta_{\mathsf{S}}(y)} f - f \big\|_p^p \, dy$$
$$\lesssim_p m^p \left(\frac{k}{n} \sum_{j=1}^{n} \int_{\widehat{\mathrm{H}}} \big\| \partial_j \mathrm{M}_{2\eta(y)} f \big\|_p^p \, dy + \left(\frac{k}{n}\right)^{\frac{p}{2}} \int_{\widehat{\mathrm{H}}} \big\| \mathrm{M}_{\eta(y)} f - f \big\|_p^p \, dy \right).$$

Metric X_p inequalities from [60, 63] apply for functions $f : \mathbb{Z}_{8m}^n \to \mathbb{X}$. Theorem 2.0.1 is an inequality for $f \in L_p(\mathcal{L}(\mathrm{G}))$, so that it should be regarded as a generalization where we replace $\mathbb{Z}_{8m}^n$ by the dual group of G. Note that its dual is a quantum group rather than a classical group when G is nonabelian. Additionally, the role of H is merely to provide an index set over which translations are performed, which generalizes the former role of $\mathbb{Z}_2^n$. In particular, Theorem 2.0.1 generalizes (MX_p) with $(\mathrm{H}, \mathrm{G}) = (\mathbb{Z}_2^n, \mathbb{Z}_{8m}^n)$, auxiliary group $\Gamma = \mathbb{Z}_{8m}^n$ and η being the natural inclusion defined coordinatewise by $1 \mapsto 1$ and $(-1) \mapsto 8m - 1$. Note that

$$(\partial_j) \qquad \int_{\mathbb{Z}_2^n} \big\| \partial_j \mathrm{M}_{2\eta(y)} f \big\|_p^p \, dy \lesssim_p \|\mathrm{M}_{e_j} f - f\|_p^p$$

holds when ∂_j is the aforementioned discrete derivative in the j-th direction. The role of Γ was hidden in [60, 63]. This is natural since it is possible to formulate Theorem 2.0.1 without appealing to the existence of η and Γ. Indeed, one can find axioms on a family of multipliers $\{\mathrm{M}_y\}_{y \in \mathrm{H}}$ and their powers —so that they act like translations— which are formally weaker than (RP-1) and (RP-2) and suffice to get the conclusion of Theorem 2.0.1. We have chosen to formulate our result in this manner because in all of our examples we may identify the auxiliary objects η and Γ, which is an efficient way of checking that we can apply Theorem 2.0.1. However the existence of weaker conditions can be useful for future constructions and we will detail them in Section 2.1. In the general case, the improved statement that one gets when an inequality like (∂_j) holds is a true metric inequality, while the conclusion of Theorem 2.0.1 is not in general. In our other examples, we shall choose derivatives whose relation with the multipliers M_v yields the stronger, metric statement.

The proof of Theorem 2.0.1 follows the strategy in [60]. Our main contribution is the identification of the right conditions under which said strategy can be extended to numerous contexts, including noncommutative ones. We illustrate Theorem 2.0.1 with several concrete scenarios, some of which we describe next. In each case, (RP-1) is checked using the results in chapter 1 and we focus in getting fully metric statements.

i) **Continuous metric X_p inequalities.** Taking $\mathrm{H} = \mathrm{G} = \mathbb{Z}^n$, we obtain two continuous statements in the n-dimensional torus using two different choices for the inclusion η. One of them can be obtained from [60] by an approximation procedure, while the other seems to require Theorem 2.0.1.

ii) **Cyclic extensions of metric X_p inequalities.** Let η_ℓ be the inclusion that maps $\mathbb{Z}_{2\ell}^n$ into $\{k : k \in [\ell] \text{ or } 8m\ell - k \in [\ell]\} \subseteq \mathbb{Z}_{8m\ell}^n$. We prove a metric X_p inequality which reduces to (MX_p) for $\ell = 1$. It is sharp in terms of the scaling parameter m and entails the same metric consequences as [60].

ii) **Transferred X_p inequalities.** When G is nonabelian, we first observe that our results in chapter 1 hold with values on noncommutative L_p spaces over QWEP algebras (see [69]), so that (RP-1) is satisfied. Next, using an appropriate corepresentation for (H, G) we obtain X_p inequalities on the free group $\mathbb{F}_n$.

The above scenarios show the applicability of Theorem 2.0.1. In metric terms we observe that noncommutative L_p spaces are metric X_p spaces. This implies that the distortion estimates for ℓ_q-grids and L_p-snowflakes in [60] still hold when considering embeddings into noncommutative L_p. Beyond this, we have not found significantly new applications of our main results in the context of metric geometry so far. It would be very interesting to generalize Theorem 2.0.1 so as to include X_p-representable pairs (H, G) admitting *nonabelian translations* in the index set H.

2.1 Metric X_p inequalities

We emphasize again that we follow the strategy of proof in [60] for an X_p-representable pair (H, G), as defined at the beginning of this chapter. Given $\mathsf{S} \subseteq [n]$ consider the auxiliary operator T_S on $\mathcal{L}(\mathrm{G})$ given by

$$T_\mathsf{S} f = \int_{\widehat{\mathrm{H}}} \mathrm{M}_{2\eta_\mathsf{S}(y)} f \, dy, \quad \text{for} \quad f \in L_p(\mathcal{L}(\mathrm{G})).$$

Proof of Theorem 2.0.1 For each $\mathsf{S} \subseteq [n]$ with $|\mathsf{S}| = k$ and $y \in \mathrm{H}$

$$\begin{aligned}\|\mathrm{M}_{4m\eta_{\mathsf{S}}(y)}f - f\|_p^p &\lesssim_p \|T_{[n]\setminus\mathsf{S}}f - f\|_p^p \\ &+ \|\mathrm{M}_{4m\eta_{\mathsf{S}}(y)}T_{[n]\setminus\mathsf{S}}f - T_{[n]\setminus\mathsf{S}}f\|_p^p \\ &+ \|\mathrm{M}_{4m\eta_{\mathsf{S}}(y)}f - \mathrm{M}_{4m\eta_{\mathsf{S}}(y)}T_{[n]\setminus\mathsf{S}}f\|_p^p =: \mathrm{A}+\mathrm{B}+\mathrm{C}.\end{aligned}$$

First, we claim that

$$(2.1)\qquad \|T_{\mathsf{S}}f - f\|_p^p \lesssim_p \int_{\widehat{\mathrm{H}}} \|\mathrm{M}_{\eta(y)}f - f\|_p^p \, dy.$$

Indeed, by convexity of the L_p norm

$$\begin{aligned}\|T_{\mathsf{S}}f - f\|_p^p &= \Big\|\int_{\widehat{\mathrm{H}}} \mathrm{M}_{2\eta_{\mathsf{S}}(y)}f\, dy - f\Big\|_p^p \lesssim_p \int_{\widehat{\mathrm{H}}} \|\mathrm{M}_{2\eta_{\mathsf{S}}(y)}f - f\|_p^p\, dy \\ &\le 2^{p-1}\Big(\int_{\widehat{\mathrm{H}}} \|\mathrm{M}_{2\eta_{\mathsf{S}}(y)}f - \mathrm{M}_{\eta(y)}f\|_p^p\, dy + \int_{\widehat{\mathrm{H}}} \|\mathrm{M}_{\eta(y)}f - f\|_p^p\, dy\Big)\end{aligned}$$

Now, $\eta(y) = \eta_S(y) + \eta_{[n]\setminus S}(y)$ and $\eta_S(y) - \eta_{[n]\setminus S}(y)$ are identically distributed by (RP-2). Therefore, by the properties of the family $\{\mathrm{M}_\varepsilon\}_\varepsilon$ —also in (RP-2)— the first term above satisfies

$$\begin{aligned}\int_{\widehat{\mathrm{H}}} \|\mathrm{M}_{2\eta_{\mathsf{S}}(y)}f - \mathrm{M}_{\eta(y)}f\|_p^p\, dy &= \int_{\widehat{\mathrm{H}}} \|\mathrm{M}_{\eta(y)}\mathrm{M}_{\eta_{\mathsf{S}}(y)-\eta_{[n]\setminus\mathsf{S}}(y)}f - \mathrm{M}_{\eta(y)}f\|_p^p\, dy \\ &\lesssim \int_{\widehat{\mathrm{H}}} \|\mathrm{M}_{\eta_{\mathsf{S}}(y)-\eta_{[n]\setminus\mathsf{S}}(y)}f - f\|_p^p\, dy \\ &= \int_{\widehat{\mathrm{H}}} \|\mathrm{M}_{\eta(y)}f - f\|_p^p\, dy,\end{aligned}$$

which implies the claim. This proves that

$$\mathrm{A}+\mathrm{C} \lesssim \mathrm{A} \lesssim \int_{\widehat{\mathrm{H}}} \|\mathrm{M}_{\eta(y)}f - f\|_p^p\, dy.$$

For the term B, our claim is the following

$$\begin{aligned}(2.2)\qquad & \frac{m^{-p}}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\ |\mathsf{S}|=k}} \int_{\widehat{\mathrm{H}}} \|\mathrm{M}_{4m\eta_{\mathsf{S}}(y)}T_{[n]\setminus\mathsf{S}}f - T_{[n]\setminus\mathsf{S}}f\|_p^p\, dy \\ &\lesssim_p \frac{k}{n}\sum_{j=1}^n \int_{\widehat{\mathrm{H}}} \|\partial_j \mathrm{M}_{2\eta(y)}f\|_p^p\, dy + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \int_{\widehat{\mathrm{H}}} \|\mathrm{M}_{\eta(y)}f - f\|_p^p\, dy.\end{aligned}$$

To prove (2.2), we start once more with (RP-2) and the triangle inequality

$$
\begin{aligned}
(2.3)\qquad & \Big(\int_{\widehat{\mathrm{H}}} \big\| \mathrm{M}_{4m\eta_{\mathsf{S}}(y)} T_{[n]\setminus\mathsf{S}} f - T_{[n]\setminus\mathsf{S}} f \big\|_p^p \, dy \Big)^{\frac{1}{p}} \\
\leq \ & \sum_{j=1}^{m} \Big(\int_{\widehat{\mathrm{H}}} \big\| \mathrm{M}_{4j\eta_{\mathsf{S}}(y)} T_{[n]\setminus\mathsf{S}} f - \mathrm{M}_{4(j-1)\eta_{\mathsf{S}}(y)} T_{[n]\setminus\mathsf{S}} f \big\|_p^p \, dy \Big)^{\frac{1}{p}} \\
= \ & \sum_{j=1}^{m} \Big(\int_{\widehat{\mathrm{H}}} \big\| \mathrm{M}_{(4j-2)\eta_{\mathsf{S}}(y)} [\mathrm{M}_{2\eta_{\mathsf{S}}(y)} T_{[n]\setminus\mathsf{S}} f - \mathrm{M}_{-2\eta_{\mathsf{S}}(y)} T_{[n]\setminus\mathsf{S}} f] \big\|_p^p \, dy \Big)^{\frac{1}{p}} \\
\lesssim \ & m \Big(\int_{\widehat{\mathrm{H}}} \big\| \underbrace{\mathrm{M}_{2\eta_{\mathsf{S}}(y)} T_{[n]\setminus\mathsf{S}} f - \mathrm{M}_{-2\eta_{\mathsf{S}}(y)} T_{[n]\setminus\mathsf{S}} f}_{F_{\mathsf{S}}(y)} \big\|_p^p \, dy \Big)^{\frac{1}{p}}.
\end{aligned}
$$

We claim that $F_{\mathsf{S}}(y) = \mathsf{E}_{[n]\setminus\mathsf{S}} h(y)$ with

$$
\begin{aligned}
h &: \widehat{\mathrm{H}} \to L_p(\mathcal{L}(\mathrm{G})), \\
h(y) &= \mathrm{M}_{2\eta(y)} f - \mathrm{M}_{-2\eta(y)} f.
\end{aligned}
$$

Indeed, recall that

$$\mathsf{E}_{[n]\setminus\mathsf{S}} g(x) = \int_{\widehat{\mathrm{H}}} g(x_{\mathsf{S}} + z_{[n]\setminus\mathsf{S}}) \, dz.$$

Also, denote the symbol of M_ε by m_ε, so that

$$\mathrm{M}_\varepsilon f = \sum_{w \in \mathrm{G}} \mathrm{m}_\varepsilon(w) \widehat{f}(w) \lambda(w),$$

where $\lambda(g)$ denotes the left regular representation of G. Then, given that $\mathsf{E}_{[n]\setminus\mathsf{S}}$ is linear, and using again the equidistribution property in (RP-2), we deduce that

$$
\begin{aligned}
\mathsf{E}_{[n]\setminus\mathsf{S}} h(y) &= \sum_{w \in \mathrm{G}} \widehat{f}(w) \lambda(w) \, [\mathsf{E}_{[n]\setminus\mathsf{S}} \mathrm{m}_{2\eta(y)}(w) - \mathsf{E}_{[n]\setminus\mathsf{S}} \mathrm{m}_{-2\eta(y)}(w)] \\
&= \sum_{w \in \mathrm{G}} \widehat{f}(w) \lambda(w) \, [\mathrm{m}_{2\eta_{\mathsf{S}}(y)}(w) - \mathrm{m}_{-2\eta_{\mathsf{S}}(y)}(w)] \int_{\widehat{\mathrm{H}}} \mathrm{m}_{2\eta_{[n]\setminus\mathsf{S}}(z)}(w) \, dz \\
&= \mathrm{M}_{2\eta_{\mathsf{S}}(y)} T_{[n]\setminus\mathsf{S}} f - \mathrm{M}_{-2\eta_{\mathsf{S}}(y)} T_{[n]\setminus\mathsf{S}} f = F_{\mathsf{S}}(y).
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
\int_{\widehat{\mathrm{H}}} h(y) \, dy &= \int_{\widehat{\mathrm{H}}} \mathsf{E}_{[n]\setminus\mathsf{S}} h(y) \, dy \\
&= \int_{\widehat{\mathrm{H}}} \big[\mathrm{M}_{2\eta_{\mathsf{S}}(y)} T_{[n]\setminus\mathsf{S}} f - \mathrm{M}_{-2\eta_{\mathsf{S}}(y)} T_{[n]\setminus\mathsf{S}} f \big] \, dy \\
&= \int_{\widehat{\mathrm{H}}} \mathrm{M}_{2\eta_{\mathsf{S}}(y)} T_{[n]\setminus\mathsf{S}} f \, dy - \int_{\widehat{\mathrm{H}}} \mathrm{M}_{2\eta_{\mathsf{S}}(y)} T_{[n]\setminus\mathsf{S}} f \, dy = 0
\end{aligned}
$$

since $2\eta_S(y)$ and $-2\eta_S(y)$ are identically distributed. Therefore, $h \in L_p^\circ(\widehat{\mathrm{H}}; L_p(\mathcal{L}(\mathrm{G})))$, so we raise to the power p and average over $\mathsf{S} \subseteq [n]$ in (2.3), and by (RP-1) we get

$$\begin{aligned}
&\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\ |\mathsf{S}|=k}} \int_{\widehat{\mathrm{H}}} \big\| \mathrm{M}_{4m\eta_{\mathsf{S}}(y)} T_{[n]\setminus\mathsf{S}} f - T_{[n]\setminus\mathsf{S}} f \big\|_p^p \, dy \\
&\quad \lesssim_p \frac{m^p}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\ |\mathsf{S}|=k}} \int_{\widehat{\mathrm{H}}} \big\| \mathsf{E}_{[n]\setminus\mathsf{S}} h(y) \big\|_p^p \, dy \\
&\quad \lesssim_p m^p \Big(\frac{k}{n} \sum_{j=1}^n \int_{\widehat{\mathrm{H}}} \big\| (\partial_j \otimes \mathrm{Id}) h(y) \big\|_p^p \, dy + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \int_{\widehat{\mathrm{H}}} \|h(y)\|_p^p \, dy \Big) \\
&\quad \lesssim m^p \Big(\frac{k}{n} \sum_{j=1}^n \int_{\widehat{\mathrm{H}}} \big\| \partial_j \mathrm{M}_{2\eta(y)} f \big\|_p^p \, dy + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \int_{\widehat{\mathrm{H}}} \|\mathrm{M}_{\eta(y)} f - f\|_p^p \, dy \Big).
\end{aligned}$$

Therefore, putting all together we get

$$\begin{aligned}
&\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\ |\mathsf{S}|=k}} \int_{\widehat{\mathrm{H}}} \big\| \mathrm{M}_{4m\eta_{\mathsf{S}}(y)} f - f \big\|_p^p \, dy \\
&\quad \lesssim m^p \Big(\frac{k}{n} \sum_{j=1}^n \int_{\widehat{\mathrm{H}}} \big\| \partial_j \mathrm{M}_{2\eta(y)} f \big\|_p^p \, dy + \Big(\Big(\frac{k}{n}\Big)^{\frac{p}{2}} + \frac{1}{m^p} \Big) \int_{\widehat{\mathrm{H}}} \big\| \mathrm{M}_{\eta(y)} f - f \big\|_p^p \, dy \Big).
\end{aligned}$$

This completes the proof for any $m \geq \sqrt{n/k}$, as imposed in the statement. □

Remark 2.1.1. A close inspection of the proof of Theorem 2.0.1 shows that one can exchange requirement (RP-2) by the existence of a family of invertible operators $\{\mathrm{M}_y\}_{y\in\widehat{\mathrm{H}}}$ satisfying the following properties:

- *Product structure.* $\mathrm{M}_y = \mathrm{M}_{y_{\mathsf{S}}} \mathrm{M}_{y_{[n]\setminus\mathsf{S}}}$.
- *Uniform boundedness.* $\max_{|k|\leq 4m} \sup_{y\in\widehat{\mathrm{H}}} \big\| \mathrm{M}_y^k \big\|_{p\to p} < \infty$.
- *Symmetry.* $\mathrm{M}_y^k f$ and $\mathrm{M}_y^{-k} f$ have the same distribution function for each f.

The above conditions are formally weaker than (RP-2), and more importantly, they clarify the fact that the auxiliary group Γ is not necessary in the definition of X_p-representable pairs. However, in this chapter we have chosen to work with the stronger condition (RP-2) because in all of our examples we are able to identify the associated group Γ and this makes checking the conditions of Theorem 2.0.1 simpler.

Remark 2.1.2. In many of the cases of interest there holds

$$\int_{\widehat{\mathrm{H}}} \big\| \partial_j \mathrm{M}_{2\eta(y)} f \big\|^p_{L_p(\mathcal{L}(\mathrm{G}))} \, dy \lesssim_p \big\| \mathrm{M}_{e_j} f - f \big\|^p_{L_p(\mathcal{L}(\mathrm{G}))}, \tag{2.4}$$

for each $j = 1, \ldots, n$. This improves the conclusion of Theorem 2.0.1 to a truly metric inequality, which resembles the one in [60] more closely. This highlights the importance of the choice of the family of derivatives when checking conditions (RP-1)-(RP-2). One case in which this holds is when derivatives and conditional expectations are related by

$$\partial_j = \mathrm{Id} - \mathsf{E}_{\{j\}} \quad \text{for} \quad 1 \leq j \leq n, \tag{2.5}$$

when ∂_j is also known as the *j-th coordinate Laplacian operator* [64]. In turn, (2.5) holds when $\partial_j \chi_w(x) = \delta_{w_j \neq 0} \, \chi_w(x)$ for any character χ_w on $\widehat{\mathrm{H}}$.

2.2 Intrinsic X_p inequalities

In this section we explore intrinsic X_p inequalities in the framework provided by Theorem 2.0.1. The term "intrinsic" means here that there exists a natural inclusion of the index group H into the X_p-group G, so there is no need to use an auxiliary group Γ other than G. In our setting, this unfortunately forces us to work with $\mathrm{G} = \Gamma$ abelian. An extension of Theorem 2.0.1 including noncommutative translations indexed by nonabelian groups H would open a door to potential applications in the metric geometry of noncommutative L_p-spaces. Intrinsic X_p inequalities will be complemented below with "transferred X_p inequalities" which refer to those which admit some $\Gamma \neq \mathrm{G}$, including nonabelian X_p-groups in the picture. We will illustrate this scenario in the context of free groups below. In the language of quantum group theory, intrinsic H-translations are given by G-comultiplications, while transferred H-translations require more general G-corepresentations in terms of Γ.

2.2.1 Continuous X_p inequalities

Let $\mathrm{H} = \mathrm{G} = \mathbb{Z}^n$, so that the dual groups are n-dimensional tori, which we identify with $[-1/2, 1/2)^n$ for convenience. We also pick $\Gamma = \mathbb{Z}^n$, and we explore two different choices for $\eta : \mathbb{T}^n \to \mathbb{T}^n$. First, we take the map $\eta(y)_j = y_j/4m$, so each component of η maps $\mathbb{T}^n$ to $[-1/(8m), 1/(8m))$. The following encodes what follows from Theorem 2.0.1 in this case.

Proposition 2.2.1. *If $p \geq 2$, $k \in [n]$ and $m \geq \sqrt{n/k}$, every $f : \mathbb{T}^n \to \mathbb{C}$ satisfies*

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subseteq [n] \\ |\mathsf{S}| = k}} \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} \big| f(x + y_{\mathsf{S}}) - f(x) \big|^p \, dy dx$$
$$\lesssim_p m^p \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} \bigg\{ \frac{k}{n} \sum_{j=1}^n \Big| f\Big(x + \frac{y_j e_j}{4m}\Big) - f(x) \Big|^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \Big| f\Big(x + \frac{y}{4m}\Big) - f(x) \Big|^p \bigg\} dy dx.$$

Proof. We start checking that $(\mathbb{Z}^n, \mathbb{Z}^n)$ is an X_p-representable pair. Taking the group $\Gamma = \mathbb{Z}^n$, it is clear that (RP-2) holds with our choice of η and multipliers given by the usual translations $\mathrm{M}_\gamma f(x) = f(x+\gamma)$ which are unitary modulations at the Fourier side. Second, if $\chi_w = \exp(2\pi i \langle w, \cdot\rangle)$ is the character associated with $w \in \mathbb{Z}^n$, we consider the differential operators

$$\partial_j \chi_w = \delta_{w_j \neq 0}\, \chi_w \quad \text{for} \quad j \in [n].$$

By Subsection 1.2.1, we have

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\ |\mathsf{S}|=k}} \Big\| \mathsf{E}_{[n]\setminus \mathsf{S}} f \Big\|_{L_p(\mathbb{T}^n)}^p \lesssim_p \frac{k}{n} \sum_{j=1}^n \|\partial_j f\|_{L_p(\mathbb{T}^n)}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_{L_p(\mathbb{T}^n)}^p.$$

In particular, by Fubini's theorem we get (RP-1). Then, Theorem 2.0.1 yields

$$\begin{aligned}
(2.6)\qquad & \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\ |\mathsf{S}|=k}} \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \big| f(x+y_{\mathsf{S}}) - f(x) \big|^p \, dy dx \\
& \lesssim_p m^p \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \bigg(\frac{k}{n} \sum_{j=1}^n \Big| \partial_j^y f\Big(x + \frac{y}{2m}\Big) \Big|^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \Big| f\Big(x + \frac{y}{4m}\Big) - f(x) \Big|^p \bigg) dy dx,
\end{aligned}$$

using the superscripts in the partial derivatives to indicate the variable over which differentiation is performed. It only remains to estimate (2.6) to establish the result. In order to do that, observe that (2.5) holds in this case. Applying it to $h(y) = f(x + y/2m)$ and using the properties of translations we get

$$\begin{aligned}
& \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \Big| \partial_j^y f\Big(x + \frac{y}{2m}\Big) \Big|^p dy dx \\
& = \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \Big| f\Big(x + \frac{y}{2m}\Big) - \mathsf{E}_{\{j\}}^y f\Big(x + \frac{y}{2m}\Big) \Big|^p dy dx \\
& = \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \Big| f\Big(x + \frac{y}{2m}\Big) - \int_{\mathbb{T}} f\Big(x + \frac{y_{[n]\setminus\{j\}} + te_j}{2m}\Big) dt \Big|^p dy dx \\
& \leq \int_{\mathbb{T}^n}\int_{\mathbb{T}^n}\int_{\mathbb{T}} \Big| f\Big(x + \frac{y}{2m}\Big) - f\Big(x + \frac{y_{[n]\setminus\{j\}} + te_j}{2m}\Big) \Big|^p dt dy dx \\
& = \int_{\mathbb{T}^n}\int_{\mathbb{T}^n}\int_{\mathbb{T}} \Big| f\Big(x + \frac{y_j e_j}{2m}\Big) - f\Big(x + \frac{te_j}{2m}\Big) \Big|^p dt dy dx \\
& = \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \Big| f\Big(x + \frac{y_j e_j}{2m}\Big) - f(x) \Big|^p dy dx \leq 2 \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \Big| f\Big(x + \frac{y_j e_j}{4m}\Big) - f(x) \Big|^p dy dx.
\end{aligned}$$

Inserting that in the outcome of Theorem 2.0.1 yields the assertion. $\square$

Remark 2.2.2. A different choice of $\eta : \mathbb{T}^n \to \mathbb{T}^n$ is given by $\eta(y)_j = \mathrm{sgn}(y_j)/8m$ which leads to an inequality closer to (MX_p). Recalling that we identify $\mathbb{T}$ with $[-1/2, 1/2)$, it turns out that $4m\eta(y) \equiv -e/2$ for $e = (1,1,\ldots,1)$, and one can prove the following statement for every $f : \mathbb{T}^n \to \mathbb{C}$

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \int_{\mathbb{T}^n} \Big| f\Big(x + \frac{e_{\mathsf{S}}}{2}\Big) - f(x)\Big|^p dx \tag{2.7}$$
$$\lesssim_p m^p \int_{\mathbb{T}^n} \Big(\frac{k}{n} \sum_{j=1}^n \Big| f\Big(x + \frac{e_j}{8m}\Big) - f(x)\Big|^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \frac{1}{2^n} \sum_{\varepsilon\in\Omega_n} \Big| f\Big(x + \frac{\varepsilon}{8m}\Big) - f(x)\Big|^p \Big) dx.$$

The proof of (2.7) follows the same lines as that of Proposition 2.2.1. It is also worth noting that inequality (2.7) can be discretized to recover (MX_p) and a limiting procedure allows one to pass from (MX_p) to (2.7), we omit the details. This means that this *semi-continuous* inequality is equivalent to Naor's original X_p inequality.

Remark 2.2.3. It is also possible to consider the usual differential structure on $\mathbb{T}^n$ which can be thought of as the most natural. In that case, L_p valued versions of estimates for balanced truncations of Fourier series hold by Fubini's theorem and one gets the following

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} |f(x + y_{\mathsf{S}}) - f(x)|^p \, dydx$$
$$\lesssim_p m^p \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} \Big(\frac{k}{n} \sum_{j=1}^n \Big| \partial_{y_j} f\Big(x + \frac{y}{2m}\Big)\Big|^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \Big| f\Big(x + \frac{y}{4m}\Big) - f(x)\Big|^p \Big) dydx,$$

which can be seen to be weaker than those from Proposition 2.2.1, see chapter 1 for details.

2.2.2 Cyclic groups with the word length

We now focus on a discrete inequality that is a strict generalization of (MX_p). Consider the inclusion $\eta_\ell : \mathbb{Z}_{2\ell}^n \to \mathbb{Z}_{8\ell m}^n$ given by

$$\eta_\ell(x) = \big(\beta_\ell(x_1), \ldots, \beta_\ell(x_n)\big), \tag{2.8}$$

where $\beta_\ell(y) = y - \ell$ when $0 \le y \le \ell - 1$ and $\beta_\ell(y) = y - (\ell - 1)$ when $\ell \le y \le 2\ell - 1$.

Proposition 2.2.4. *If $p \ge 2$, $k \in [n]$ and $m \ge \sqrt{n/k}$, every $f : \mathbb{Z}_{8\ell m}^n \to \mathbb{C}$ satisfies*

$$\frac{m^{-p}}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \Big(\frac{1}{(8\ell m)^n (2\ell)^n} \sum_{x\in\mathbb{Z}_{8\ell m}^n} \sum_{y\in\mathbb{Z}_{2\ell}^n} \big| f\big(x + 4m\eta_\ell(y)_{\mathsf{S}}\big) - f(x)\big|^p \Big)$$
$$\lesssim_p \quad (4\ell)^{p-1} \frac{k}{n} \sum_{j=1}^n \frac{1}{(8\ell m)^n} \sum_{x\in\mathbb{Z}_{8\ell m}^n} |f(x + e_j) - f(x)|^p$$
$$+ \quad \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \frac{1}{(8\ell m)^n (2\ell)^n} \sum_{x\in\mathbb{Z}_{8\ell m}^n} \sum_{y\in\mathbb{Z}_{2\ell}^n} |f(x + \eta_\ell(y)) - f(x)|^p.$$

Proof. Choose $\ell \geq 1$ and take $(\mathrm{H}, \mathrm{G}) = (\mathbb{Z}_{2\ell}^n, \mathbb{Z}_{8\ell m}^n)$ with auxiliary group $\Gamma = \mathrm{G}$. We want to check that this is an X_p-representable pair. It is clear that the range of β_ℓ is $-[\ell] \cup [\ell] \subseteq \mathbb{Z}_{8\ell m}^n$. When $\ell = 1$, it recovers the map that sends 0 to -1 and 1 to 1. In addition, a simple observation yields that $-\beta_\ell(y) = \beta_\ell(2\ell - 1 - y)$ which implies condition (RP-2) i). Next, denoting again characters by χ_w for each $w \in \mathbb{Z}_{2\ell}^n$, we choose conditional expectations given by

$$\mathsf{E}_{[n]\setminus \mathsf{S}} \chi_w = \delta_{w \in \mathbb{Z}_{2\ell}^{\mathsf{S}}} \chi_w.$$

By Subsection 1.2.2 and Fubini's theorem, (RP-1) holds with derivatives given again by $\partial_j : \chi_w \mapsto \delta_{w_j \neq 0} \chi_w$. Finally, the map $\mathrm{M}_\gamma : \chi_w \mapsto \chi_w(\gamma)\chi_w$ is a completely bounded multiplier on $L_p(\mathbb{Z}_{8\ell m}^n)$, so the family $\{\mathrm{M}_\gamma\}_{\gamma \in \Gamma}$ satisfies (RP-2) ii). The application of Theorem 2.0.1 yields

$$\begin{aligned} \text{(2.9)} \qquad \frac{m^{-p}}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subseteq [n] \\ |\mathsf{S}| = k}} \Big(\frac{1}{(8\ell m)^n (2\ell)^n} \sum_{x \in \mathbb{Z}_{8\ell m}^n} \sum_{y \in \mathbb{Z}_{2\ell}^n} \Big| f\big(x + 4m\eta_\ell(y)_{\mathsf{S}}\big) - f(x) \Big|^p \Big) \\ \lesssim_p \frac{k}{n} \sum_{j=1}^n \Big(\frac{1}{(8\ell m)^n (2\ell)^n} \sum_{x \in \mathbb{Z}_{8\ell m}^n} \sum_{y \in \mathbb{Z}_{2\ell}^m} \Big| \partial_j^y f\big(x + 2\eta_\ell(y)\big) \Big|^p \Big) \\ + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \frac{1}{(8\ell m)^n (2\ell)^n} \sum_{x \in \mathbb{Z}_{8\ell m}^n} \sum_{y \in \mathbb{Z}_{2\ell}^n} \Big| f(x + \eta_\ell(y)) - f(x) \Big|^p. \end{aligned}$$

Since (2.5) holds in this setting too, we get

$$\begin{aligned} &\sum_{x \in \mathbb{Z}_{8\ell m}^n} \sum_{y \in \mathbb{Z}_{2\ell}^n} \Big| \partial_j^y f(x + 2\eta_\ell(y)) \Big|^p \\ &\qquad = \sum_{x \in \mathbb{Z}_{8\ell m}^n} \sum_{y \in \mathbb{Z}_{2\ell}^n} \Big| f(x + 2\eta_\ell(y)) - \mathsf{E}_{\{j\}}^y f(x + 2\eta_\ell(y)) \Big|^p \\ &\qquad = \sum_{x \in \mathbb{Z}_{8\ell m}^n} \sum_{y \in \mathbb{Z}_{2\ell}^n} \Big| \frac{1}{2\ell} \sum_{t \in \mathbb{Z}_{2\ell}} \Big(f(x + 2\eta_\ell(y)) - f(x + 2\eta_\ell(y_{[n]\setminus\{j\}} + t e_j)) \Big) \Big|^p \\ &\qquad \leq \sum_{x \in \mathbb{Z}_{8\ell m}^n} \sum_{y \in \mathbb{Z}_{2\ell}^n} \frac{1}{2\ell} \sum_{t \in \mathbb{Z}_{2\ell}} \Big| f(x + 2\eta_\ell(y)) - f(x + 2\eta_\ell(y_{[n]\setminus\{j\}} + t e_j)) \Big|^p \\ &\qquad = (2\ell)^{n-2} \sum_{x \in \mathbb{Z}_{8\ell m}^n} \Big(\sum_{y_j \in \mathbb{Z}_{2\ell}} \sum_{t \in \mathbb{Z}_{2\ell}} \Big| f(x + 2\beta_\ell(y_j) e_j) - f(x + 2\beta_\ell(t) e_j) \Big|^p \Big), \end{aligned}$$

by the definition of $\mathsf{E}_{[n]\setminus \mathsf{S}}$ and convexity. Since $|\beta_\ell(y_j) - \beta_\ell(t)| \leq 2\ell$, the translation invariance of the Haar measure yields

$$\begin{aligned} (2\ell)^{n-2} \sum_{x \in \mathbb{Z}_{8\ell m}^n} \sum_{y_j, t \in \mathbb{Z}_{2\ell}} \Big| f(x + 2\beta_\ell(y_j) e_j) - f(x + 2\beta_\ell(t) e_j) \Big|^p \\ \leq (2\ell)^n (4\ell)^{p-1} \sum_{x \in \mathbb{Z}_{8\ell m}^n} \Big| f(x + e_j) - f(x) \Big|^p, \end{aligned}$$

which yields the result when inserted in (2.9). This completes the proof. $\square$

Remark 2.2.5. In our cyclic metric X_p inequalities, a similar computation as the one in [63, Proposition 1.4] shows that the condition $m \geq \sqrt{n/k}$ is necessary in Proposition 2.2.4.

2.3 Transferred X_p inequalities

Now we apply Theorem 2.0.1 to some pairs (H, G) with $\mathrm{H} = \mathbb{Z}_2^n$ and nonabelian G. Clearly, the choices of the auxiliary group Γ and G in each case must be related. We give the details for one particular choice and comment on a few more later. We choose $\Gamma = \mathbb{Z}_{8m}^n$ and

$$\mathrm{G} = \mathbb{Z}_{8m}^{*n}.$$

Given $u \in \mathbb{Z}_{8m}^n$, we consider the character $\chi_u : \mathbb{Z}_{8m}^{*n} \to \mathbb{C}$ determined by

$$\chi_u(w) = \exp\Big(\frac{2\pi i}{8m} \sum_{j=1}^{n} u_j \sum_{t:k_t=j} \ell_t \Big) \quad \text{for} \quad w = g_{k_1}^{\ell_1} g_{k_2}^{\ell_2} \dots g_{k_r}^{\ell_r}.$$

It is clear that $\chi_u(w)\chi_v(w) = \chi_{u+v}(w)$. Next, we define the Fourier multipliers $\mathrm{M}_u : \mathcal{L}(\mathbb{Z}_{8m}^{*n}) \to \mathcal{L}(\mathbb{Z}_{8m}^{*n})$ determined by $\lambda(w) \mapsto \chi_u(w)\lambda(w)$ for the left regular representation λ on $\mathbb{Z}_{8m}^{*n}$.

Lemma 2.3.1. M_u *is a normal unital trace-preserving* $*-$*homomorphism on* $\mathcal{L}(\mathbb{Z}_{8m}^{*n})$.

Proof. M_u is clearly unital and

$$\mathrm{M}_u(\lambda(w)^*) = \mathrm{M}_u(\lambda(w^{-1})) = \overline{\chi_u(w)}\, \lambda(w)^* = \mathrm{M}_u(\lambda(w))^*.$$

Next we claim that $\mathrm{M}_u(\lambda(ww')) = \mathrm{M}_u(\lambda(w))\mathrm{M}_u(\lambda(w'))$. It is clear that this identity holds when both w and w' are powers of a fixed generator g_j. More precisely, we have

$$\mathrm{M}_u(\lambda(g_j^{\ell_1+\ell_2})) = \mathrm{M}_u(\lambda(g_j^{\ell_1}))\, \mathrm{M}_u(\lambda(g_j^{\ell_2})).$$

Now, let us consider two reduced words given by

$$w = g_{k_1}^{\ell_1} g_{k_2}^{\ell_2} \dots g_{k_r}^{\ell_r} \quad \text{and} \quad w' = g_{k'_1}^{\ell'_1} g_{k'_2}^{\ell'_2} \dots g_{k'_s}^{\ell'_s}.$$

If $k_r \neq k'_1$ the claim trivially holds. If $k_r = k'_1$ and $\ell_r + \ell'_1 \not\equiv 0 \ (\mathrm{mod}\ 8m)$, then

$$\mathrm{M}_u(\lambda(ww')) = \mathrm{M}_u(\lambda(g_{k_1}^{\ell_1} \dots g_{k_{r-1}}^{\ell_{r-1}}))\, \mathrm{M}_u(\lambda(g_{k_r}^{\ell_r+\ell'_1}))\, \mathrm{M}_u(\lambda(g_{k'_2}^{\ell'_2} \dots g_{k'_s}^{\ell'_s}))$$

which yields the same conclusion. For the remaining case, we may write ww' as

$$ww' = \underbrace{g_{k_1}^{\ell_1} \dots g_{k_{r-1}}^{\ell_{r-1}}}_{\rho} \underbrace{g_{k'_2}^{\ell'_2} \dots g_{k'_s}^{\ell'_s}}_{\rho'}.$$

Arguing as above, if $k_{r-1} \neq k'_2$ or $\ell_{r-1} + \ell'_2 \not\equiv 0 \pmod{8m}$ we get

$$\begin{aligned} \mathrm{M}_u(\lambda(ww')) &= \mathrm{M}_u(\lambda(\rho))\mathrm{M}_u(\lambda(\rho')) \\ &= \mathrm{M}_u(\lambda(\rho))\mathrm{M}_u(\lambda(g_{k_r}^{\ell_r+\ell'_1}))\mathrm{M}_u(\lambda(\rho')) = \mathrm{M}_u(\lambda(w))\mathrm{M}_u(\lambda(w')). \end{aligned}$$

One can iterate this process to deduce the claim. Thus, M_u is a $*-$homomorphism on $\mathrm{span}\{\lambda(w) : w \in \mathbb{Z}_{8m}^{*n}\}$. In particular, it is a completely positive unital map. Moreover, it is trace-preserving since $\tau(\mathrm{M}_u(\lambda(w))) = \chi_u(w)\tau(\lambda(w))$ and $\chi_u(e) = 1$. Finally, note that for any $f, g \in \mathcal{L}(\mathbb{Z}_{8m}^{*n})$ there holds

$$\begin{aligned} \tau(\mathrm{M}_u(f)g^*) &= \tau\Big[\Big(\sum_{w\in\mathbb{Z}_{8m}^{*n}} \widehat{f}(w)\chi_u(w)\lambda(w)\Big)\Big(\sum_{\eta\in\mathbb{Z}_{8m}^{*n}} \widehat{g}(\eta)\lambda(\eta)\Big)^*\Big] \\ &= \sum_{w\in\mathbb{Z}_{8m}^{*n}} \widehat{f}(w)\chi_u(w)\overline{\widehat{g}(w)} = \sum_{w\in\mathbb{Z}_{8m}^{*n}} \widehat{f}(w)\overline{\chi_{-u}(w)\widehat{g}(w)} = \tau(f(\mathrm{M}_{-u}g)^*). \end{aligned}$$

Since M_{-u} extends to a bounded map on $L_1(\mathcal{L}(\mathbb{Z}_{8m}^{*n}))$, M_u is w^*-continuous. □

Before exploring X_p inequalities for free groups, we shall need to use Theorem 1.0.2 with values in a noncommutative L_p-space over a QWEP von Neumann algebra $\mathcal{M}$. More precisely, given a mean-zero $f : \Omega_n \to \mathcal{M}$ we claim that

$$\begin{aligned} \text{(2.10)} \qquad & \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n] \\ |\mathsf{S}|=k}} \big\|(\mathsf{E}_{[n]\setminus\mathsf{S}} \otimes \mathrm{Id})f\big\|_{L_p(\Omega_n;L_p(\mathcal{M}))}^p \\ & \lesssim_p \frac{k}{n}\sum_{j=1}^n \big\|(\partial_j \otimes \mathrm{Id})f\big\|_{L_p(\Omega_n;L_p(\mathcal{M}))}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \|f\|_{L_p(\Omega_n;L_p(\mathcal{M}))}^p, \end{aligned}$$

where $\mathsf{E}_{[n]\setminus\mathsf{S}}$ and ∂_j stand for the original conditional expectation and directional derivative used by Naor in [60]. To check that (2.10) holds, one can see that the proof of Theorem 1.0.2 goes through in the operator valued setting as long as dimension-free bounds hold for operator-valued Riesz transforms [40], which just requires to use the same argument together with Fubini's theorem from [36].

Proposition 2.3.2. *If $p \geq 2$, $k \in [n]$ and $m \geq \sqrt{n/k}$, every $f \in L_p(\mathcal{L}(\mathbb{Z}_{8m}^{*n}))$ satisfies*

$$\begin{aligned} & \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n] \\ |\mathsf{S}|=k}} \frac{1}{2^n}\sum_{\varepsilon\in\Omega_n} \big\|\mathrm{M}_{4m\varepsilon_\mathsf{S}} f - f\big\|_{L_p(\mathcal{L}(\mathbb{Z}_{8m}^{*n}))}^p \\ & \lesssim_p m^p \Big(\frac{k}{n}\sum_{j=1}^n \big\|\mathrm{M}_{e_j} f - f\big\|_p^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \frac{1}{2^n}\sum_{\varepsilon\in\Omega_n} \big\|\mathrm{M}_\varepsilon f - f\big\|_p^p\Big). \end{aligned}$$

Proof. We apply Theorem 2.0.1 to $(\mathrm{H},\mathrm{G}) = (\mathbb{Z}_2^n, \mathbb{Z}_{8m}^{*n})$, so we need to check that it is an X_p-representable pair. Condition (RP-1) follows from (2.10) since $\mathcal{L}(\mathbb{Z}_{8m}^{*n})$ is a QWEP von

Neumann algebra. Consider the auxiliary group $\Gamma = \mathbb{Z}_{8m}^n$ with the inclusion η defined as in (2.8) for $\ell = 1$. Condition (RP-2) i) is then automatically satisfied. Finally, according to Lemma 2.3.1 we know that M_γ extends to a completely contractive map on $L_p(\mathcal{L}(\mathbb{Z}_{8m}^{*n}))$ for any $1 \leq p \leq \infty$. Therefore, the family $\{\mathrm{M}_\gamma\}_{\gamma \in \mathbb{Z}_{8m}^n}$ can be used to check (RP-2) ii) and Theorem 2.0.1 leads to

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subseteq [n] \\ |\mathsf{S}| = k}} \frac{1}{2^n} \sum_{\varepsilon \in \Omega_n} \big\| \mathrm{M}_{4m\varepsilon_{\mathsf{S}}} f - f \big\|_{L_p(\mathcal{L}(\mathbb{Z}_{8m}^{*n}))}^p$$
$$\lesssim_p m^p \left(\frac{k}{n} \sum_{j=1}^n \frac{1}{2^n} \sum_{\varepsilon \in \Omega_n} \big\| (\partial_j \otimes \mathrm{Id}) \mathrm{M}_{2\varepsilon} f \big\|_p^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \frac{1}{2^n} \sum_{\varepsilon \in \Omega_n} \big\| \mathrm{M}_\varepsilon f - f \big\|_p^p \right),$$

for every $f \in L_p(\mathbb{Z}_{8m}^n; L_p(\mathbb{Z}_{8m}^{*n}))$. Finally, the computations for discrete derivatives from Subsection 2.2.2 imply that condition (2.4) holds, yielding the result. □

The example above is clearly not the only possible choice that we can make for the groups Γ and G in the pair. In general, given abelian groups $\Gamma_1, \ldots, \Gamma_n$ and $\Gamma = \Gamma_1 \times \Gamma \times \cdots \times \Gamma_n$ so that there is a nice map

$$\eta : \Omega_n \to \widehat{\Gamma},$$

one can take $\mathrm{G} = \Gamma_1 * \ldots * \Gamma_n$ and construct a family of multipliers $\{\mathrm{M}_\gamma\}_{\gamma \in \widehat{\Gamma}}$ as above. Once the properties of the family are checked, the above scheme yields a metric X_p inequality for the pair $(\mathbb{Z}_2^n, \Gamma, \mathrm{G})$. For example, the interested reader can check that a representation in the spirit of Lemma 2.3.1 for $\Gamma = \mathbb{Z}^n$ and $\mathrm{G} = \mathbb{F}_n$ (the free group of n generators) holds. This yields the following free metric X_p inequality in the free group algebra

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S} \subseteq [n] \\ |\mathsf{S}| = k}} \frac{1}{2^n} \sum_{\varepsilon \in \Omega_n} \big\| \mathrm{M}_{4m\varepsilon_{\mathsf{S}}} f - f \big\|_{L_p(\mathcal{L}(\mathbb{F}_n))}^p$$
$$\lesssim_p m^p \left(\frac{k}{n} \sum_{j=1}^n \big\| \mathrm{M}_{e_j} f - f \big\|_{L_p(\mathcal{L}(\mathbb{F}_n))}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \frac{1}{2^n} \sum_{\varepsilon \in \Omega_n} \big\| \mathrm{M}_\varepsilon f - f \big\|_{L_p(\mathcal{L}(\mathbb{F}_n))}^p \right).$$

2.4 Metric consequences

We conclude by collecting a few metric consequences of the results in the previous sections. In [60], Naor gave two sufficient hypothesis for a Banach space $\mathbb{X}$ to be a metric X_p space: a Banach X_p inequality and operator valued dimension-free estimates for Riesz transforms. In particular, any mean-zero $f : \Omega_n \to \mathbb{X}$ must satisfy

$$\frac{1}{2^n} \sum_{\varepsilon \in \Omega_n} \Big\| \sum_{j=1}^n \varepsilon_j (\partial_j \otimes \mathrm{Id}) f \Big\|_{L_p(\Omega_n; \mathbb{X})}^p \simeq_p \big\| (\Delta^{1/2} \otimes \mathrm{Id}) f \big\|_{L_p(\Omega_n; \mathbb{X})}^p.$$

When $\mathbb{X} = L_p(\mathcal{M})$, the Banach X_p inequality is Theorem 1.1.2, while the Riesz transform estimates can be found in [40] when $\mathcal{M}$ is QWEP, as explained in the previous section.

This latter fact had previously been announced for the Schatten classes S_p in [60, 61]. The result that we get is the following:

Theorem 2.4.1 (Noncommutative L_p spaces are metric X_p spaces)**.** *Let $\mathcal{M}$ be a* QWEP *von Neuman algebra. Then, if $p \geq 2$, $k \in [n]$ and $m \geq \sqrt{n/k}$, every $f : \mathbb{Z}_{8m}^n \to L_p(\mathcal{M})$ satisfies*

$$\frac{m^{-p}}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\ |\mathsf{S}|=k}} \frac{1}{(16m)^n} \sum_{x\in\mathbb{Z}_{8m}^n} \sum_{\varepsilon\in\Omega_n} \big\| f(x+4m\varepsilon_{\mathsf{S}}) - f(x) \big\|_{L_p(\mathcal{M})}^p$$

$$\lesssim_p \frac{k}{n} \sum_{j=1}^n \frac{1}{(8m)^n} \sum_{x\in\mathbb{Z}_{8m}^n} \big\| f(x+e_j) - f(x) \big\|_{L_p(\mathcal{M})}^p + \Big(\frac{k}{n}\Big)^{\frac{p}{2}} \frac{1}{(16m)^n} \sum_{x\in\mathbb{Z}_{8m}^n} \sum_{\varepsilon\in\Omega_n} \big\| f(x+\varepsilon) - f(x) \big\|_{L_p(\mathcal{M})}^p.$$

Theorem 2.4.1 provides a large class of examples of metric X_p spaces beyond the class of (commutative) L_p spaces. Also, as shown in [60], being a metric X_p space implies lower estimates on the distorsion of bi-Lipschitz embeddings of nonlinear sets. Indeed, let $c_{L_p(\mathcal{M})}(\mathbf{X})$ denote the smallest norm of a bi-Lipschitz map $\mathbf{X} \to L_p(\mathcal{M})$. Then we have the following:

Corollary 2.4.2. *Let $\mathcal{M}$ be* QWEP *and $2 < q < p$. Then*

i) $c_{L_p(\mathcal{M})}\big([m]_q^n\big) \gtrsim_{p,q} \min\Big\{ n^{\frac{(p-q)(q-2)}{q^2(p-2)}}, m^{1-\frac{2}{q}} \Big\}.$

ii) *If $c_{L_p(\mathcal{M})}\big((L_q, \|x-y\|_q^\theta)\big) < \infty$, then necessarily $\theta \leq q/p$.*

Here $[m]_q^n$ denotes the grid $\{1,\ldots,m\}^n$ equipped with the distance $d_{\ell_q^n}(x,y) = \|x-y\|_{\ell_q^n}$. We refer to [60, 63] for a precise definition on θ-snowflakes $(L_q, \|x-y\|_q^\theta)$. We end with a note about the optimality of the discrete cyclic inequalities from Section 2.2. Given $m, n \geq 2$, we know from [63, Lemma 3.1] that there exists $h_m^n : \mathbb{Z}_m^n \to \{0,\ldots,4m\}^{2n}$ such that

$$\Big(\sum_{j=1}^n \Big| \exp\Big(\frac{2\pi i x_j}{m}\Big) - \exp\Big(\frac{2\pi i y_j}{m}\Big) \Big|^q \Big)^{\frac{1}{q}} \sim \big\| h_m^n(x) - h_m^n(y) \big\|_q$$

holds (up to absolute constants) for any $x, y \in \mathbb{Z}_m^n$ and $q \geq 2$. Given a bi-Lipschitz map $g : [16m]_q^{2n} \to L_p$ with bi-Lipschitz norm D, set $F = g \circ h_{4m,n}$. Then, arguing as in [63, Theorem 1.14] we get

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\ |\mathsf{S}|=k}} \frac{1}{(4m)^{2n}} \sum_{x,y\in\mathbb{Z}_{4m}^n} \big\| F(x+y_{\mathsf{S}}) - F(x) \big\|_{L_p}^p \gtrsim_p m^p k^{\frac{p}{q}} \big| e^{\frac{2\pi i}{4m}} - 1 \big|.$$

Next, we make the choices $k = \lceil n^{\frac{p(q-2)}{q(p-2)}} \rceil$ and $m = \lceil n^{\frac{p-q}{q(p-2)}} \rceil$, which ensure that $m \gtrsim \sqrt{n/k}$, and so we can apply Proposition 2.2.4 to F in order to get the following statement: if $2 < q < p$ and $m, n, \ell \in \mathbb{N}$, there holds

$$c_{L_p}([m]_q^n) \gtrsim_{p,q} \min\Big\{ |e^{i\pi/\ell} - 1| \, n^{\frac{(p-q)(q-2)}{q^2(p-2)}}, m^{1-\frac{2}{q}} \Big\} \gtrsim_{p,q,\ell} \min\Big\{ n^{\frac{(p-q)(q-2)}{q^2(p-2)}}, m^{1-\frac{2}{q}} \Big\}.$$

Thus, up to constants (which get worse as $\ell \to \infty$) we recover the optimal distortions found in [60]. Also, Naor's result for θ-snowflakes in [60] follow from Proposition 2.2.4.

Problem 2.4.3. *According to [61, Theorem 12]*

$$c_{S_p}(\mathrm{X}) \leq \dim(\mathrm{X})^{\frac{q}{2}(\frac{1}{q}-\frac{1}{p})} \tag{2.11}$$

for every linear subspace $\mathrm{X} \subseteq S_q^n$ *and any* $2 < q < p < \infty$. *Is there an embedding of* X *into* $S_p(\ell_q^m)$ *with* $m = \dim(\mathrm{X})$ *and constants independent of* m*? This is trivially true for row, column or diagonal subspaces and we know from [43, Theorem 2] that it holds for the full space* $\mathrm{X} = S_q^n$ *with optimal* m *being* $\dim(\mathrm{X}) = n^2$. *If this was the case for general* X, *we would get* $c_{S_p}(\mathrm{X}) \lesssim c_{S_p}(S_p(\ell_q^m))$ *whose rightmost term is dominated by an operator space (matrix amplification) form of* $c_{S_p}(\ell_q^m)$. *Using that* S_p *is a metric* X_p*-space and arguing as above*

$$c_{S_p}(\ell_q^m) \asymp_{p,q} m^{\frac{(p-q)(q-2)}{q^2(p-2)}}. \tag{2.12}$$

Does $c_{S_p}(S_p(\ell_q^m))$ *behave like* $c_{S_p}(\ell_q^m)$*? If so, one could guess that*

$$c_{S_p}(\mathrm{X}) \leq_{p,q} \dim(\mathrm{X})^{\frac{(p-q)(q-2)}{q^2(p-2)}} \quad \text{for every} \quad \mathrm{X} \subseteq S_q^n? \tag{2.13}$$

This would already improve (2.11) *and it is best possible for* $\mathrm{X} = \ell_q^n = \mathrm{diag}(S_q^n)$. *It is still open to find the optimal distortion* $c_{S_p}(\mathrm{X})$ *for the full space* $\mathrm{X} = S_q^n$. *In this case, we have the following bounds*

$$\mathrm{A} := n^{\frac{(p-q)(q-2)}{q^2(p-2)}} \leq c_{S_p}(S_q^n) \leq \min\Big\{ n^{\frac{1}{2}-\frac{1}{q}}, n^{\frac{1}{q}-\frac{1}{p}} \Big\}. \tag{2.14}$$

Indeed, the lower bound follows from (2.12) *and the inclusion* $\ell_q^n \subseteq S_q^n$. *The upper bound follows easily using Hölder inequality. Note that* (2.13) *gives* $c_{S_p}(S_q^n) \leq \mathrm{A}^2$ *but* (2.14) *implies that no bound* $c_{S_p}(S_q^n) \leq \mathrm{A}^\beta$ *could be optimal for any* $\beta > 1$.

Chapter 3

Spin chaos and the Pisier inequality

In another attempt of extending X_p inequalities, one could consider different generalizations for the boolean cube apart from group cocycles as studied in chapters 1 and 2: for instance, *general spin systems*.

We will follow the approach by Lust-Piquard [49]. Let U, Q, P be the Pauli 2×2 matrices given by

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; P = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

These operators are self-adjoint and unitary and, together with the identity $\mathbf{1}$, they linearly span the $*-$algebra $\mathbb{M}_{2\times 2}(\mathbb{C})$. It is easy to check that they satisfy the following relations:

$$QP = -PQ = -iU, \qquad \mathrm{Tr}(Q) = \mathrm{Tr}(P) = \mathrm{Tr}(PQ) = 0.$$

For $1 \leq j \leq n$, we define the $2^n \times 2^n$ matrix

$$Q'_j = \mathbf{1} \otimes \ldots \otimes \mathbf{1} \otimes \underbrace{Q}_{j-th} \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1}.$$

It is worth mentioning that the $*-$algebra generated by these operators is $*-$isomorphic to $L_\infty(\Omega_n)$. On the other hand, we define analogously P'_j and U_j. Let $\mathcal{M}$ be the von Neumann algebra generated by the sequences $\{P'_j\}_{j=1}^n$ and $\{Q'_j\}_{j=1}^n$. They are clearly unitary self-adjoint operators and they satisfy

$$\begin{aligned} &Q'_jQ'_k = Q'_kQ'_j, \quad P'_jP'_k = P'_kP'_j, \quad 1 \leq j,k \leq n \\ &Q'_jP'_j = -P'_jQ'_j, \\ &Q'_jP'_k = P'_kQ'_j, \quad 1 \leq j \neq k \leq n. \end{aligned}$$

Now for $1 \leq j,k \leq n$, and setting a family of signs $\{\varepsilon(j,k)\}_{j,k\in[n]}$ satisfing $\varepsilon(j,k) = \varepsilon(k,j)$ if j is different from k, and $\varepsilon(j,j) = -1$, we can construct a *general spin system* constituted by the operators

$$Q_j^\varepsilon = U_1^{\frac{1}{2}(1-\varepsilon(j,1))} U_2^{\frac{1}{2}(1-\varepsilon(j,2))} \dots U_{j-1}^{\frac{1}{2}(1-\varepsilon(j,j-1))} Q_j' \tag{3.1}$$

$$P_j^\varepsilon = U_1^{\frac{1}{2}(1-\varepsilon(j,1))} U_2^{\frac{1}{2}(1-\varepsilon(j,2))} \dots U_{j-1}^{\frac{1}{2}(1-\varepsilon(j,j-1))} P_j'. \tag{3.2}$$

Recall that the case $\varepsilon(j,k) = 1$ whenever $j \neq k$ corresponds to the *Walsh system*, i.e. the dual group of the boolean cube, as long as the choice $\varepsilon(j,k) = -1$ corresponds to the *fermion system*. In general, they fulfil the following relations

$$\begin{array}{ll} Q_j^\varepsilon Q_k^\varepsilon = \varepsilon(j,k) Q_k^\varepsilon Q_j^\varepsilon, \quad P_j^\varepsilon P_k^\varepsilon = \varepsilon(j,k) P_k^\varepsilon P_j^\varepsilon, & 1 \leq j \neq k \leq n \\ Q_j^\varepsilon P_k^\varepsilon = \varepsilon(j,k) P_k^\varepsilon Q_j^\varepsilon, \quad 1 \leq j, k \leq n. & \end{array}$$

For a general family of signs ε as above, let $\mathcal{R}_n^\varepsilon$ be the von Neumann subalgebra generated by $\{Q_j^\varepsilon\}_{j\in[n]}$ in $\mathbb{M}_{2^n\times 2^n}(\mathbb{C})$, i.e.,

$$\mathcal{R}_n^\varepsilon = \mathrm{span}\{Q_{\mathsf{A}}^\varepsilon \ : \ \mathsf{A} \subseteq [n]\},$$

where, if $\mathsf{A} = \{j_1 < j_2 < \ldots < j_r\} \subseteq [n]$, we set $Q_{\mathsf{A}}^\varepsilon := Q_{j_1}^\varepsilon Q_{j_2}^\varepsilon \ldots Q_{j_r}^\varepsilon$. It is easy to check that the operators $\{Q_j^\varepsilon,\ P_j^\varepsilon\}_{j\in[n]}$ generate $\mathbb{M}_{2^n} := \mathbb{M}_{2^n\times 2^n}(\mathbb{C})$. We will equip both algebras $\mathcal{M}$ and $\mathcal{R}_n^\varepsilon$ with the tensor product of n copies of the usual trace in $\mathbb{M}_{2\times 2}(\mathbb{C})$, i.e.

$$\tau = \frac{1}{2^n} \bigotimes_{j=1}^n \mathrm{Tr}$$

In the following, and whenever it does not lead to confusion, we will omit the superscript ε when referring to the algebra $\mathcal{R}_n^\varepsilon$ or the elements of general spin systems.

In analogy with the Fourier theory developed on the boolean cube, the construction of $\mathcal{R}_n$ as the linear span of $\{Q_{\mathsf{A}}\}_{\mathsf{A}\subseteq[n]}$ leads to consider a similar expansion by replacing Walsh functions by fermions: if $f \in \mathcal{R}_n$, we can write

$$f = \sum_{\mathsf{A}\subseteq[n]} \widehat{f}(\mathsf{A})\, Q_{\mathsf{A}},$$

where $\widehat{f}(\mathsf{A}) = \tau(Q_{\mathsf{A}}^* f)$. Moreover, a family of conditional expectations can be considered: given $\mathsf{S} \subseteq [n]$, then $\mathcal{R}_{\mathsf{S}} = \mathrm{span}\{Q_{\mathsf{A}} \ : \ A \subseteq \mathsf{S}\}$ is a subalgebra of $\mathcal{R}_n$, so the adjoint operator of the inclusion $i_{\mathsf{S}} : L_1(\mathcal{R}_{\mathsf{S}}) \longrightarrow L_1(\mathcal{R}_n)$ which will be denoted by $\mathsf{E}_{[n]\setminus\mathsf{S}} : \mathcal{R}_n \longrightarrow \mathcal{R}_{\mathsf{S}}$ satisfies

$$\mathsf{E}_{[n]\setminus\mathsf{S}} f = \sum_{\mathsf{A}\subseteq\mathsf{S}} \widehat{f}(\mathsf{A})\, Q_{\mathsf{A}} \quad \text{for any } f \in \mathcal{R}_n.$$

This operator extends to a completely contractive map from $L_p(\mathcal{R}_n)$ onto $L_p(\mathcal{R}_{\mathsf{S}})$. On the other hand, as a first approach to the analogous concept of discrete derivative on the boolean cube, one can consider in $L_2(\mathcal{R}_n, \tau)$ for $1 \leq j \leq n$ the operators of *annihilation*, whose action on the elements of the basis of $\mathcal{R}_n$ is

$$\begin{array}{ll} D_j(Q_{\mathsf{A}}) = 0, & \text{if } j \notin \mathsf{A}, \\ D_j(Q_{\mathsf{A}}) = Q_j Q_{\mathsf{A}}, & \text{if } j \in \mathsf{A}, \end{array}$$

and the operators of *creation*, which satisfy

$$D_j^*(Q_{\mathsf{A}}) = 0, \qquad \text{if } j \in \mathsf{A},$$
$$D_j^*(Q_{\mathsf{A}}) = Q_j Q_{\mathsf{A}}, \quad \text{if } j \notin \mathsf{A}.$$

These maps allow us to define the *number operator* on $L_2(\mathcal{R}_n, \tau)$,

$$N = \sum_{j=1}^{n} D_j^* D_j.$$

which satisfies $N(Q_{\mathsf{A}}) = |\mathsf{A}|\ Q_{\mathsf{A}}$ for any subset A of $[n]$. Indeed, one can consider, for any $j \in [n]$, the *j-th discrete derivative*

$$\partial_j = 2Q_j D_j,$$

and check that the usual concept of the Laplacian operator coincides with the number operator, i.e.,

$$\Delta = \frac{1}{4}\sum_{j=1}^{n} \partial_j^* \partial_j = \sum_{j=1}^{n} D_j^* D_j = N.$$

Moreover, given $\mathsf{S} \subseteq [n]$ and a real number α, we can consider a family of *fractional Laplacians* on $\mathcal{R}_n$:

$$\Delta_{\mathsf{S}}^{\alpha} f = \sum_{\substack{\mathsf{A}\subseteq[n] \\ \mathsf{A}\cap\mathsf{S}\neq\emptyset}} |\mathsf{A}\cap\mathsf{S}|^{\alpha}\ \widehat{f}(\mathsf{A})\, Q_{\mathsf{A}} \quad \text{for any } f \in L_2(\mathcal{R}_n, \tau).$$

This yields a suitable expression for the *Riesz transforms* for general spin systems: given $j \in [n]$, the *j-th Riesz transform* R_j is defined by

$$\begin{cases} R_j = D_j N^{-1/2} = \frac{1}{2} Q_j \partial_j \Delta_{[n]}^{-1/2} \text{ on } L_2^{\circ}(\mathcal{R}_n, \tau) \\ R_j(\mathbf{1}_{\mathcal{R}_n}) = 0 \end{cases}$$

where $L_p^{\circ}(\mathcal{R}_n) = \{f \in L_p(\mathcal{R}_n) \ :\ \tau_{\mathcal{R}_n}(f) = 0\} = \{f \in L_p(\mathcal{R}_n) \ :\ \widehat{f}(\emptyset) = 0\}$.

In parallel to Theorem 1.1.1, the Riesz transforms estimates for spin systems are contained in Lust-Piquard's work [49]: given $2 \le p < \infty$, for any $T \in L_p^{\circ}(\mathcal{R}_n)$,

$$c_p^{-1}\|T\|_p \le \max\Big\{\Big\|\Big(\sum_{j=1}^{n} |R_j(T)|^2\Big)^{1/2}\Big\|_p, \Big\|\Big(\sum_{j=1}^{n} |R_j(T^*)|^2\Big)^{1/2}\Big\|_p\Big\} \le K_p\ \|T\|_p \tag{3.3}$$

where $c_p = O(p^2)$, $K_p = O(p^{3/2})$. On the other hand, as well as for the boolean cube setting [60], we consider the *Rademacher projections*: given $k \in [n]$, the k-th Rademacher projection of $f \in \mathcal{R}_n$ is

$$\mathrm{Rad}_k f = \sum_{\substack{\mathsf{A}\subseteq[n] \\ |\mathsf{A}|=k}} \widehat{f}(\mathsf{A})\, Q_{\mathsf{A}}.$$

Proposition 3.0.1. *Given $2 \leq p < \infty$, $k \in [n]$, the norm of the operator*

$$\mathrm{Rad}_k : L_p(\mathcal{R}_n) \longrightarrow L_p(\mathcal{R}_n)$$

is less or equal than $p^{k/2}$.

Proof. Consider the noncommutative Markov semigroup $S_t : Q_{\mathsf{A}} \mapsto e^{-t|\mathsf{A}|}\, Q_{\mathsf{A}}$. If $p \geq 2$, $\|S_t\|_{2\to p} = 1$ if and only if $e^{-2t} \leq \frac{p-1}{q-1}$ ([10, Theorem 4] for fermions, and [2, Theorem 3] for general spin systems). Then, if we take $t = \frac{1}{2}\log(p-1)$,

$$\begin{aligned}
\|\mathrm{Rad}_k f\|_p &= \Big\| \sum_{|\mathsf{A}|=k} \widehat{f}(\mathsf{A})\, Q_{\mathsf{A}} \Big\|_p = e^{tk} \Big\| S_t\Big(\sum_{|\mathsf{A}|=k} \widehat{f}(\mathsf{A})\, Q_{\mathsf{A}}\Big) \Big\|_p \\
&\leq (p-1)^{k/2} \Big\| \sum_{|\mathsf{A}|=k} \widehat{f}(\mathsf{A})\, Q_{\mathsf{A}} \Big\|_2 \leq p^{k/2} \Big\| \sum_{|\mathsf{A}|=k} \widehat{f}(\mathsf{A})\, Q_{\mathsf{A}} \Big\|_2 \\
&= p^{k/2} \Big(\sum_{|\mathsf{A}|=k} |\widehat{f}(\mathsf{A})|^2 \Big)^{1/2} \leq p^{k/2} \|f\|_2 \leq p^{k/2} \|f\|_p.
\end{aligned}$$

□

The above proposition shows that boundedness of Rademacher projections for spin systems is the same as for the Walsh system. Indeed, as well as in the latter case, it also provides a bound for the norm of the fractional Laplacian of negative exponent, whose proof follows verbatim as in [60, Lemma 10].

Lemma 3.0.2. *Suppose that $p \in [2,\infty)$ and $\alpha \in (0,\infty)$ satisfy $\alpha \leq \frac{5+\log p}{4}$. Then*

$$\sup_{n\in\mathbb{N}} \|\Delta_{[n]}^{-\alpha}\|_{p\to p} \lesssim \frac{(\log p)^\alpha}{2^\alpha \Gamma(1+\alpha)}.$$

3.1 X_p inequality for spin chaos

We are interested in getting an operator-valued X_p inequality for Rademacher chaos, via Riesz transforms estimates for spin systems with values on an arbitrary QWEP von Neumann algebra. Our effort will give rise to an operator-valued X_p inequality for Rademacher chaos, which already follows from Theorem 2.0.1 from chapter 2. Despite that, we include a different proof in this section. However, we first include the proof of the X_p inequality for spin chaos.

Theorem 3.1.1 (X_p inequality for spin chaos)**.** *Let $p \in [2,\infty)$, $n \in \mathbb{N}$ and $k \in [n]$. Given $f \in L_p^\circ(\mathcal{R}_n)$, there holds*

$$\Big(\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n] \\ |\mathsf{S}|=k}} \|\mathsf{E}_{[n]\setminus\mathsf{S}} f\|_p^p \Big)^{1/p} \lesssim_p \Big(\frac{k}{n} \sum_{j=1}^{n} \|\partial_j f\|_p^p + \Big(\frac{k}{n}\Big)^{p/2} \|f\|_p^p \Big)^{1/p}.$$

Proof. For every $\mathsf{S} \subseteq [n]$, by (3.3), it holds

$$\begin{aligned}
\|\mathsf{E}_{[n]\setminus \mathsf{S}} f\|_p &\lesssim_p \Big\|\Big(\sum_{j=1}^n |R_j \mathsf{E}_{[n]\setminus \mathsf{S}} f|^2\Big)^{1/2}\Big\|_p + \Big\|\Big(\sum_{j=1}^n |R_j (\mathsf{E}_{[n]\setminus \mathsf{S}} f)^*|^2\Big)^{1/2}\Big\|_p \\
&= \Big\|\Big(\sum_{j\in \mathsf{S}} |\mathsf{E}_{[n]\setminus \mathsf{S}} R_j f|^2\Big)^{1/2}\Big\|_p + \Big\|\Big(\sum_{j\in \mathsf{S}} |\mathsf{E}_{[n]\setminus \mathsf{S}} R_j (f^*)|^2\Big)^{1/2}\Big\|_p \\
&= \Big\|(\mathsf{E}_{[n]\setminus \mathsf{S}} \otimes \mathrm{Id})\Big(\sum_{j\in \mathsf{S}} R_j f \otimes e_{j,1}\Big)\Big\|_{S_p[L_p(\mathcal{R}_n)]} \\
&\quad + \Big\|(\mathsf{E}_{[n]\setminus \mathsf{S}} \otimes \mathrm{Id})\Big(\sum_{j\in \mathsf{S}} R_j (f^*) \otimes e_{j,1}\Big)\Big\|_{S_p[L_p(\mathcal{R}_n)]} \\
&\leq \Big\|\sum_{j\in \mathsf{S}} R_j f \otimes e_{j,1}\Big\|_{S_p[L_p(\mathcal{R}_n)]} + \Big\|\sum_{j\in \mathsf{S}} R_j (f^*) \otimes e_{j,1}\Big\|_{S_p[L_p(\mathcal{R}_n)]} \\
&= \Big\|\Big(\sum_{j\in \mathsf{S}} |R_j f|^2\Big)^{1/2}\Big\|_p + \Big\|\Big(\sum_{j\in \mathsf{S}} |R_j (f^*)|^2\Big)^{1/2}\Big\|_p,
\end{aligned}$$

since the conditional expectation $\mathsf{E}_{[n]\setminus \mathsf{S}}$ is completely contractive for any subset $\mathsf{S} \subseteq [n]$. Therefore, for any $f \in L_p^\circ(\mathcal{R}_n)$,

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq [n]\\ |\mathsf{S}|=k}} \|\mathsf{E}_{[n]\setminus \mathsf{S}} f\|_p^p \lesssim_p \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq [n]\\ |\mathsf{S}|=k}} \Big\|\Big(\sum_{j\in \mathsf{S}} |R_j f|^2\Big)^{1/2}\Big\|_p^p + \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq [n]\\ |\mathsf{S}|=k}} \Big\|\Big(\sum_{j\in \mathsf{S}} |R_j (f^*)|^2\Big)^{1/2}\Big\|_p^p$$

Note that we can concentrate on bounding the first summand of the previous line, since $D_j(f^*) = (D_j f)^*$. Now, as an application of noncommutative Khintchine inequality and Theorem 1.1.2, we obtain

$$\begin{aligned}
\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq [n]\\ |\mathsf{S}|=k}} \Big\|\Big(\sum_{j\in \mathsf{S}} |R_j f|^2\Big)^{1/2}\Big\|_p^p &= \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq [n]\\ |\mathsf{S}|=k}} \Big\|\sum_{j\in \mathsf{S}} R_j f \otimes e_{j,1}\Big\|_{S_p[L_p(\mathcal{R}_n)]}^p \\
&\lesssim_p \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq [n]\\ |\mathsf{S}|=k}} \Big\|\sum_{j\in \mathsf{S}} \varepsilon_j\, R_j f\Big\|_{L_p(\Omega_n; L_p(\mathcal{R}_n))}^p \\
&\lesssim_p \Big(\frac{p}{\sqrt{\log p}}\Big)^p \Big[\frac{k}{n} \sum_{j=1}^n \|R_j f\|_{L_p(\mathcal{R}_n)}^p + \Big(\frac{k}{n}\Big)^{p/2} \Big\|\sum_{j=1}^n \varepsilon_j R_j f\Big\|_{L_p(\Omega_n; L_p(\mathcal{R}_n))}^p\Big].
\end{aligned}$$

Another application of these results yields

$$\begin{aligned}
\Big\|\sum_{j=1}^n \varepsilon_j R_j f\Big\|_{L_p(\Omega_n; L_p(\mathcal{R}_n))}^p &\leq_p \Big\|\Big(\sum_{j=1}^n |R_j f|^2\Big)^{1/2}\Big\|_{L_p(\mathcal{R}_n)}^p + \Big\|\Big(\sum_{j=1}^n |(R_j f)^*|^2\Big)^{1/2}\Big\|_{L_p(\mathcal{R}_n)}^p \\
&\lesssim_p \|f\|_{L_p(\mathcal{R}_n)}^p,
\end{aligned}$$

and by Lemma 3.0.2, $\|R_j f\|_{L_p(\mathcal{R}_n)}^p \leq \|D_j f\|_{L_p(\mathcal{R}_n)}^p = \|\partial_j f\|_{L_p(\mathcal{R}_n)}^p$, since Q_j is a unitary operator. In conclusion,

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\ |\mathsf{S}|=k}} \Big\| \Big(\sum_{j\in\mathsf{S}} |R_j f|^2 \Big)^{1/2} \Big\|_p^p \lesssim \frac{k}{n} \sum_{j=1}^n \|\partial_j f\|_p^p + \Big(\frac{k}{n}\Big)^{p/2} \|f\|_p^p,$$

what yields the statement of the theorem. □

The aforementioned estimates for operator-valued Riesz transforms will follow from an adaptation of Lust-Piquard work for spin systems [49]. In particular, it will be sufficient to check that some assumptions from Definition 1.1 and Propositions 1.3-1.5 from that paper hold. For that purpose, recall that, given a QWEP von Neumann algebra $\mathcal{N}$ with a trace $\tau_{\mathcal{N}}$, any $f \in \mathcal{R}_n \overline{\otimes} \mathcal{N}$ and $g \in \mathbb{M}_{2^n} \overline{\otimes} \mathcal{N}$ admit respectively the following decomposition

$$f = \sum_{\mathsf{A}\subseteq[n]} Q_{\mathsf{A}} \otimes \widehat{f}(\mathsf{A}), \quad \text{where } \widehat{f}(\mathsf{A}) \in \mathcal{N}, \tag{3.4}$$

$$g = \sum_{A,B\subseteq[n]} P_{\mathsf{A}} Q_{\mathsf{B}} \otimes \widehat{g}(\mathsf{A},\mathsf{B}), \quad \text{where } \widehat{g}(\mathsf{A},\mathsf{B}) \in \mathcal{N}. \tag{3.5}$$

Proposition 3.1.2. *Let n be a natural number, and let $(\mathcal{N}, \tau_{\mathcal{N}})$ be a QWEP von Neumann algebra. Consider a general spin system generated by the operators $\{P_j, Q_j\}_{j\in[n]}$. Then the following conditions hold:*

1. *For any $j \in [n]$, it holds $(P_j \otimes \mathbf{1})^*(P_j \otimes \mathbf{1}_{\mathcal{N}}) = \mathbf{1}_{\mathbb{M}_{2^n}} \otimes \mathbf{1}_{\mathcal{N}}$.*
2. *If $j, k \in [n]$ satisfy $k \neq j$, then $\mathsf{E}_{\mathcal{R}_n \overline{\otimes} \mathcal{N}}((P_k \otimes \mathbf{1}_{\mathcal{N}})^* \, P_j \otimes \mathbf{1}_{\mathcal{N}}) = 0$.*
3. *Let $1 \leq p \leq \infty$ and δ be a vector of signs. Then any sequence $\{f_j\}_{j\in[n]}$ in $\mathcal{R}_n \overline{\otimes} \mathcal{N}$ satisfies the norm identity*

$$\Big\| \sum_{j=1}^n \delta_j (P_j \otimes \mathbf{1}_{\mathcal{N}}) f_j \Big\|_{L_p(\mathcal{R}_n \overline{\otimes} \mathcal{N})} = \Big\| \sum_{j=1}^n (P_j \otimes \mathbf{1}_{\mathcal{N}}) f_j \Big\|_{L_p(\mathcal{R}_n \overline{\otimes} \mathcal{N})}.$$

4. *For any $f \in \mathcal{R}_n \overline{\otimes} \mathcal{N}$ and $j \in [n]$, the matrix $(P_j \otimes \mathbf{1}_{\mathcal{N}}) f (P_j \otimes \mathbf{1}_{\mathcal{N}})^*$ is also an element of $\mathcal{R}_n \overline{\otimes} \mathcal{N}$.*
5. *Fixed $j \in [n]$, let Π_j be the orthogonal projection from $L_2(\mathbb{M}_{2^n})$ onto $P_j L_2(\mathbb{M}_{2^n})$. Then, for any $f \in \mathcal{R}_n \overline{\otimes} \mathcal{N}$, and $g \in \mathbb{M}_{2^n} \overline{\otimes} \mathcal{N}$,*

$$\begin{aligned} \|(\Pi_j \otimes \mathrm{Id})(fg)\|_{L_2(\mathbb{M}_{2^n} \overline{\otimes} \mathcal{N})} &= \|f (\Pi_j \otimes \mathrm{Id})(g)\|_{L_2(\mathbb{M}_{2^n} \overline{\otimes} \mathcal{N})}, \\ \|(\Pi_j \otimes \mathrm{Id})(gf)\|_{L_2(\mathbb{M}_{2^n} \overline{\otimes} \mathcal{N})} &= \|f (\Pi_j \otimes \mathrm{Id})(g)\|_{L_2(\mathbb{M}_{2^n} \overline{\otimes} \mathcal{N})}. \end{aligned}$$

(6) The sequence of operators $\{R_j \otimes \mathrm{Id}_{L_2(\mathcal{N})}\}_{j\in[n]}$ *satisfy the following identity:*

$$\sum_{j=1}^{n} (R_j \otimes \mathrm{Id}_{L_2(\mathcal{N})})^* (R_j \otimes \mathrm{Id}_{L_2(\mathcal{N})}) = \mathrm{Id}_{L_2^\circ(\mathcal{R}_n \overline{\otimes} \mathcal{N})}.$$

(7) For every $1 < p < \infty$*, and every* $f \in L_p^\circ(\mathcal{R}_n \overline{\otimes} \mathcal{N})$*, it holds*

$$\| \sum_{j=1}^{n} (P_j \otimes \mathbf{1})(R_j \otimes \mathbf{1}_{\mathcal{N}}) f \|_{L_p(\mathbb{M}_{2^n} \overline{\otimes} \mathcal{N})} \le c(p) \, \|f\|_{L_p(\mathcal{R}_n \overline{\otimes} \mathcal{N})}.$$

Proof. Conditions (1), (2) and (4) can be reduced to statements about the sequence $\{P_j\}_{j\in[n]}$ which are verified at Proposition 2.1 (b) from Lust-Piquard work [49]. On the other hand, fixed $j \in [n]$, the map T_j on $\mathbb{M}_{2^n}$ given by

$$g \mapsto Q_j' U_{j+1}^{(1/2)(1-\varepsilon(j,j+1))} \dots U_n^{(1/2)(1-\varepsilon(j,n))} \; g \; U_n^{(1/2)(1-\varepsilon(j,n))} \dots U_{j+1}^{(1/2)(1-\varepsilon(j,j+1))} Q_j',$$

is a $*-$inner isomorphism. Recall the relations $UQ = -QU$ and $UP = -PU$ and that $\varepsilon(i,j) = \varepsilon(j,i)$. Then, it is easy to see that T_j fixes Q_k^ε. If $j < k$ then

$$T_j Q_k^\varepsilon = Q_j' U_k^{(1/2)(1-\varepsilon(j,k))} Q_k^\varepsilon U_k^{(1/2)(1-\varepsilon(j,k))} Q_j' = Q_k^\varepsilon,$$

if $j = k$ then $T_j Q_k^\varepsilon = Q_k' Q_k^\varepsilon Q_k' = Q_k^\varepsilon$, and if $j > k$ then $T_j Q_k^\varepsilon = Q_k^\varepsilon$.

On the other hand, T_j fixes P_k^ε if $k \neq j$ due to a similar argument as above, but it sends P_j^ε to $-P_j^\varepsilon$ since $T_j P_j^\varepsilon = Q_j' P_j^\varepsilon Q_j' = -Q_j' Q_j' P_j^\varepsilon = -P_j^\varepsilon$. In particular, it is a trace-preserving isometry in $\mathbb{M}_{2^n}$. Therefore, $T_j \otimes \mathrm{Id}_{\mathcal{N}}$ is a trace-preserving isometry on $\mathbb{M}_{2^n} \overline{\otimes} \mathcal{N}$, which induces an isometry on $L_p(\mathbb{M}_{2^n} \overline{\otimes} \mathcal{N})$. By iteration and an application of the suitable maps depending on the choice of δ, this implies (3).

The decompositions (3.4) and (3.5) imply that (5) can be reduced to checking the identities without norms for $f \in \mathcal{R}_n$ and $g \in \mathbb{M}_{2^n}$, what is included in the proof of Proposition 2.1 from [49].

Moreover, (6) is a direct consequence of the properties of Riesz transforms on general spin systems. In order to prove (7), consider the unitary representation on the torus $R : [0, 2\pi) \longrightarrow \mathbb{M}_{2\times 2}$ given by

$$R_\theta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

which induces a unitary representation $\mathcal{R} : \mathbb{T} \longrightarrow \mathbb{M}_{2^n}$ given by

$$\mathcal{R}_\theta = \underbrace{R_\theta \times \dots \times R_\theta}_{n \text{ times}}.$$

Now, we can argue as in [49, Lemma 3.1 (b)] and apply [49, Lemma 1.1]: given a simple tensor $Q_{\mathsf{A}} \otimes \widehat{f}(\mathsf{A}) \in \mathbb{M}_{2^n}\overline{\otimes}\mathcal{N}$ such that $A \neq \emptyset$, it holds

$$\frac{1}{\sqrt{2\pi}}(\Pi_j \otimes \mathrm{Id})\Big(\text{p.v.} \int_{-\pi/2}^{\pi/2} (\mathcal{R}_\theta \otimes \mathbf{1})(Q_{\mathsf{A}} \otimes \widehat{f}(\mathsf{A}))(\mathcal{R}_\theta \otimes \mathbf{1})\frac{sgn(\theta)}{\sqrt{-\log(\cos^2(\theta))}}\, d\theta\Big)$$
$$= \frac{1}{\sqrt{2\pi}}\text{p.v.} \int_{-\pi/2}^{\pi/2} \cos(\theta)^{|\mathsf{A}|-1}\sin(\theta)\frac{\mathrm{sgn}(\theta)}{\sqrt{-\log(\cos^2(\theta))}}d\theta\ (P_j \otimes \mathbf{1})(R_j \otimes \mathrm{Id})(Q_{\mathsf{A}} \otimes \widehat{f}(\mathsf{A}))$$
$$= |\mathsf{A}|^{-1/2}(P_j \otimes \mathbf{1})(D_j \otimes \mathrm{Id})(Q_{\mathsf{A}} \otimes \widehat{f}(\mathsf{A})) = (P_j \otimes \mathbf{1})(R_j \otimes \mathrm{Id})(Q_{\mathsf{A}} \otimes \widehat{f}(\mathsf{A})).$$

Therefore, by summing in j and setting $\Pi = \sum_{1\le j\le n} \Pi_j$, we obtain

$$\sum_{j=1}^{n} (P_j \otimes \mathbf{1})(R_j \otimes \mathrm{Id})f =$$
$$= \frac{1}{\sqrt{2\pi}}(\Pi \otimes \mathrm{Id}) \circ \text{p.v.} \int_{-\pi/2}^{\pi/2} (\mathcal{R}_\theta \otimes \mathbf{1})^* f(\mathcal{R}_\theta \otimes \mathbf{1})\frac{\mathrm{sgn}(\theta)}{\sqrt{-\log(\cos^2(\theta))}}\, d\theta.$$

Since $\mathcal{R} \otimes \mathrm{Id}$ is a unitary representation of $\mathbb{T}$ on $\mathcal{R}_n\overline{\otimes}\mathcal{N}$, an application of [49, Lemma 1.2] is possible. Along with the previous identity, this yields, for $1 < p < \infty$ and $f \in \mathcal{R}_n\overline{\otimes}\mathcal{N}$

$$\|\sum_{j=1}^{n} (P_j \otimes \mathbf{1})(R_j \otimes \mathrm{Id})f\|_{L_p(\mathbb{M}_{2^n}\overline{\otimes}\mathcal{N})} \le C(p)\|f\|_{L_p(\mathcal{R}_n\overline{\otimes}\mathcal{N})},$$

with constant $C(p) = \|\Pi \otimes \mathrm{Id}\|_{p\to p}\|H\|_{L_p(\mathbb{T})\to L_p(\mathbb{T})} = O((p/(p-1))^{3/2})$ where H is the Hilbert transform on the torus. Notice that $\Pi \otimes \mathrm{Id}$ is bounded by a constant that only depends on p as a consequence [49, Proposition 1.5 (b)], since conditions (4) an (5) hold. □

Although validity of the previous conditions is sufficient to establish operator-valued Riesz transform estimates in a similar way that the original Lust-Piquard's estimates are proved, and outline of the proof is included below.

Theorem 3.1.3. *Given $n \in \mathbb{N}$, let $(Q_j)_{1\le j\le n}$, $(P_j)_{1\le j\le n}$ the generators of a general spin system, $\mathcal{R}_n$ the von Neumann algebra generated by $(Q_j)_{1\le j\le n}$, $(R_j)_{1\le j\le n}$ the associated Riesz transforms, and $\mathcal{N}$ an arbitrary QWEP von Neumann algebra.*

(1) Let $2 \le p < \infty$. Then for every $f \in L_p^\circ(\mathcal{R}_n\overline{\otimes}\mathcal{N})$,

$$c(p)^{-1}\|f\|_p \le \max\Big\{\Big\|\Big(\sum_{j=1}^{n}|(R_j \otimes \mathrm{Id})f|^2\Big)^{1/2}\Big\|_{L_p(\mathbb{M}_{2^n}\overline{\otimes}\mathcal{N})},$$
$$\Big\|\Big(\sum_{j=1}^{n}(P_j \otimes \mathbf{1})|(R_j \otimes \mathrm{Id})(f)^*|^2(P_j \otimes \mathbf{1})\Big)^{1/2}\Big\|_{L_p(\mathbb{M}_{2^n}\overline{\otimes}\mathcal{N})}\Big\} \le K(p)\|f\|_p$$

with $c(p) = K(p)K(p') = O(p^2)$*, and* $K(p) = O(p^{3/2})$*.*

(2) Let $1 < p \le 2$*. Then for every* $f \in L_p^\circ(\mathcal{R}_n \overline{\otimes} \mathcal{N})$

$$K(p')\|f\|_p \le \inf_{(R_j \otimes \mathrm{Id})f = a_j + b_j} \Big\{ \Big\| \Big(\sum_{j=1}^n a_j^* a_j\Big)^{1/2} \Big\|_p, \Big\| \Big(\sum_{j=1}^n (P_j \otimes \mathbf{1}) b_j b_j^* (P_j \otimes \mathbf{1})\Big)^{1/2} \Big\|_p \Big\} \le C(p)\|f\|_p$$

where the infimum runs over all decompositions of $(R_j \otimes \mathrm{Id})f$ *in* $\mathcal{R}_n$ *and* $K(p') = O(p'^{3/2}) = O(1/(p-1)^{3/2})$*,* $C(p) = K(p')K(p) = O(1/(p-1)^2)$*.*

Proof. Since conditions (1), (2) and (3) from Proposition 3.1.2 hold, an application of [49, Proposition 1.4] yields: for $2 \le p < \infty$ and $f \in L_p^\circ(\mathcal{R}_n \overline{\otimes} \mathcal{N})$,

$$\begin{aligned}\Big\| \sum_{j=1}^n (P_j \otimes \mathbf{1})(R_j \otimes \mathrm{Id})f \Big\|_{L_p(\mathcal{M}_n \overline{\otimes} \mathcal{N})} &\le K(p) \max\Big\{ \Big\| \Big(\sum_{j=1}^n |(R_j \otimes \mathrm{Id})f|^2\Big)^{1/2} \Big\|_p, \\ &\qquad \Big\| \Big(\sum_{j=1}^n (P_j \otimes \mathbf{1})|(R_j \otimes \mathrm{Id})(f)^*|^2 (P_j \otimes \mathbf{1})\Big)^{1/2} \Big\|_p \Big\} \\ &\le K(p) \Big\| \sum_{j=1}^n (P_j \otimes \mathbf{1})(R_j \otimes \mathrm{Id})f \Big\|_p\end{aligned}$$

with $K(p) = O(p^{1/2})$, and for $1 < p < 2$,

$$\begin{aligned}\Big\| \sum_{j=1}^n (P_j \otimes \mathbf{1})(R_j \otimes \mathrm{Id})f \Big\|_p &\le \inf_{(R_j \otimes \mathrm{Id})f = a_j + b_j} \Big\{ \Big\| \Big(\sum_{j=1}^n a_j^* a_j\Big)^{1/2} \Big\|_p, \Big\| \Big(\sum_{j=1}^n P_j b_j b_j^* P_j\Big)^{1/2} \Big\|_p \\ &\le K(p') \Big\| \sum_{j=1}^n (P_j \otimes \mathbf{1})(R_j \otimes \mathrm{Id})f \Big\|_{L_p(\mathbb{M}_{2^n} \overline{\otimes} \mathcal{N})}.\end{aligned}$$

Now the conditions (6) and (7) imply the statement from [49, Proposition 1.3] so for $2 \le p < \infty$,

$$\begin{aligned}c(p)^{-1}\|f\|_{L_p(\mathcal{R}_n \overline{\otimes} \mathcal{N})} \le \ &\max\Big\{ \Big\| \Big(\sum_{j=1}^n |(R_j \otimes \mathrm{Id})f|^2\Big)^{1/2} \Big\|_p, \\ &\Big\| \Big(\sum_{j=1}^n (P_j \otimes \mathbf{1})|(R_j \otimes \mathrm{Id})(f)^*|^2 (P_j \otimes \mathbf{1})\Big)^{1/2} \Big\|_p \Big\} \le K(p)\|f\|_p,\end{aligned}$$

and, for $1 < p \le 2$,

$$\begin{aligned}K(p')^{-1}\|f\|_p &\le \inf_{(R_j \otimes \mathrm{Id})f = a_j + b_j} \Big\{ \Big\| \Big(\sum_{j=1}^n a_j^* a_j\Big)^{1/2} \Big\|_p, \Big\| \Big(\sum_{j=1}^n (P_j \otimes \mathbf{1}) b_j b_j^* (P_j \otimes \mathbf{1})\Big)^{1/2} \Big\|_p \\ &\le C(p)\|f\|_p.\end{aligned}$$

□

Corollary 3.1.4. *Let $p \in [2,\infty)$ and $\mathcal{N}$ be an arbitrary QWEP von Neumann algebra. Then, for any $f \in L_p^\circ(\Omega_n; L_p(\mathcal{N}))$,*

$$\|f\|_p \lesssim_p \Big\| \Big(\sum_{j=1}^n |(R_j \otimes \mathrm{Id})(f)|^2\Big)^{1/2} \Big\|_p + \Big\| \Big(\sum_{j=1}^n |(R_j \otimes \mathrm{Id})(f^*)|^2\Big)^{1/2} \Big\|_p \lesssim_p \|f\|_p.$$

Proof. As a consequence of the anticommutation relations, there holds $P_j Q_{\mathsf{A}} = Q_{\mathsf{A}} P_j$ whenever $j \notin \mathsf{A}$. Moreover, since $R_j(f)$ does not depend on Q_j, it follows $R_j(f)P_j = P_j R_j(f)$, so P_j dissapears from the estimate in Theorem 3.1.3 for $2 \leq p < \infty$. □

Given a QWEP von Neumann algebra $\mathcal{N}$ with a normal semifinite faithful trace, their L_p spaces, when $2 \leq p < \infty$, have type 2. Therefore, since they have non-trivial type, these spaces are K-convex [67],i.e. the operators

$$\{\mathrm{Rad_k} \otimes \mathrm{Id}_{L_p(\mathcal{N})}\}_{k\in\mathbb{N}}$$

are uniformly bounded. Moreover, the operator $\Delta_{[n]}^{-1/2} \otimes \mathrm{Id}_{L_p(\mathcal{N})}$ is uniformly bounded for any natural number n [62, Theorem 5]. As an open question, we ask ourselves if a similar statement can be proved for Laplacians on general spin systems and, as a consequence, the theorem below can be obtained in this context.

Theorem 3.1.5 (X_p inequality for operator-valued Rademacher chaos)**.** *Suppose that $p \in [2,\infty)$, $n \in \mathbb{N}$, and $k \in [n]$. Let $\mathcal{N}$ a QWEP von Neumann algebra. Then every $f \in L_p^\circ(\Omega_n; L_p(\mathcal{N}))$ satisfies*

$$\Bigg(\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \|(\mathsf{E}_{[n]\setminus\mathsf{S}} \otimes \mathrm{Id})f\|_p^p\Bigg)^{1/p} \lesssim_p \Bigg(\frac{k}{n}\sum_{j=1}^n \|(\partial_j \otimes \mathrm{Id})f\|_p^p + \Big(\frac{k}{n}\Big)^{p/2} \|f\|_p^p\Bigg)^{1/p}.$$

Proof. By Corollary 3.1.4, it holds

$$\frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \|(\mathsf{E}_{[n]\setminus\mathsf{S}} \otimes \mathrm{Id})f\|_p^p \lesssim_p \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \Big\| \Big(\sum_{j=1}^n |(R_j \otimes \mathrm{Id})(\mathsf{E}_{[n]\setminus\mathsf{S}} \otimes \mathrm{Id})f|^2\Big)^{1/2} \Big\|_p^p$$
$$+ \frac{1}{\binom{n}{k}} \sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \Big\| \Big(\sum_{j=1}^n |(R_j \otimes \mathrm{Id})(\mathsf{E}_{[n]\setminus\mathsf{S}} \otimes \mathrm{Id})f^*|^2\Big)^{1/2} \Big\|_p^p.$$

Moreover, since the conditional expectations are completely contractive,

$$\begin{aligned}
&\frac{1}{\binom{n}{k}}\sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \|(\mathsf{E}_{[n]\setminus\mathsf{S}}\otimes \mathrm{Id})f\|_p^p \\
&\quad= \frac{1}{\binom{n}{k}}\sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \Big\|(\mathsf{E}_{[n]\setminus\mathsf{S}}\otimes \mathrm{Id}_{L_p(\mathcal{N})}\otimes \mathrm{Id}_{S_p})\Big(\sum_{j\in\mathsf{S}}(R_j\otimes\mathrm{Id})f\otimes e_{j,1}\Big)\Big\|_p^p \\
&\quad+ \frac{1}{\binom{n}{k}}\sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \Big\|(\mathsf{E}_{[n]\setminus\mathsf{S}}\otimes \mathrm{Id}_{L_p(\mathcal{N})}\otimes \mathrm{Id}_{S_p})\Big(\sum_{j\in\mathsf{S}}(R_j\otimes\mathrm{Id})f\otimes e_{j,1}\Big)\Big\|_p^p \\
&\quad\le \frac{1}{\binom{n}{k}}\sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \Big\|\sum_{j\in\mathsf{S}}(R_j\otimes\mathrm{Id})f\otimes e_{j,1}\Big\|_p^p + \frac{1}{\binom{n}{k}}\sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \Big\|\sum_{j\in\mathsf{S}}(R_j\otimes\mathrm{Id})f^*\otimes e_{j,1}\Big\|_p^p \\
&\quad= \frac{1}{\binom{n}{k}}\sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \Big\|\Big(\sum_{j\in\mathsf{S}}|(R_j\otimes\mathrm{Id})f|^2\Big)^{1/2}\Big\|_p^p + \frac{1}{\binom{n}{k}}\sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \Big\|\Big(\sum_{j\in\mathsf{S}}|(R_j\otimes\mathrm{Id})f^*|^2\Big)^{1/2}\Big\|_p^p.
\end{aligned}$$

Note that we can concentrate on bounding the first summand of the previous line, since $(D_j\otimes\mathrm{Id})(f^*) = ((D_j\otimes\mathrm{Id})f)^*$. Now, by noncommutative Khintchine inequality and Theorem 1.1.2 ,

$$\begin{aligned}
&\frac{1}{\binom{n}{k}}\sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \Big\|\Big(\sum_{j\in\mathsf{S}}|(R_j\otimes\mathrm{Id})f|^2\Big)^{1/2}\Big\|_p^p \\
&\qquad\lesssim_p \frac{1}{\binom{n}{k}}\sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}} \mathbb{E}_\delta \Big\|\sum_{j\in\mathsf{S}}\delta_j(R_j\otimes\mathrm{Id})f\Big\|_p^p \\
&\qquad\lesssim_p \frac{k}{n}\sum_{j=1}^n \|(R_j\otimes\mathrm{Id})f\|_p^p + \Big(\frac{k}{n}\Big)^{p/2}\mathbb{E}_\delta\Big\|\sum_{j=1}^n\delta_j(R_j\otimes\mathrm{Id})f\Big\|_p^p.
\end{aligned}$$

Another application of noncommutative Khintchine inequality and Theorem 3.1.4, implies that

$$\mathbb{E}_\delta\Big\|\sum_{j=1}^n\delta_j(R_j\otimes\mathrm{Id})f\Big\|_p^p \lesssim_p \Big\|\Big(\sum_{j=1}^n|(R_j\otimes\mathrm{Id})f|^2\Big)^{1/2}\Big\|_p^p \lesssim_p \|f\|_p^p.$$

Therefore, the boundedness of $\Delta_{[n]}^{-1/2}\otimes\mathrm{Id}_{L_p(\mathcal{N})}$ implies the inequality

$$\|(R_j\otimes\mathrm{Id})f\|_{L_p(\Omega_n;L_p(\mathcal{N}))} \lesssim_p \big(\mathbb{E}_\varepsilon\|(\varepsilon_j\partial_j\otimes\mathrm{Id})f\|_p^p\big)^{1/p} = \|(\partial_j\otimes\mathrm{Id})f\|_p.$$

In conclusion,

$$\left(\frac{1}{\binom{n}{k}}\sum_{\substack{\mathsf{S}\subseteq[n]\\|\mathsf{S}|=k}}\|(\mathsf{E}_{[n]\setminus\mathsf{S}}\otimes\mathrm{Id})f\|_p^p\right)^{1/p}\lesssim_p\left(\frac{k}{n}\sum_{j=1}^n\|(\partial_j\otimes\mathrm{Id})f\|_p^p+\left(\frac{k}{n}\right)^{p/2}\|f\|_p^p\right)^{1/p}.$$

□

3.2 Dimension-free Pisier's inequality for spin chaos

In this section, we study the extension of a dimension-free Pisier's inequality on the Hamming cube Ω_n which was recently introduced by Ivanisvili, van Handel and Volberg [33]. This estimate constituted the key fact for solving a long-standing problem in the metric geometry of Banach spaces: Rademacher and Enflo type coincide (see [16, 52]). The aforementioned inequality can be stated as follows: given a Banach space $\mathbb{X}$ and $p\in[1,\infty)$, there holds

$$\left(\mathbb{E}_\varepsilon\|f(\varepsilon)-\mathbb{E}_\delta f\|_{\mathbb{X}}^p\right)^{1/p}\le C\int_0^1\left(\mathbb{E}_\varepsilon\mathbb{E}_\eta\left\|\sum_{j=1}^n\frac{\eta_j^{(r)}}{1+r\eta_j^{(r)}}\partial_j f\right\|_{\mathbb{X}}^p\right)^{1/p}dt \tag{3.6}$$

where $\mathbb{E}_\varepsilon$ and $\mathbb{E}_\delta$ respectively denote the expectation over $\varepsilon\in\Omega_n$ and $\delta\in\Omega_n$ chosen uniformly at random. However, the vector

$$\eta^{(r)}=(\eta_1^{(r)},\ldots,\eta_n^{(r)})$$

is set to be a vector of independent identically distributed *biased* Rademacher variables so that

$$\Pr(\eta_j=1)=\frac{1+r}{2}\ \text{and}\ \Pr(\eta_j=-1)=\frac{1-r}{2}.$$

In that case, $\mathbb{E}_\eta\eta_j=r\neq 0$. Alternatively, (3.6) can be expresed as

$$\|f-(\mathsf{E}_{[n]}\otimes\mathrm{Id})f\|_{L_p(\Omega_n;\mathbb{X})}\le C\int_0^1\left(\mathbb{E}_\eta\left\|\sum_{j=1}^n\frac{\eta_j^{(r)}}{1+r\eta_j^{(r)}}(\partial_j\otimes\mathrm{Id})f\right\|_{L_p(\Omega_n;\mathbb{X})}^p\right)^{1/p}dr.$$

Our effort goes in the direction of replacing the Hamming cube Ω_n by the spin algebra $\mathcal{R}_n$. For that purpose, it will be convenient to consider operator spaces instead of Banach spaces.

Let $\mathcal{M}$ be a hyperfinite von Neumann algebra, i.e. $\mathcal{M}=\overline{\cup M_\alpha}^{w^*}$ where $(M_\alpha)_\alpha$ is a net of finite-dimensional von Neumann algebras directed by inclusion. Also, assume that $\mathcal{M}$ is equipped with a normal semifinite faithful trace, and let $E\subseteq B(\mathcal{H})$ be an *operator space*, that is, a closed subspace of $B(\mathcal{H})$. Set

$$L_1(\mathcal{M};E)=L_1(\mathcal{M})\widehat{\otimes}E$$

where $\widehat{\otimes}$ denotes the operator space version of the projective tensor product. Then, given $1 < p < \infty$, the *noncommutative vector valued L_p-space on $\mathcal{M}$ with values in E* is defined via complex interpolation as

$$L_p(\mathcal{M}; E) = (\mathcal{M} \otimes_{\min} E, L_1(\mathcal{M}; E))_{1/p}.$$

Then $L_p(\mathcal{M}; E)$ can be equipped with the norm

$$\|x\|_{L_p(\mathcal{M};E)} = \inf\{\|a\|_{L_{2p}(\mathcal{M})}\|y\|_{\mathcal{M}\otimes_{\min}E}\|b\|_{L_{2p}(\mathcal{M})} \ : \ x = a \cdot y \cdot b\},$$

where $y \in \mathcal{M} \otimes E$, and

$$a, b \in V = \bigcup_{\substack{e\in\mathcal{P}(\mathcal{M})\\ \tau_{\mathcal{M}}(e)<\infty}} e\mathcal{M}e.$$

Here $\mathcal{P}(\mathcal{M})$ denotes the lattice of orthogonal projections in $\mathcal{M}$. Moreover, if $\mathcal{M}$ is finite dimensional, the following duality identity holds completely isometrically for $1 \leq p < \infty$,

$$L_p(\mathcal{M}; E)^* = L_{p'}(\mathcal{M}; E^*). \tag{3.7}$$

Take $\mathcal{M} = \mathcal{R}_n$. Then, every $f \in L_p(\mathcal{R}_n; E)$ admits the expansion

$$f = \sum_{\mathsf{A}\subseteq[n]} Q_{\mathsf{A}} \otimes \widehat{f}(\mathsf{A}),$$

for some coefficients $\widehat{f}(\mathsf{A}) \in E$.

Theorem 3.2.1. *Fix $1 \leq p < \infty$ and $n \in \mathbb{N}$. Then for any operator space $E \subseteq B(\mathcal{H})$ and $f \in L_p(\mathcal{R}_n, E)$, there holds*

$$\|f - (\mathsf{E}_{[n]} \otimes \mathrm{Id})f\|_{L_p(\mathcal{R}_n;E)} \leq C \int_0^1 \left(\mathbb{E}_\eta \left\| \sum_{j=1}^n \frac{\eta_j^{(r)}}{1 + r\eta_j^{(r)}} (\partial_j \otimes \mathrm{Id})f \right\|^p_{L_p(\mathcal{R}_n;E)} \right)^{1/p} dr,$$

where the random variables $\{\eta_j^{(r)}\}_{j=1}^n$ are independent, identically distributed, and take the value 1 with probability $\frac{1+r}{2}$ and the value -1 with probability $\frac{1-r}{2}$.

Before proving the main result of this section, let's introduce some auxiliar maps. Given an operator

$$h = \sum_{\mathsf{A}\subseteq[n]} \widehat{h}(\mathsf{A}) Q_{\mathsf{A}},$$

let $r > 0$, $k \in [n]$ and η a biased Rademacher vector as considered in Theorem 3.2.1. Then, consider the maps

- $T_r h = \sum_{\mathsf{A}\subseteq[n]} r^{|\mathsf{A}|}\, \widehat{h}(\mathsf{A}) Q_{\mathsf{A}}$,
- $A_\eta h = \sum_{\mathsf{A}\subseteq[n]} \Big(\prod_{j\in\mathsf{A}} \eta_j \Big)\, \widehat{h}(\mathsf{A})\, Q_{\mathsf{A}}$,
- $S_k h = \sum_{\mathsf{A}\subseteq[n]} (-1)^{\delta_{A\ni k}}\, \widehat{h}(\mathsf{A})\, Q_{\mathsf{A}}$.

Recall that A_η can be understood as a spin version of a *translation operator*, while S_k represents flipping the sign of coordinate ε_j since $\partial_k = \mathrm{Id} - S_k$. These three operators can be related via the following identities.

Lemma 3.2.2. *Let $1 \leq p < \infty$ and let E be an operator space. Then, any $f \in L_p(\mathcal{R}_n; E)$ satisfies*

$$f - (\mathsf{E}_{[n]} \otimes \mathrm{Id})(f) = \frac{1}{r}(\Delta_{[n]} \otimes \mathrm{Id})(T_r \otimes \mathrm{Id})f, \tag{3.8}$$

$$\frac{1}{r}(\partial_k \otimes \mathrm{Id})(T_r \otimes \mathrm{Id})(f) = 2\, \mathbb{E}_\eta \Big[\frac{\eta_k^{(r)}}{1 + r\eta_k^{(r)}} (A_\eta \otimes \mathrm{Id}) f \Big] \tag{3.9}$$

for any $k \in [n]$ and $r > 0$.

Proof. Let $h \in L_p(\mathcal{R}_n)$. It is easy to check that $T_0 h = \mathsf{E}_{[n]} h$ and $T_1 h = h$ so, given $f \in L_p(\mathcal{R}_n; E)$, there holds

$$f - (\mathsf{E}_{[n]} \otimes \mathrm{Id})(f) = (T_1 \otimes \mathrm{Id})f - (T_0 \otimes \mathrm{Id})f = \int_0^1 \frac{d}{dr}(T_r \otimes \mathrm{Id})f \; dr.$$

Then, we recover the first identity in the statement since

$$\begin{aligned} \frac{d}{dr}(T_r \otimes \mathrm{Id})f &= \sum_{\mathsf{A}\subseteq[n]} |\mathsf{A}| r^{|\mathsf{A}|-1}\, Q_{\mathsf{A}} \otimes \widehat{f}(\mathsf{A}) = \frac{1}{r} \sum_{\mathsf{A}\subseteq[n]} |\mathsf{A}| r^{|\mathsf{A}|}\, Q_{\mathsf{A}} \otimes \widehat{f}(\mathsf{A}) \\ &= \frac{1}{r}(\Delta_{[n]} \otimes \mathrm{Id})(T_r \otimes \mathrm{Id})f. \end{aligned}$$

On the other hand, recall that

$$\mathbb{E}_\eta (A_\eta \otimes \mathrm{Id}) f = \sum_{\mathsf{A}\subseteq[n]} r^{|\mathsf{A}|}\, Q_{\mathsf{A}} \otimes \widehat{f}(\mathsf{A}) = (T_r \otimes \mathrm{Id}) f, \tag{3.10}$$

since the variables $\{\eta_j\}_{j=1}^n$ are independent and $\mathbb{E}_\eta \eta_j = r$. Then, fixed $k \in [n]$, regarding the distribution of η, and setting $\eta^{[k]} = \eta - 2\eta_k e_k$,

$$\begin{aligned}
\mathbb{E}_\eta (A_\eta \otimes \mathrm{Id})(S_k \otimes \mathrm{Id}) f &= \sum_{\eta \in \Omega_n} (A_\eta \otimes \mathrm{Id})(S_k \otimes \mathrm{Id}) f \prod_{j=1}^n \frac{1 + \eta_j r}{2} \\
&= \sum_{\eta \in \Omega_n} \sum_{\mathsf{A} \subseteq [n]} (-1)^{\delta_{A \ni k}} \Big(\prod_{j \in \mathsf{A}} \eta_j \Big) \Big(Q_{\mathsf{A}} \otimes \widehat{f}(\mathsf{A}) \prod_{j=1}^n \frac{1 + \eta_j r}{2} \\
&= \sum_{\eta \in \Omega_n} (A_{\eta^{[k]}} \otimes \mathrm{Id}) f \prod_{j=1}^n \frac{1 + \eta_j r}{2} \\
&= \sum_{\eta \in \Omega_n} (A_\eta \otimes \mathrm{Id}) f \, \frac{1 - \eta_k r}{1 + \eta_k r} \prod_{j=1}^n \frac{1 + \eta_j r}{2}.
\end{aligned}$$

Therefore, as a consequence of (3.10),

$$\begin{aligned}
\frac{1}{r} (\partial_k \otimes \mathrm{Id})(T_r \otimes \mathrm{Id}) f = \frac{1}{r} (\partial_k \otimes \mathrm{Id}) \mathbb{E}_\eta (A_\eta \otimes \mathrm{Id}) f &= \frac{1}{2} \frac{1}{r} ((\mathrm{Id} - S_k) \otimes \mathrm{Id}) \mathbb{E}_\eta (A_\eta \otimes \mathrm{Id}) f \\
&= \frac{1}{r} \sum_{\eta \in \Omega_n} (A_\eta \otimes \mathrm{Id}) f \, \Big(1 - \frac{1 - \eta_k r}{1 + \eta_k r} \Big) \Big(\prod_{j=1}^n \frac{1 + \eta_j r}{2} \\
&= \frac{1}{r} \sum_{\eta \in \Omega_n} (A_\eta \otimes \mathrm{Id}) f \, \frac{2 \eta_k r}{1 + \eta_k r} \prod_{j=1}^n \frac{1 + \eta_j r}{2} \\
&= 2 \sum_{\eta \in \Omega_n} (A_\eta \otimes \mathrm{Id}) f \, \frac{\eta_k}{1 + \eta_k r} \prod_{j=1}^n \frac{1 + \eta_j r}{2} \\
&= 2 \, \mathbb{E}_\eta \Big[\frac{\eta_k}{1 + \eta_k r} (A_\eta \otimes \mathrm{Id}) f \Big],
\end{aligned}$$

what completes the proof of the statement. □

Proof of Theorem 3.2.1. As a consequence of the duality identity (3.7) and (3.8), it follows that

$$\begin{aligned}
\| f - (\mathsf{E}_{[n]} \otimes \mathrm{Id}) f \|_{L_p(\mathcal{R}_n; E)} &= \sup_{\substack{g \in L_{p'}(\mathcal{R}_n; E^*) \\ \|g\|_{L_{p'}(\mathcal{R}_n; E^*)} \leq 1}} | \langle f - (\mathsf{E}_{[n]} \otimes \mathrm{Id}) f, \; g \rangle | \\
&= \sup_g \int_0^1 \Big(\langle \frac{1}{r} (T_r \otimes \mathrm{Id})(\Delta_{[n]} \otimes \mathrm{Id}) f, g \rangle \, dr \\
&= \sup_g \int_0^1 \Big(\sum_{j=1}^n \Big(\langle \frac{1}{r} (\partial_j \otimes \mathrm{Id}) f, \; (\partial_j \otimes \mathrm{Id})(T_r \otimes \mathrm{Id}) g \rangle \, dr
\end{aligned}$$

since T_r is self-adjoint. Now, the identity (3.9) from Lemma 3.2.2 implies

$$\begin{aligned}\|f - (\mathsf{E}_{[n]} - \mathrm{Id})f\|_{L_p(\mathcal{R}_n;E)} &\leq \sup_g \int_0^1 \sum_{j=1}^n \langle (\partial_j \otimes \mathrm{Id})f, \frac{1}{r}(\partial_j \otimes \mathrm{Id})(T_r \otimes \mathrm{Id})g\rangle \; dr \\ &= 2 \sup_g \int_0^1 \sum_{j=1}^n \langle (\partial_j \otimes \mathrm{Id})f, \mathbb{E}_\eta \Big[\frac{\eta_j}{1+\eta_j r}(A_\eta \otimes \mathrm{Id})g\Big]\rangle \; dr \\ &= 2 \sup_g \int_0^1 \mathbb{E}_\eta \langle \sum_{j=1}^n \frac{\eta_j}{1+\eta_j r}(\partial_j \otimes \mathrm{Id})f, (A_\eta \otimes \mathrm{Id})g\rangle \; dr,\end{aligned}$$

so the duality identity

$$L_p(\mathcal{R}_n \otimes L_\infty(\Omega_n, d\eta); E)^* = L_{p'}(\mathcal{R}_n \otimes L_\infty(\Omega_n, d\eta); E^*)$$

yields

$$\|f - (\mathsf{E}_{[n]} - \mathrm{Id})f\|_{L_p(\mathcal{R}_n;E)}$$

(3.11)

$$\lesssim \sup_g \int_0^1 \Big[\mathbb{E}_\eta \Big\| \sum_{j=1}^n \frac{\eta_j}{1+\eta_j r}(\partial_j \otimes \mathrm{Id})f\Big\|^p_{L_p(\mathcal{R}_n;E)}\Big]^{1/p} \Big[\mathbb{E}_\eta \|(A_\eta \otimes \mathrm{Id})g\|^{p'}_{L_{p'}(\mathcal{R}_n;E^*)}\Big]^{1/p'} \; dr.$$

We claim that

$$\|(A_\eta \otimes \mathrm{Id})g\|_{L_{p'}(\mathcal{R}_n;E^*)} \leq \|g\|_{L_{p'}(\mathcal{R}_n;E^*)}$$

holds for any $\eta \in \Omega_n$, what yields the statement of the theorem.

The norm on $L_{p'}(\mathcal{R}_n;E^*)$ admits the expression

$$\|(A_\eta \otimes \mathrm{Id})g\|_{L_{p'}(\mathcal{R}_n;E^*)} = \inf\{\|a'\|_{L_{2p'}(\mathcal{R}_n)} \|v'\|_{\mathcal{R}_n \otimes_{\min} E^*} \|b'\|_{L_{2p'}(\mathcal{R}_n)}\}$$

where $a', b' \in \mathcal{R}_n$ and $v' \in \mathcal{R}_n \otimes E^*$ satisfy $g = a' \cdot v' \cdot b'$. An arbitrary decomposition for g, say, $g = a \cdot v \cdot b$ where

$$a = \sum_{\mathsf{A} \subseteq [n]} \alpha_A \; Q_{\mathsf{A}}, \;\; v = \sum_{C \subseteq [n]} Q_C \otimes \widehat{v}(C), \;\; b = \sum_{B \subseteq [n]} \beta_B \; Q_{\mathsf{B}},$$

yields a valid decomposition for $(A_\eta \otimes \mathrm{Id})g$ by taking $a' = A_\eta a$, $b' = A_\eta b$ and $v' = (A_\eta \otimes \mathrm{Id})v$. Indeed, if $p' \in \mathbb{N}$,

$$\begin{aligned}\|A_\eta a\|^{2p'}_{L_{2p'}(\mathcal{R}_n)} &= \tau_{\mathcal{R}_n}\Bigg[\sum_{\substack{A_i \subseteq [n] \\ 1 \leq i \leq 2p'}} \Big(\prod_{j \in \mathbb{I}(\{A_i\}_{i=1}^{2p'})} \eta_j\Big) \overline{\alpha_1}\alpha_2 \ldots \overline{\alpha_{2p'-1}}\alpha_{2p'} \; Q^*_{A_1} Q_{A_2} \ldots Q^*_{A_{2p'-1}} Q_{A_{2p'}}\Bigg] \\ &= \sum_{\substack{A_i \subseteq [n] \\ 1 \leq i \leq 2p' \\ \mathbb{I}(\{A_i\}_{i=1}^{2p'}) = \emptyset}} \overline{\alpha_1}\alpha_2 \ldots \overline{\alpha_{2p'-1}}\alpha_{2p'} = \|a\|^{2p'}_{L_{2p'}(\mathcal{R}_n)}\end{aligned}$$

where $\mathbb{I}(\{A_i\}_{i=1}^{2p'}) = \{j \in [n] \,:\, |\{i \in [2p'] : j \in A_i\}| \text{ is odd}\}$. An analogous identity holds for $A_\eta b$. Hence, by interpolation and duality, we infer that

$$\|A_\eta a\|_{L_{p'}(\mathcal{R}_n)} = \|a\|_{L_{p'}(\mathcal{R}_n)}, \quad \|A_\eta b\|_{L_{p'}(\mathcal{R}_n)} = \|b\|_{L_{p'}(\mathcal{R}_n)}$$

holds for $1 < p' \leq \infty$. On the other hand, according to the formula for the minimal tensor product of operator spaces [69], and recalling that $E^* \subseteq B(K)$ for some Hilbert space K, it follows that

$$\begin{aligned}\|(A_\eta \otimes \mathrm{Id})v\|_{\mathcal{R}_n\otimes E^*} &= \|\sum_{C\subseteq[n]} \Big(\prod_{j\in C}\eta_j\Big) Q_C \otimes \widehat{v}(C)\|_{\mathcal{R}_n\otimes_{\min}E^*}\\ &= \sup_{m,K_m} \|\sum_{C\subseteq[n]} \Big(\prod_{j\in C}\eta_j\Big) Q_C \otimes P_{K_m}\widehat{v}(C)\restriction_{K_m}\|_{\mathcal{R}_n\otimes B(K_m)},\end{aligned}$$

so that the supremum is taken on any m natural and any $m-$dimensional Hilbert space $K_m \subseteq K$, so that $x\restriction_{K_m}$ denotes the restriction to K_m of an operator $x \in B(K)$ and P_{K_m} is the orthogonal projection from K into K_m. Supposing an orthonormal basis is chosen in K, we can identify $B(K_m)$ with M_m, so the norms on the right hand side are just norm of operators on some (finite dimensional) von Neumann algebra. Then, given $q \in \mathbb{N}$, for any K_m,

$$\begin{aligned}&\|\sum_{C\subseteq[n]} \Big(\prod_{j\in C}\eta_j\Big) Q_C \otimes P_{K_m}\widehat{v}(C)\restriction_{K_m}\|_{L_{2q}(\mathcal{R}_n\otimes M_m)}^{2q}\\ &= \tau_{\mathcal{R}_n\otimes M_m}\Bigg[\sum_{\substack{C_i\subseteq[n]\\1\leq i\leq 2q}} \Big(\prod_{j\in\mathbb{I}(\{C_i\}_{i=1}^{2q})}\eta_j\Big) Q_{C_1}^*\ldots Q_{C_{2q}} \otimes (P_{K_m}\widehat{v}(C_1)\restriction_{K_m})^*\ldots P_{K_m}\widehat{v}(C_n)\restriction_{K_m}\Bigg]\\ &= \sum_{\substack{C_i\subseteq[n]\\1\leq i\leq 2q\\ \mathbb{I}(\{C_i\}_{i=1}^{2q})=\emptyset}} \tau_{M_m}[(P_{K_m}\widehat{v}(C_1)\restriction_{K_m})^*\ldots P_{K_m}\widehat{v}(C_n)\restriction_{K_m}]\\ &= \|\sum_{C\subseteq[n]} Q_C \otimes P_{K_m}\widehat{v}(C)\restriction_{K_m}\|_{L_{2q}(\mathcal{R}_n\otimes M_m)}^{2q}.\end{aligned}$$

By interpolation between $2q$ and $2(q+1)$ for $q \geq 1$, if follows that

$$\begin{aligned}&\|\sum_{C\subseteq[n]} \Big(\prod_{j\in C}\eta_j\Big) Q_C \otimes P_{K_m}\widehat{v}(C)\restriction_{K_m}\|_{L_{p'}(\mathcal{R}_n\otimes M_m)}\\ &\qquad = \|\sum_{C\subseteq[n]} Q_C \otimes P_{K_m}\widehat{v}(C)\restriction_{K_m}\|_{L_{p'}(\mathcal{R}_n\otimes M_m)}\end{aligned}$$

holds for any $2 < p' < \infty$. Therefore, for any m natural and $K_m \subseteq K$,

$$\begin{aligned}\| \sum_{C\subseteq[n]} \Big(\prod_{j\in C} \eta_j\Big) Q_C \otimes P_{K_m} \widehat{v}(C) \restriction_{K_m} \|_{\mathcal{R}_n \otimes B(K_m)} &= \lim_{p'\to\infty} \| \sum_{C\subseteq[n]} \Big(\prod_{j\in C} \eta_j\Big) Q_C \otimes P_{K_m} \widehat{v}(C) \restriction_{K_m} \|_{L_{p'}(\mathcal{R}_n \otimes M_m)} \\ &= \lim_{p'\to\infty} \| \sum_{C\subseteq[n]} Q_C \otimes P_{K_m} \widehat{v}(C) \restriction_{K_m} \|_{L_{p'}(\mathcal{R}_n \otimes M_m)} \\ &= \| \sum_{C\subseteq[n]} Q_C \otimes P_{K_m} \widehat{v}(C) \restriction_{K_m} \|_{\mathcal{R}_n \otimes B(K_m)}.\end{aligned}$$

Taking supremum over m and K_m, this identity yields

$$\|(A_\eta \otimes \mathrm{Id})v\|_{\mathcal{R}_n \otimes_{\min} E^*} = \|v\|_{\mathcal{R}_n \otimes_{\min} E^*}.$$

In conclusion,

$$\begin{aligned}\|(A_\eta \otimes \mathrm{Id})g\|_{L_{p'}(\mathcal{R}_n;E^*)} &\leq \inf_{a\cdot v\cdot b=g} \{\|A_\eta a\|_{L_{2p'}(\mathcal{R}_n)} \ \|(A_\eta \otimes \mathrm{Id})v\|_{\mathcal{R}_n \otimes_{\min} E^*} \ \|A_\eta b\|_{L_{2p'}(\mathcal{R}_n)}\} \\ &= \inf_{a\cdot v\cdot b=g} \{\|a\|_{L_{2p'}(\mathcal{R}_n)} \ \|v\|_{\mathcal{R}_n \otimes_{\min} E^*} \ \|b\|_{L_{2p'}(\mathcal{R}_n)}\} \\ &= \|g\|_{L_{p'}(\mathcal{R}_n;E^*)},\end{aligned}$$

which, when implemented in (3.11) implies the statement. □

Chapter 4

Calderón-Zygmund operators with operator-valued kernel

This chapter is related to the theory of semicommutative Calderón-Zygmund operators. This is a research line that takes advantage of the hybrid nature of certain vector-valued L_p spaces. Let $(\mathcal{M}, \tau)$ be a von Neumann algebra of operators on a separable Hilbert space, equipped with a normal semifinite faithful trace τ. Denote by $\mathcal{A}$ the weak operator closure of the space of essentially bounded (strongly measurable) functions $f : \mathbb{R}^n \longrightarrow \mathcal{M}$. For the sake of exposition, we will restrict ourselves to the case $n = 1$, even though our arguments extend trivially to any finite dimension. The von Neumann algebra $\mathcal{A}$ can be identified with the tensor product $L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M}$ equipped with the trace

$$\varphi(f) = \int_{\mathbb{R}} \tau(f(x))\, dx.$$

The noncommutative L_p spaces associated with $\mathcal{A}$ are indeed vector-valued L_p spaces: indeed, [68, Chapter 3]

$$L_p(\mathcal{A}) = L_p(\mathbb{R}; L_p(\mathcal{M})),$$

for $1 \leq p < \infty$. We are interested in endpoint estimates for operators acting on $L_p(\mathcal{A})$, and in particular in the boundedness of operators from the operator-valued version of the Hardy space H_1 into L_1. This question was widely studied in the classical setting for scalar-valued functions [56, 57] as well as for vector-valued functions [18, 32], where the existence of the atomic decomposition plays an essential role. This technique does not seem to have been exploited as often in the noncommutative setting. Mei [54] was the first to introduce the so-called *operator-valued Hardy space* $H_1(\mathbb{R}, \mathcal{M})$ in this context via noncommutative equivalents of the Poisson integral, the Lusin area integral and the Littlewood-Paley g function. These techniques allowed Mei to compute the dual space of $H_1(\mathbb{R}, \mathcal{M})$, which is denoted by $\mathsf{BMO}(\mathbb{R}, \mathcal{M})$, in the spirit of the classical argument by Fefferman and Stein [17]. Moreover, some maximal inequalities, and several interpolation results via a martingale approach were established. Mei's fundamental contribution has been key in the development of noncommutative forms of Calderón-Zygmund theory, both in the semicommutative context and in fully noncommutative ones via transference techniques. For the first one, the semicommutative Calderón-Zygmund theory was initiated in [66] with the obtention of weak L_1 endpoint inequalities for singular integrals, with an argument that was simplified in recent years [6, 7]. The second line has many instances, among which are [25, 41].

The initial motivation for this chapter was obtaining new interpolation consequences of endpoint estimates of the type $L_\infty(\mathcal{A}) - \mathsf{BMO}(\mathbb{R}, \mathcal{M})$ which rely, by duality, on the structure of the Hardy space $H_1(\mathbb{R}, \mathcal{M})$. Our goal led to two main tasks to tackle: a completely explicit description of $\mathsf{BMO}(\mathbb{R}, \mathcal{M})$, and the study of the boundedness of Calderón-Zygmund operators on the Hardy space via its atomic decomposition. The operator-valued BMO space introduced by Mei, $\mathsf{BMO}(\mathbb{R}, \mathcal{M})$, is defined as the intersection of a column space and a row space, $\mathsf{BMO}_c(\mathbb{R}, \mathcal{M})$ and $\mathsf{BMO}_r(\mathbb{R}, \mathcal{M})$ respectively. The reason for considering both a column and a row space is a ubiquitous phenomenon in noncommutative analysis (see [48] for an outstanding example), and by symmetry we shall limit our discussion to the column case. $\mathsf{BMO}_c(\mathbb{R}, \mathcal{M})$ is set to be the subspace of the column Hilbert-valued space [37] $L_\infty(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dt}{1+t^2}))$ for which the seminorm

(MBMO) $$\|g\|_{\mathsf{BMO}_c} = \sup_{\substack{I \subseteq \mathbb{R} \\ |I| \text{ finite}}} \Big\| \Big(\frac{1}{|I|} \int_I \big| g - g_I \big|^2 \Big)^{1/2} \Big\|_{\mathcal{M}}$$

is finite, where $g_I = \frac{1}{|I|} \int_I g$. The $\mathsf{BMO_r}$ seminorm is $\|g\|_{\mathsf{BMO}_r} = \|g^*\|_{\mathsf{BMO}_c}$. The space $L_\infty(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dt}{1+t^2}))$ denotes the closure of $\mathcal{M} \otimes L_2(\mathbb{R}, \frac{dt}{1+t^2})$ with respect to the weak* topology of the von Neumann algebra $\mathcal{M} \overline{\otimes} B(L_2(\mathbb{R}, \frac{dt}{1+t^2}))$. There is no guarantee that $\mathsf{BMO}_c(\mathbb{R}, \mathcal{M})$ is a space of $\mathcal{M}-$valued functions (in the Bochner sense), and so the integral in (MBMO) may not be well-defined. Indeed,

$$L_2\Big(\mathbb{R}, \frac{dt}{1+t^2}; \mathcal{M}\Big) \subseteq L_\infty\Big(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dt}{1+t^2})\Big),$$

but the reverse inclusion fails in general, *a priori* preventing us from defining BMO_c as a space of functions. In Section 4.1 and the first part of Section 4.2, we propose a general construction of $\mathsf{BMO}_c(\mathbb{R}, \mathcal{M})$ which recovers Mei's description given by (MBMO). We will also study a predual of $\mathsf{BMO}_c(\mathbb{R}, \mathcal{M})$ (resp. $\mathsf{BMO}_r(\mathbb{R}, \mathcal{M})$). The novelty here is that this predual space, which we denote as $H_1^r(\mathcal{A})$ (resp. $H_1^c(\mathcal{A})$), will be a row (resp. column) Hardy space which is exclusively constructed in terms of "new" atomic decompositions, which extend the work in Ricard's Ph.D. Thesis [72].

The key to our approach is the $H_1 - \mathsf{BMO}$ duality product when elements in the former space are described in terms of atoms. In the classical case, it is a well-known fact that the norm of $g \in \mathsf{BMO}(\mathbb{R})$ can be characterized through the expression

(atBMO) $$\|g\|_{\mathsf{BMO}} = \sup_a \Big| \int g a \Big|,$$

so that the supremum is taken over L_2-atoms [20, 56, 57]. An analogous formula for $\mathsf{BMO}_r(\mathbb{R},\mathcal{M})$ may shed light on the structure of atoms in $H_1^c(\mathcal{A})$. This is exactly what we achieve. Let $g \in \mathcal{A} \cap \mathsf{BMO}_r(\mathbb{R},\mathcal{M})$. The expression (MBMO) is meaningful for g, and so duality yields

$$\begin{aligned}
\|g\|_{\mathsf{BMO}_r} &= \|g^*\|_{\mathsf{BMO}_c} = \sup_I \Big\| \Big(\frac{1}{|I|}\int_I \big|g^* - (g^*)_I\big|^2\Big)^{1/2}\Big\|_{\mathcal{M}} \\
&= \sup_{I,h} \Big(\frac{1}{|I|}\int_I \|(g^* - (g^*)_I)h\|_{L_2(\mathcal{M})}^2\Big)^{1/2} = \sup_{I,h} \Big(\int_I \|\frac{1}{\sqrt{|I|}}h(g-g_I)\|_{L_2(\mathcal{M})}^2\Big)^{1/2}.
\end{aligned}$$

with the supremum taken over h in the unit ball of $L_2(\mathcal{M})$. Now, recalling that $g - g_I$ has zero integral over I and considering the Hilbert space

$$L_2^\circ(I; L_2(\mathcal{M})) = \{f \in L_2(I; L_2(\mathcal{M})) \ : \ \int_I f = 0\},$$

it follows that

$$\begin{aligned}
\|g\|_{\mathsf{BMO}_r} &= \sup_{I,h} \Big(\int_I \|\frac{1}{\sqrt{|I|}}h(g-g_I)\|_{L_2(\mathcal{M})}^2\Big)^{1/2} = \sup_{I,h} \Big\|\frac{1}{\sqrt{|I|}}h(g-g_I)\chi_I\Big\|_{L_2^\circ(I;L_2(\mathcal{M}))} \\
&= \sup_{I,h} \sup_{\|f\|_{L_2^\circ}\le 1} \Big|\big(\tau\circ \int\big)\Big(h(g-g_I)\frac{f\chi_I}{\sqrt{|I|}}\Big)\Big| = \sup_{I,h,f} \Big|\big(\tau\circ\int\big)\Big((g-g_I)\frac{f\chi_I}{\sqrt{|I|}}h\Big)\Big| \\
&= \sup_{I,h,f} \Big|\big(\tau\circ \int\big)\Big(g\frac{f\chi_I}{\sqrt{|I|}}h\Big)\Big|.
\end{aligned}$$

Comparing the latter expression with (atBMO) suggests that an atom in $H_1^c(\mathbb{R},\mathcal{M})$ should be an operator of the form $a = bh$ in $L_1(\mathcal{A})$, where $h \in L_2(\mathcal{M})$ with $\|h\|_{L_2(\mathcal{M})} \le 1$ and $b \in L_2(\mathcal{A})$ is supported on some interval I and has additional cancellation over I that we will make precise later. In what follows, a will be called a *$c-$atom*. Then, define the column Hardy space $H_1^c(\mathcal{A})$ as the Banach subspace of $L_1(\mathcal{A})$ of those operators f such that

$$f = \sum_{i=0}^{\infty} \lambda_i a_i \ \text{in } L_1(\mathcal{A}), \ \text{for some } (a_i)_i \ c-\text{atoms}, \ (\lambda_i)_i \in \ell_1,$$

which becomes a Banach space with respect to the norm

$$\|f\|_{H_1^c} = \inf\Big\{ \sum_{i=0}^{\infty} |\lambda_i| \ : \ f = \sum_{i=0}^{\infty}\lambda_i a_i\Big\}.$$

The row space $H_1^r(\mathcal{A})$ is defined analogously and $H_1(\mathcal{A}) = H_1^c(\mathcal{A}) + H_1^r(\mathcal{A})$. By symmetry, it suffices to show $H_1^c(\mathcal{A})^* = \mathsf{BMO}_r(\mathbb{R},\mathcal{M})$. The proof of this duality result strongly relies on the extension of a well-known argument by Meyer [56]: the space

$$L_2^\circ(\mathbb{R},(1+t^2)dt) = \Big\{ f \in L_2(\mathbb{R},(1+t^2)dt) \ : \ \int_{\mathbb{R}} f = 0 \Big\}$$

is a dense subspace of the classical atomic Hardy space $H_1(\mathbb{R})$. A further characterization of the column Hilbert-valued spaces $L_\infty(\mathcal{M};\mathcal{H}^c)$ will be the key to establish an analogous result in our context. On the other hand, it is still an open question whether the column Hardy space $H_1^c(\mathcal{A})$ coincides with the one introduced by Mei in [54].

It is worth noting that Mei's work already contained a description of $H_1^c(\mathbb{R},\mathcal{M})$ in terms of certain atomic decompositions. However, the one included in the present chapter is more useful to establish estimates for Calderón-Zygmund operators. Indeed, it allows us to consider singular integrals with noncommuting kernels, something that is usually not a possibility at the weak L_1 level [7]. Let $\mathcal{M}$ be a von Neumann algebra over a separable Hilbert space, and let T be a bounded operator on $L_2(\mathcal{A})$ for which there exists a kernel $K:\mathbb{R}\times\mathbb{R}\setminus\{x=y\}\longrightarrow\mathcal{M}$ such that

$$\int T(f)(x)g(x)\,dx = \int\int K(x,y)f(y)g(x)\,dx\ dy$$

holds for any compactly supported $f,g\in L_\infty(\mathcal{A})\cap L_2(\mathcal{A})$ satisfying that the distance between the supports $\mathrm{supp}_{\mathbb{R}}\|f\|_{L_2(\mathcal{M})}$ and $\mathrm{supp}_{\mathbb{R}}\|g\|_{L_2(\mathcal{M})}$ is strictly greater than zero. In that case, we will say that T is a *Calderón-Zygmund operator*. Also, assume that K and T fulfil

- $T(fh)=T(f)h$ for any $f\in L_2(\mathcal{A})$ with compact support and $h\in\mathcal{M}$, and
- the *Hörmander condition*

$$\int_{|x-y|\geq 2|y'-y|}\|K(x,y)-K(x,y')\|_{\mathcal{M}}\ dx<\infty.$$

Under these assumptions, T extends to a bounded map $T:H_1^c(\mathcal{A})\longrightarrow L_1(\mathcal{A})$. The proof of this statement is inspired by [57] and is divided in two steps. The first one consists of obtaining a universal constant $C>0$ such that

$$\|T(a)\|_{L_1(\mathcal{A})}\leq C \text{ for any } c-atom\ a.$$

In order to establish this statement, we take advantage of the modularity identity

$$T(bh)=T(b)h \text{ for any } c\text{−atom } a=bh,$$

which follows from our definition of Calderón-Zygmund operators, and allows us to exploit the boundedness of T on $L_2(\mathcal{A})$. On the other hand, showing that T extends to the whole $H_1^c(\mathcal{A})$ requires the approximation of K by certain bounded kernels. Once we have done that, we obtain that whenever K is scalar, T extends to a bounded operator from $H_1(\mathcal{A})$ into $L_1(\mathcal{A})$.

The rest of the chapter is organized as follows. In Section 1, we shall introduce the column and row Hilbert-valued noncommutative L_p spaces. This enables us to define $\mathsf{BMO}(\mathbb{R},\mathcal{M})$, as well as to identify a predual $H_1^c(\mathcal{A})$ in Section 2. Finally, in Section 3 we will see how the atomic decomposition provides a boundedness result for Calderón-Zygmund operators with noncommuting kernels.

4.1 Column/row Hilbert-valued L_p spaces

Let $\mathcal{H}$ be a separable Hilbert space. $B(\mathcal{H})$ can be identified as the space of bounded infinite matrices acting on $\mathcal{H}$, and when equipped with the usual trace for matrices Tr, it gives rise to the Schatten classes $S_p(\mathcal{H}) = L_p(B(\mathcal{H}), \mathrm{Tr})$ for any $0 < p \leq \infty$. Along this chapter, the inner product in $\mathcal{H}$ is assumed to be linear in the first variable and antilinear in the second one. Moreover, elements in the dual Hilbert space $\mathcal{H}^* \simeq \overline{\mathcal{H}}$ will be represented with an overlined letter. For instance, given $h \in \mathcal{H}$, $\overline{h}$ will denote the continuous functional

$$\overline{h} : k \mapsto \langle k, h \rangle \text{ for any } k \in \mathcal{H}.$$

Given two elements ξ and η of $\mathcal{H}$, we consider the *rank-one operator* $\xi \otimes \eta$ acting on $\mathcal{H}$ as follows

$$(\xi \otimes \eta)(h) = \langle h, \eta \rangle \; \xi \text{ for any } h \in \mathcal{H}.$$

Lemma 4.1.1. *Let $0 < p \leq \infty$. Then, for any $\xi, \xi', \eta, \eta' \in \mathcal{H}$, the following properties hold:*

(1) $(\xi \otimes \eta)^* = \eta \otimes \xi$.

(2) $(\xi \otimes \eta)(\xi' \otimes \eta') = \langle \xi', \eta \rangle \; \xi \otimes \eta'$,

(3) For any $0 < p \leq \infty$, $\|\xi \otimes \eta\|_{S_p(\mathcal{H})} = \|\xi\|_{\mathcal{H}} \|\eta\|_{\mathcal{H}}$,

(4) $\mathrm{Tr}(\eta \otimes \xi) = \langle \eta, \xi \rangle$,

(5) $\mathrm{Tr}((\xi \otimes \eta)(\xi' \otimes \eta)) = \langle \xi', \eta \rangle \; \langle \xi, \eta' \rangle$.

Proof. Given two elements h, h' in $\mathcal{H}$, there holds

$$\langle (\xi \otimes \eta)h, h' \rangle = \langle \langle h, \eta \rangle \; \xi, h' \rangle = \langle h, \langle h', \xi \rangle \; \eta \rangle = \langle h, (\eta \otimes \xi) h' \rangle,$$

what yields (1). Then, it easily follows that

$$\begin{aligned}(\xi \otimes \eta)(\xi' \otimes \eta')(h) &= (\xi \otimes \eta)\langle h, \eta' \rangle \; \xi' = \langle \xi', \eta \rangle \; \langle h, \eta' \rangle \; \xi \\ &= \langle \xi', \eta \rangle \; (\xi \otimes \eta')(h).\end{aligned}$$

In order to prove (3), consider an orthonormal basis for $\mathcal{H}$ such that $\eta / \|\eta\|_{\mathcal{H}}$ belongs to it. Then, it is clear that $(\xi \otimes \eta)^*(\xi \otimes \eta)$ is a self-adjoint compact operator and its range coincides with the subspace generated by $\eta / \|\eta\|_{\mathcal{H}}$. Therefore,

$$\|\xi \otimes \eta\|^2_{S_p(\mathcal{H})} = \| |\xi \otimes \eta|^2 \|_{S_{p/2}(\mathcal{H})} = |\langle |\xi \otimes \eta|^2 \frac{\eta}{\|\eta\|}, \frac{\eta}{\|\eta\|} \rangle| = \|\xi\|^2_{\mathcal{H}} \|\eta\|^2_{\mathcal{H}},$$

since

$$|\xi \otimes \eta|^2 \Big(\frac{\eta}{\|\eta\|_{\mathcal{H}}} \Big) = \|\xi\|^2_{\mathcal{H}} \|\eta\|^2_{\mathcal{H}} \; \frac{\eta}{\|\eta\|_{\mathcal{H}}}.$$

Under these assumptions, it follows $(\eta \otimes \xi)(\frac{\eta}{\|\eta\|_{\mathcal{H}}}) = \langle \frac{\eta}{\|\eta\|_{\mathcal{H}}}, \xi\rangle \eta = \langle \eta, \xi\rangle \frac{\eta}{\|\eta\|_{\mathcal{H}}}$, so the only eigenvalue with respect to the fixed basis is $\langle \eta, \xi\rangle$. This proves (4), and (5) is a straightforward consequence. □

In the following, let $\mathbb{1}$ denote a fixed element of $\mathcal{H}$ with $\|\mathbb{1}\|_{\mathcal{H}} = 1$, and let $p_{\mathbb{1}} = \mathbb{1} \otimes \mathbb{1}$ denote the rank one projection onto $\operatorname{span}\{\mathbb{1}\}$. Assume that $\mathcal{M}$ is an arbitrary semifinite von Neumann algebra equipped with a normal semifinite faithful trace τ. Then, we define the *column Hilbert-valued L_p space*

$$L_p(\mathcal{M};\mathcal{H}^c) = L_p(\mathcal{M}\overline{\otimes}B(\mathcal{H}))(\mathbf{1}_{\mathcal{M}} \otimes p_{\mathbb{1}}).$$

for any $0 < p \leq \infty$. Identify $L_p(\mathcal{M})$ as a subspace of $L_p(\mathcal{M}\overline{\otimes}B(\mathcal{H}))$ via the map $m \mapsto m \otimes p_{\mathbb{1}}$. This is equivalent to the identity

$$L_p(\mathcal{M}) = (\mathbf{1}_{\mathcal{M}} \otimes p_{\mathbb{1}})L_p(\mathcal{M}\overline{\otimes}B(\mathcal{H}))(\mathbf{1}_{\mathcal{M}} \otimes p_{\mathbb{1}}).$$

Then, given an element f in $L_p(\mathcal{M};\mathcal{H}^c)$,

$$f^*f \in (\mathbf{1}_{\mathcal{M}} \otimes p_{\mathbb{1}})L_{p/2}(\mathcal{M}\overline{\otimes}B(\mathcal{H}))(\mathbf{1}_{\mathcal{M}} \otimes p_{\mathbb{1}}) = L_{p/2}(\mathcal{M}).$$

This justifies defining, up to some identifications,

$$\|f\|_{L_p(\mathcal{M};\mathcal{H}^c)} = \|(f^*f)^{1/2}\|_{L_p(\mathcal{M})}$$

on $L_p(\mathcal{M};\mathcal{H}^c)$. Analogously, we will consider the *row Hilbert-valued L_p space* associated to $\mathcal{H}^* = \overline{\mathcal{H}}$,

$$L_p(\mathcal{M};\overline{\mathcal{H}}^r) = (\mathbf{1}_{\mathcal{M}} \otimes p_{\mathbb{1}})L_p(\mathcal{M}\overline{\otimes}B(\mathcal{H})).$$

equipped with the norm

$$\|f\|_{L_p(\mathcal{M};\overline{\mathcal{H}}^r)} = \|(ff^*)^{1/2}\|_{L_p(\mathcal{M})},$$

so that $\|f\|_{L_p(\mathcal{M};\overline{\mathcal{H}}^r)} = \|f^*\|_{L_p(\mathcal{M};\mathcal{H}^c)}$. In fact, column and row Hilbert-valued L_p spaces satisfy the expected duality relations expressed via the bracket

$$\begin{aligned}(m_1 \otimes (h_1 \otimes \mathbb{1}), m_2 \otimes (\mathbb{1} \otimes h_2))_{c,r} &= \operatorname{Tr}((\mathbb{1} \otimes h_2)(h_1 \otimes \mathbb{1}))\, \tau_{\mathcal{M}}(m_2m_1)\\ &= \operatorname{Tr}(\langle h_1, h_2\rangle_{\mathcal{H}}\ p_{\mathbb{1}})\, \tau_{\mathcal{M}}(m_2m_1)\\ &= \langle h_1, h_2\rangle_{\mathcal{H}}\ \tau_{\mathcal{M}}(m_2m_1). \qquad (4.1)\end{aligned}$$

In particular, it holds

$$L_p(\mathcal{M};\mathcal{H}^c)^* = L_{p'}(\mathcal{M};\overline{\mathcal{H}}^r) \text{ and } L_p(\mathcal{M};\overline{\mathcal{H}}^r)^* = L_{p'}(\mathcal{M};\mathcal{H}^c).$$

for any $1 \leq p < \infty$ whenever $1/p + 1/p' = 1$.

Lemma 4.1.2. *Given $0 < p \leq \infty$ and an operator*

$$f = \sum_{i=1}^{n} m_i \otimes h_i \in L_p(\mathcal{M}) \otimes \mathcal{H},$$

f can be interpreted as an element of $L_p(\mathcal{M};\mathcal{H}^c)$ and $L_p(\mathcal{M};\overline{\mathcal{H}}^r)$ with the respective norms

$$\|f\|_{L_p(\mathcal{M};\mathcal{H}^c)} = \Big\| \sum_{i=1}^{n} m_i \otimes (h_i \otimes \mathbb{1}) \Big\|_{L_p(\mathcal{M};\mathcal{H}^c)} = \Big\| \Big(\sum_{i,j=1}^{n} \langle h_j, h_i \rangle_{\mathcal{H}} \; m_i^* m_j \Big)^{1/2} \Big\|_{L_p(\mathcal{M})}$$

and

$$\|f\|_{L_p(\mathcal{M};\overline{\mathcal{H}}^r)} = \Big\| \sum_{i=1}^{n} m_i \otimes (\mathbb{1} \otimes h_i) \Big\|_{L_p(\mathcal{M};\overline{\mathcal{H}}^r)} = \Big\| \Big(\sum_{i,j=1}^{n} \langle h_j, h_i \rangle_{\mathcal{H}} \; m_i m_j^* \Big)^{1/2} \Big\|_{L_p(\mathcal{M})}.$$

Therefore, whenever p is finite, $L_p(\mathcal{M};\mathcal{H}^c)$ and $L_p(\mathcal{M};\overline{\mathcal{H}}^r)$ can be regarded as the completion of $L_p(\mathcal{M}) \otimes \mathcal{H}$ with respect to these norms above.

Proof. Given an element f in $L_p(\mathcal{M}) \otimes \mathcal{H}$ as above, then

$$f^* f = \Big(\sum_{i=1}^{n} m_i^* \otimes (\mathbb{1} \otimes h_i) \Big) \Big(\sum_{j=1}^{n} m_j \otimes (h_j \otimes \mathbb{1}) \Big) = \sum_{i,j=1}^{n} \langle h_j, h_i \rangle \; m_i^* m_j \otimes (\mathbb{1} \otimes \mathbb{1}).$$

Then, the statement follows from Lemma 4.1.1 (3). The norm for the row space can be computed analogously, and the last claims follows from [37, Lemma 2.2]. □

Corollary 4.1.3. *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces, and let $T : \mathcal{H} \longrightarrow \mathcal{K}$ be a bounded linear operator. Then $\mathrm{Id}_{\mathcal{M}} \otimes T$ admits a unique weak*-continuous bounded extension $\widetilde{T}$ from $L_\infty(\mathcal{M};\mathcal{H}^c)$ into $L_\infty(\mathcal{M};\mathcal{K}^c)$ satisfying*

$$\|\widetilde{T}\| = \|T\|.$$

Analogously, $\mathrm{Id}_{\mathcal{M}} \otimes T$ extends to a weak-continuous bounded map from $L_\infty(\mathcal{M}, \mathcal{H}^r)$ into $L_\infty(\mathcal{M}, \mathcal{K}^r)$.*

Proof. The statement above follows from [37, Lemma 2.4]: given $1 \leq p < \infty$, the map $\mathrm{Id}_{L_p(\mathcal{M})} \otimes T$ defined on $L_p(\mathcal{M}) \otimes \mathcal{H}$ extends uniquely to a bounded operator from $L_p(\mathcal{M};\mathcal{H}^c)$ into $L_p(\mathcal{M};\mathcal{K}^c)$ with norm $\|\mathrm{Id}_{L_p(\mathcal{M})} \otimes T\| = \|T\|$. In particular, if $T^* : \mathcal{K} \to \mathcal{H}$ is the adjoint map for T, then $\mathrm{Id}_{L_1(\mathcal{M})} \otimes T^*$ extends to a bounded map from $L_1(\mathcal{M};\overline{\mathcal{K}}^r)$ into $L_1(\mathcal{M};\overline{\mathcal{H}}^r)$ satisfying

$$\|\mathrm{Id}_{L_1(\mathcal{M})} \otimes T^* : L_1(\mathcal{M};\overline{\mathcal{K}}^r) \to L_1(\mathcal{M};\overline{\mathcal{H}}^r)\| = \|T^*\| = \|T\|.$$

Then, by duality, we can define $\widetilde{T}$ as the adjoint operator of the extension to $L_1(\mathcal{M};\overline{\mathcal{K}}^r)$ of $\mathrm{Id}_{L_1(\mathcal{M})} \otimes T^*$, so that $\widetilde{T}$ coincides with $\mathrm{Id}_{\mathcal{M}} \otimes T$ on $\mathcal{M} \otimes \mathcal{H}$. □

Some properties of these extension maps will be crucial in the following sections.

Lemma 4.1.4. *Let $\mathcal{H}$ be a Hilbert space, let S, T, and $(T_j)_{j=1}^{\infty}$ be some bounded operators on $\mathcal{H}$ and let $\widetilde{S}, \widetilde{T}, (\widetilde{T}_j)_{j=1}^{\infty}$ be the corresponding extensions from $L_\infty(\mathcal{M}, \mathcal{H}^t)$ to $L_\infty(\mathcal{M}, \mathcal{H}^t)$ for $t = c, r$. Then the following holds.*

(1) $\widetilde{ST} = \widetilde{S}\widetilde{T}$,

(2) If S and T commute, then $\widetilde{S}$ and $\widetilde{T}$ also commute,

(3) whenever $\sum_{j=1}^{\infty} T_j$ converges in the norm of $B(\mathcal{H})$, there holds $\widetilde{\sum_{j=1}^{\infty} T_j} = \sum_{j=1}^{\infty} \widetilde{T}_j$.

Proof. By symmetry, is it sufficient to consider the column case. For the first point, it is clear that the operators T^*S^* and $(ST)^*$ coincide on $L_1(\mathcal{M}) \otimes \mathcal{H}$. Then, by uniqueness, the extensions $\mathrm{Id}_{L_1(\mathcal{M})} \otimes T^*S^*$ and $\mathrm{Id}_{L_1(\mathcal{M})} \otimes (ST)^*$ coincide on $L_1(\mathcal{M}; \overline{\mathcal{H}}^r)$, yielding the identity

$$\langle (\widetilde{ST} - \widetilde{S}\widetilde{T})g, f\rangle = \langle g, (\mathrm{Id}_{L_1(\mathcal{M})} \otimes T^*S^* - \mathrm{Id}_{L_1(\mathcal{M})} \otimes (ST)^*)f\rangle = 0$$

for any f and g belonging to $L_1(\mathcal{M}; \overline{\mathcal{H}}^r)$ and $L_\infty(\mathcal{M}; \mathcal{H}^c)$ respectively. Then, (2) is an immediate consequence, while (3) follows from the linearity and continuity of the map $T \mapsto \widetilde{T}$. □

The column space $L_\infty(\mathcal{M}, \mathcal{H}^c)$ admits a interpretation as continuous functionals over the projective tensor product $(L_2(\mathcal{M})^* \otimes_2 \mathcal{H}^*)\widehat{\otimes}_\pi L_2(\mathcal{M})$ which will be useful in the study of the duality $H_1 - \mathsf{BMO}$.

Proposition 4.1.5. *Let $\mathcal{H}$ be a Hilbert space. Then the map*

$$\begin{array}{rccc} U: & \mathcal{M} \otimes \mathcal{H} & \longrightarrow & B(L_2(\mathcal{M}), L_2(\mathcal{M}) \otimes_2 \mathcal{H}) \\ & \sum_{i=1}^n m_i \otimes (h_i \otimes \mathbb{1}) & \longmapsto & \Big(k \mapsto \sum_{i=1}^n m_i k \otimes h_i\Big), \end{array}$$

extends to a weak-continuous isometry defined on $L_\infty(\mathcal{M}; \mathcal{H}^c)$. Moreover, there holds*

$$U(L_\infty(\mathcal{M}; \mathcal{H}^c)) = \overline{U(\mathcal{M} \otimes \mathcal{H})}^{w^*}$$

which is closed in norm, and the pre-adjoint map U_* is a surjective contraction.

Proof. First, we check that U is an isometry on $\mathcal{M} \otimes \mathcal{H}$. Indeed,

$$\begin{aligned} \Big\| \sum_{i=1}^n m_i \otimes (h_i \otimes \mathbb{1}) \Big\|_{L_\infty(\mathcal{M}; \mathcal{H}^c)} &= \Big\| \sum_{i=1}^n |m_i \otimes (h_i \otimes \mathbb{1})|^2 \Big\|^{1/2}_{\mathcal{M}\overline{\otimes}B(\mathcal{H})} \\ &= \Big\| \sum_{i,j=1}^n m_i^* m_j \langle h_j, h_i\rangle_{\mathcal{H}} \Big\|^{1/2}_{\mathcal{M}} \\ &= \sup_{k \in L_2(\mathcal{M})} \Big| \langle \sum_{i,j=1}^n m_i^* m_j k \, \langle h_j, h_i\rangle_{\mathcal{H}}, k\rangle_{L_2(\mathcal{M})} \Big|^{1/2} \end{aligned}$$

$$= \sup_{k\in L_2(\mathcal{M})} \Big(\sum_{i,j=1}^{n} \langle m_i^* m_j k, k\rangle_{L_2(\mathcal{M})} \langle h_j, h_i\rangle_{\mathcal{H}} \Big)^{1/2}$$

$$= \sup_{k\in L_2(\mathcal{M})} \Big(\sum_{i,j=1}^{n} \langle m_j k, m_i k\rangle_{L_2(\mathcal{M})} \langle h_j, h_i\rangle_{\mathcal{H}} \Big)^{1/2}$$

$$= \sup_{k\in L_2(\mathcal{M})} \Big(\sum_{i,j=1}^{n} \langle m_j k \otimes h_j, m_i k \otimes h_i\rangle_{L_2(\mathcal{M})\otimes_2 \mathcal{H}} \Big)^{1/2}$$

$$= \sup_{k\in L_2(\mathcal{M})} \Big\| \sum_{i=1}^{n} m_i k \otimes h_i \Big\|_{L_2(\mathcal{M})\otimes_2 \mathcal{H}}.$$

Since the dual of the projective tensor product $(L_2(\mathcal{M})^* \otimes_2 \mathcal{H}^*)\widehat{\otimes}_\pi L_2(\mathcal{M})$ coincides with $B(L_2(\mathcal{M}), L_2(\mathcal{M})\otimes_2 \mathcal{H})$ [76], then U induces a map U_* on the dense class $(L_2(\mathcal{M})^* \otimes \mathcal{H}^*) \otimes L_2(\mathcal{M})$. More clearly, given $f \otimes m' = (\sum_{j=1}^N \overline{m'_j} \otimes \overline{h'_j}) \otimes m' \in (L_2(\mathcal{M})^* \otimes_2 \mathcal{H}^*) \otimes L_2(\mathcal{M})$,

$$\begin{aligned}
\langle U(\sum_{i=1}^{n} m_i \otimes (h_i \otimes \mathbb{1})), f \otimes m'\rangle &= \sum_{i=1}^{n}\sum_{j=1}^{N} \langle m_i m' \otimes h_i, \overline{m'_j} \otimes \overline{h'_j}\rangle_{L_2(\mathcal{M})\otimes_2\mathcal{H}, L_2(\mathcal{M})^*\otimes_2\mathcal{H}^*} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{N} \tau_{\mathcal{M}}(m_i m' (m'_j)^*)\, \overline{h'_j}(h_i) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{N} \tau_{\mathcal{M}}(m_i m' (m'_j)^*)\, \langle h_i, h'_j\rangle \\
&= \sum_{i=1}^{n}\sum_{j=1}^{N} \tau_{\mathcal{M}}(m_i m' (m'_j)^*)\, \mathrm{Tr}((\mathbb{1} \otimes h'_j)(h_i \otimes \mathbb{1})) \\
&= \langle \sum_{i=1}^{n} m_i \otimes (h_i \otimes \mathbb{1}), \sum_{j=1}^{N} m'(m'_j)^* \otimes (\mathbb{1} \otimes h'_j)\rangle \\
&= \langle \sum_{i=1}^{n} m_i \otimes (h_i \otimes \mathbb{1}), U_*(f \otimes m')\rangle.
\end{aligned}$$

Now, it is clear that $U_*(f \otimes m')$ belongs to $L_1(\mathcal{M}) \otimes \mathcal{H}$ and that the map U_* is a contraction on $(L_2(\mathcal{M})^* \otimes \mathcal{H}^*) \otimes L_2(\mathcal{M})$. Indeed,

$$\begin{aligned}
\|U_*(f \otimes m')\|_{L_1(\mathcal{M};\overline{\mathcal{H}}^r)} &= \Big\| \sum_{j=1}^{N} m'(m'_j)^* \otimes (\mathbb{1} \otimes h'_j) \Big\|_{L_1(\mathcal{M};\overline{\mathcal{H}}^r)} \\
&\le \|m' \otimes (\mathbb{1} \otimes \mathbb{1})\|_{L_2(\mathcal{M}\overline{\otimes}B(\mathcal{H}))} \Big\| \sum_{j=1}^{N} (m'_j)^* \otimes (\mathbb{1} \otimes h'_j) \Big\|_{L_2(\mathcal{M};\overline{\mathcal{H}}^r)} \\
&= \|m'\|_{L_2(\mathcal{M})} \Big\| \sum_{j=1}^{N} m'_j \otimes h'_j \Big\|_{L_2(\mathcal{M})\otimes_2\mathcal{H}}
\end{aligned}$$

$$= \|m'\|_{L_2(\mathcal{M})} \, \| \sum_{j=1}^{N} \overline{n'_j} \otimes \overline{h'_j} \|_{L_2(\mathcal{M})^* \otimes_2 \mathcal{H}^*}.$$

Finally, U_* admits an extension to the whole $(L_2(\mathcal{M})^* \otimes_2 \mathcal{H}^*) \widehat{\otimes}_\pi L_2(\mathcal{M})$, so its adjoint map is a weak*-continuous contraction that will be denoted by U. Also, since simple tensors in $L_1(\mathcal{M}) \otimes \mathcal{H}$ are contained in the image of U_*, then U_* has dense range, so injectivity of U follows.

In addition, it turns out that U is an isometry. Given $g \in L_\infty(\mathcal{M}; \mathcal{H}^c)$ and $m \in L_2(\mathcal{M})$, there holds

$$U(g)m = (\pi \otimes \mathrm{Id}_{B(\mathcal{H})})(g)(m \otimes \mathbb{1}) = (\pi \otimes \mathrm{Id}_{B(\mathcal{H})})(g)(j(m))$$

where $\pi : \mathcal{M} \longrightarrow B(L_2(\mathcal{M}))$ is the canonical normal faithful representation by left multiplication, and j is the isometry from $L_2(\mathcal{M})$ to $L_2(\mathcal{M}) \otimes_2 \mathcal{H}$ sending f to $f \otimes \mathbb{1}$. Then, the adjoint map j^* is a contraction sending $m \otimes h$ to $\langle h, \mathbb{1} \rangle m$, and a straightforward verification shows that $jj^* = \mathrm{Id}_{L_2(\mathcal{M})} \otimes (\mathbb{1} \otimes \mathbb{1})$, so

$$\begin{aligned} U(g)j^* &= (\pi \otimes \mathrm{Id}_{B(\mathcal{H})})(g) \circ (\mathrm{Id}_{L_2(\mathcal{M})} \otimes (\mathbb{1} \otimes \mathbb{1})) \\ &= (\pi \otimes \mathrm{Id}_{B(\mathcal{H})})(g). \end{aligned}$$

Moreover, since $\pi \otimes \mathrm{Id}_{B(\mathcal{H})} : \mathcal{M} \overline{\otimes} B(\mathcal{H}) \longrightarrow \pi(\mathcal{M}) \overline{\otimes} B(\mathcal{H})$ is a faithful representation of von Neumann algebras, it is isometric. Then, there follows

$$\|g\|_{L_\infty(\mathcal{M};\mathcal{H}^c)} = \|(\pi \otimes \mathrm{Id}_{B(\mathcal{H})})(g)\|_{B(L_2(\mathcal{M})) \overline{\otimes} B(\mathcal{H})} = \|U(g)j^*\| \leq \|U(g)\| \, \|j^*\| \leq \|U(g)\|.$$

It only remains to check that $U(L_\infty(\mathcal{M}; \mathcal{H}^c))$ is closed with respect to the weak* topology of $B(L_2(\mathcal{M}), L_2(\mathcal{M}) \otimes_2 \mathcal{H})$. Recall that since $\mathcal{M} \otimes \mathcal{H}$ is weak*-dense in $L_\infty(\mathcal{M}; \mathcal{H}^c)$ and U is weak*-to-weak* continuous operator, there holds

$$U(L_\infty(\mathcal{M}; \mathcal{H}^c)) = U(\overline{\mathcal{M} \otimes \mathcal{H}}^{w^*}) \subseteq \overline{U(\mathcal{M} \otimes \mathcal{H})}^{w^*}.$$

Therefore, we only need to show that $U(L_\infty(\mathcal{M}; \mathcal{H}^c))$ is weak*-closed in order to get the stated result. Notice that since $L_2(\mathcal{M})$ and $L_2(\mathcal{M}) \otimes_2 \mathcal{H}$ are separable Banach spaces, then $(L_2(\mathcal{M})^* \otimes \mathcal{H}^*) \widehat{\otimes}_\pi L_2(\mathcal{M})$ is separable too. Then, a subspace of its dual $B(L_2(\mathcal{M}), L_2(\mathcal{M}) \otimes_2 \mathcal{H})$ is weak*-closed if and only if it is weakly* sequentially closed [53, Corollary 2.7.13]. Therefore, the following argument can be displayed in terms of sequences.

Consider $(g_n)_{n \geq 1} \subseteq U(L_\infty(\mathcal{M}; \mathcal{H}^c))$ such that $(U(g_n))_{n \geq 1}$ is weakly* convergent, so that it is a bounded sequence [53, Corollary 2.6.10]. Since U is an isometry, then the sequence $(g_n)_{n \geq 1}$ has the same bound. Therefore, as a consequence of the Banach-Alaoglu theorem, there exists a weakly*-convergent subsequence $(g_{n_j})_{j \geq 1}$ with limit g in $L_\infty(\mathcal{M}; \mathcal{H}^c)$. By the weak*-continuity of U, $(U(g_{n_j}))_{j \geq 1}$ converges to $U(g)$, so it follows that $G = U(g)$.

The subspace $U(L_\infty(\mathcal{M};\mathcal{H}^c))$ being weak*-closed is equivalent to being norm-closed [75, p.101], and also to U_* having closed range. Therefore, since the image of U_* is also dense, U_* has to be surjective. On the other hand, since U is a bounded injective operator with closed range, there exists a bounded inverse for U defined on $U(L_\infty(\mathcal{M};\mathcal{H}^c))$. □

So far, the map U from Proposition 4.1.5 has turned out to be a weak*-continuous isometry with bounded inverse on its image. However, this has no further consequences for the invertibility of

$$U_* : (L_2(\mathcal{M})^* \otimes_2 \mathcal{H}^*)\widehat{\otimes}_\pi L_2(\mathcal{M}) \longrightarrow L_1(\mathcal{M};\mathcal{H}^c)$$

since U is not surjective. Replacing the domain of U_* by a predual of $U(L_\infty(\mathcal{M};\mathcal{H}^c))$ allows us to obtain a bounded inverse for U_*.

Corollary 4.1.6. *Let $\mathcal{H}$ be a Hilbert space. Then*

$$\big((L_2(\mathcal{M})^* \otimes_2 \mathcal{H}^*)\widehat{\otimes}_\pi L_2(\mathcal{M}) \,/\, U(L_\infty(\mathcal{M};\mathcal{H}^c))_\perp\big)^* \simeq U(L_\infty(\mathcal{M};\mathcal{H}^c))$$

and the map $U : L_\infty(\mathcal{M};\mathcal{H}^c) \longrightarrow U(L_\infty(\mathcal{M};\mathcal{H}^c))$ induces a bounded contraction

$$U_* : (L_2(\mathcal{M})^* \otimes_2 \mathcal{H}^*)\widehat{\otimes}_\pi L_2(\mathcal{M}) \,/\, U(L_\infty(\mathcal{M};\mathcal{H}^c))_\perp \longrightarrow L_1(\mathcal{M};\mathcal{H}^c).$$

Therefore,

$$U : L_\infty(\mathcal{M};\mathcal{H}^c) \longrightarrow U(L_\infty(\mathcal{M};\mathcal{H}^c))$$

is a weak-continuous isometry with bounded inverse, and U_* has a bounded inverse too.*

Proof. The formula for a predual of $U(L_\infty(\mathcal{M};\mathcal{H}^c))$ follows because this space is closed with respect to the weak* topology of $B(L_2(\mathcal{M}), L_2(\mathcal{M})\otimes_2\mathcal{H})$. Then, given $F \in L_\infty(\mathcal{M};\mathcal{H}^c)$ and $\sum_{i=1}^\infty \overline{f_i} \otimes m_i + U(L_\infty(\mathcal{M};\mathcal{H}^c))_\perp \in U(L_\infty(\mathcal{M};\mathcal{H}^c))_*$, there holds

$$\langle U(F), \sum_{i=1}^\infty \overline{f_i} \otimes m_i\rangle = \langle F, U_*(\sum_{i=1}^\infty \overline{f_i} \otimes m_i)\rangle,$$

so U_* is well-defined on a predual of $U(L_\infty(\mathcal{M};\mathcal{H}^c))$. By the computations in the proof of Proposition 4.1.5, U_* is a bounded contraction, so U is a weak*-continuous isomorphism of Banach spaces. □

Remark 4.1.7. *Similar computations show that the linear map sending*

$$\sum_{i=1}^n m_i \otimes (\mathbb{1} \otimes h_i) \longmapsto (k \mapsto \sum_{i=1}^n \overline{m_i^* k^*} \otimes \overline{h_i})$$

extends to a injective weak-continuous isometry from $L_\infty(\mathcal{M};\overline{\mathcal{H}}^r)$ to $B(L_2(\mathcal{M}), L_2(\mathcal{M})^* \otimes_2 \mathcal{H}^*)$. Therefore, the pre-adjoint map U_* acts sending $f \otimes m'$ from $(L_2(\mathcal{M}) \otimes_2 \mathcal{H}) \otimes L_2(\mathcal{M})$ to $f\ (m' \otimes (\mathbb{1} \otimes \mathbb{1})) \in L_1(\mathcal{M};\mathcal{H}^c)$. Similarly, the map*

$$U_* : (L_2(\mathcal{M}) \otimes_2 \mathcal{H})\widehat{\otimes}_\pi L_2(\mathcal{M}) \,/\, U(L_\infty(\mathcal{M};\mathcal{H}^c))_\perp \longrightarrow L_1(\mathcal{M};\mathcal{H}^c)$$

as constructed above has a bounded inverse, and its adjoint U has it too.

4.1.1 Noncommutative spaces $L_p(\mathcal{M}; L_2^c(\Omega))$

Let (Ω, μ) be a semifinite measure space. A remarkable setting for noncommutative Hilbert-valued column/row L_p spaces is the case $\mathcal{H} = L_2(\Omega) := L_2(\Omega, \mu)$. Notice that under these conditions, the duality bracket (4.1) is given by the expression

$$\begin{aligned}(m_1 \otimes (f_1 \otimes \mathbb{1}), m_2 \otimes (\mathbb{1} \otimes f_2))_{c,r} &= \tau_{\mathcal{M}}(m_2 m_1)\, \tau_{B(L_2(\Omega,\mu))}((\mathbb{1} \otimes f_2)(f_1 \otimes \mathbb{1})) \\ &= \tau_{\mathcal{M}}(m_2 m_1) \int_\Omega f_1 \overline{f_2}\, d\mu,\end{aligned}$$

yielding the identification

$$L_{p'}(\mathcal{M}; \overline{L_2^r(\Omega, \mu)}) = L_p(\mathcal{M}; L_2^c(\Omega, \mu))^* \text{ for } 1 \leq p < \infty.$$

Notation 4.1.8. *In the following, we will refer to the space $L_p(\mathcal{M}; \overline{L_2^r(\Omega, \mu)})$ by writing instead $L_p(\mathcal{M}; L_2^r(\Omega, \mu))$ in order to avoid an overloaded notation.*

Despite $L_2(\Omega, \mu)$ being a commutative Hilbert space, the associated column and row spaces are not spaces of functions. In other words, $L_p(\mathcal{M}; L_2^c(\Omega))$ is not generally contained in $L_2(\Omega; L_p(\mathcal{M}))$, the space of strongly measurable functions $f : \Omega \to L_p(\mathcal{M})$ satisfying

$$\|f\|_{L_2(\Omega; L_p(\mathcal{M}))} = \left(\int_\Omega \|f(x)\|^2_{L_p(\mathcal{M})} d\mu(x) \right)^{1/2} < \infty.$$

However, when $1 \leq p \leq 2$, both $L_p(\mathcal{M}; L_2^c(\Omega))$ and $L_p(\mathcal{M}; L_2^r(\Omega))$ are indeed spaces of functions contained in $L_2(\Omega; L_p(\mathcal{M}))$.

In the following sections, the case $p = \infty$ will be specially relevant. For that reason, the study of the extension of some useful operators on $L_2(\Omega)$ to $L_\infty(\mathcal{M}; L_2^c(\Omega))$ will be carefully checked.

Lemma 4.1.9. *Let A, B be two measurable sets, and let w and w' be strictly positive functions belonging to $L_\infty(\Omega, \mu)$. Consider the following operators on $L_2(\Omega, \mu)$:*

$$\begin{aligned} T_w &: f \mapsto w^{1/2} f, \\ P_A &: f \mapsto \chi_A f. \end{aligned}$$

These maps extend to bounded operators on $L_\infty(\mathcal{M}; L_t^2(\Omega, \mu))$, for $t = c, r$ such that $\|\widetilde{T}_w\| = \|w\|_{L_\infty(\Omega,\mu)}$, $\|\widetilde{P}_A\| = \|\chi_A\|_{L_\infty(\Omega,\mu)}$. Moreover, they satisfy the following relations:

1. $\widetilde{T}_w \widetilde{T}_{w'} = \widetilde{T}_{w'} \widetilde{T}_w$,
2. $\widetilde{P}_A \widetilde{T}_w = \widetilde{T}_w \widetilde{P}_A$,
3. *whenever w^{-1} is bounded,* $\widetilde{T}_w \widetilde{T}_{w^{-1}} = \mathrm{Id}_{L_\infty(\mathcal{M}; L_t^2(\Omega,\mu))} = \widetilde{T}_{w^{-1}} \widetilde{T}_w$

4. $\widetilde{P}_A = \widetilde{P}_A \widetilde{P}_B = \widetilde{P}_B \widetilde{P}_A$ *whenever* $A \subseteq B$,
5. $\widetilde{P}_{B \setminus A} = \widetilde{P}_B - \widetilde{P}_A$ *whenever* $A \subseteq B$.

Proof. By Corollary 4.1.3, the extension operators $\widetilde{T}_w$, $\widetilde{P}_A$ are bounded as long as the original ones are bounded on $L^2(\Omega, \mu)$. The maps T_w and P_A are bounded with norm $\|w^{1/2}\|_{L_\infty}$ and 1 respectively, since they are pointwise multiplication operators. Claims (1)-(4) follow from Lemma 4.1.4, while linearity of the map $T \mapsto \widetilde{T}$ implies (5). □

4.2 Duality between Hardy spaces and BMO spaces

Consider the measure space $(\mathbb{R}, \frac{dt}{1+t^2})$. Then, $L_2(\mathbb{R}, \frac{dt}{1+t^2})$ is a Hilbert space with the inner product

$$\langle f, g \rangle = \int_{\mathbb{R}} f(t)\, \overline{g(t)}\, \frac{dt}{1+t^2}.$$

Then, we will consider the associated column space $L_p(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dt}{1+t^2}))$ for $0 < p \leq \infty$. We will choose $\mathbb{1}$ to be the constant function $1/\sqrt{\pi}$, which satisfies the condition

$$\|\mathbb{1}\|_{L_2(\mathbb{R}, \frac{dt}{1+t^2})} = 1.$$

For the sake of exposition, we define some operators on $L_\infty(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dt}{1+t^2}))$ which can be described in terms of the maps appearing in Lemma 4.1.9. Set $\omega(t) = 1 + t^2$, and let A be a measurable set such that $|A| \neq 0$. Then, define the map

$$R_A = \widetilde{T}_\omega \widetilde{T}_{|A|^{-1}} \widetilde{P}_A$$

so that R_A is the extension of the operator acting on $L_2(\mathbb{R}, \frac{dt}{1+t^2})$ as follows:

$$f \mapsto \frac{(1+t^2)^{1/2}}{\sqrt{|A|}} \chi_A f.$$

On the other hand, denote by a_A the extension to $L_\infty(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dt}{1+t^2}))$ of the map

$$a_A : f \mapsto f_A\, 1 = \left(\frac{1}{|A|} \int_A f \right) 1.$$

A straightforward verification shows that the operator on $L_\infty(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dt}{1+t^2}))$ given by

$$f \mapsto \sqrt{\pi}\, \left[(\mathrm{Id}_{\mathcal{M}} \otimes \mathrm{Tr}_{B(L_2(\mathbb{R}, \frac{dt}{1+t^2}))}) \widetilde{T}_{\omega^2} \widetilde{T}_{|A|^{-2}} \widetilde{P}_A f \right] \otimes (1 \otimes \mathbb{1}) =: f_A \otimes (1 \otimes \mathbb{1}) \tag{4.2}$$

is an extension of a_A. Indeed, given an operator $f = \sum_{i=1}^n m_i \otimes (f_i \otimes \mathbb{1})$ belonging to $\mathcal{M} \otimes L_2(\mathbb{R}, \frac{dt}{1+t^2})$,

$$a_A f = \sum_{i=1}^n m_i\, \sqrt{\pi}\, \mathrm{Tr}\left(\frac{\omega}{|A|} \chi_A f_i \otimes \mathbb{1} \right) \otimes (1 \otimes \mathbb{1})$$

$$= \sum_{i=1}^{n} n_i \sqrt{\pi}\, \langle \frac{\omega}{|A|} \chi_A f_i, \mathbb{1} \rangle \otimes (1 \otimes \mathbb{1})$$
$$= \sum_{i=1}^{n} n_i \Big(\frac{1}{|A|} \int_A f_i \Big) \otimes (1 \otimes \mathbb{1}).$$

Lemma 4.2.1. *Let A, B be two measurable sets such that $|A|, |B| \neq 0$, and let R_A and a_A be the operators defined above on $L_\infty(\mathcal{M}; L_2^t(\mathbb{R}, \frac{dt}{1+t^2}))$ for $t = c, r$. Then,*

$$\|R_A\| \leq \frac{\sup_{t\in A}(1+t^2)^{1/2}}{\sqrt{|A|}},$$
$$\|a_A\| \leq \min\Big\{ \pi^{1/2} \frac{\sup_{t\in A}(1+t^2)^{1/2}}{\sqrt{|A|}}, \pi \frac{\sup_{t\in A}(1+t^2)}{|A|} \Big\}.$$

Moreover, $a_A a_B = a_B$.

Proof. The first bound follows from the definition of R_A. On the other hand, given a function f in $L_2(\mathbb{R}, \frac{dt}{1+t^2})$, the following bound holds:

$$\|f_A\ 1\|_{L_2(\mathbb{R}, \frac{dt}{1+t^2})} = \pi^{1/2} \int_A f \frac{1}{|A|} \leq \pi^{1/2} \Big(\int_{\mathbb{R}} |f(t)|^2 \frac{dt}{1+t^2} \Big)^{1/2} \Big(\int_A \frac{1}{|A|^2} (1+t^2)\, dt \Big)^{1/2}$$
$$\leq \pi^{1/2}\, \|f\|_{L_2(\mathbb{R}, \frac{dt}{1+t^2})}\, \frac{\sup_{t\in I}(1+t^2)^{1/2}}{\sqrt{|A|}}.$$

Moreover, an alternative estimate yields

$$\|f_A\ 1\|_{L_2(\mathbb{R}, \frac{dt}{1+t^2})} = \pi^{1/2} \int_{\mathbb{R}} f(t) \chi_A(t) \frac{1+t^2}{|A|} \frac{dt}{1+t^2}$$
$$\leq \pi^{1/2} \frac{\sup_{t\in A}(1+t^2)}{|A|} \int_{\mathbb{R}} |f| \frac{dt}{1+t^2}$$
$$\leq \pi^{1/2} \frac{\sup_{t\in A}(1+t^2)}{|A|} \Big(\int_{\mathbb{R}} |f|^2 \frac{dt}{1+t^2} \Big)^{1/2} \Big(\int_{\mathbb{R}} \frac{dt}{1+t^2} \Big)^{1/2}$$
$$= \pi \frac{\sup_{t\in A}(1+t^2)}{|A|} \|f\|_{L_2(\mathbb{R}, \frac{dt}{1+t^2})}$$

Therefore, the norm of a_A is bounded as stated at the statement as a consequence of Corollary 4.1.3. The last claim follows from Lemma 4.1.4(1) and the identity

$$a_A a_B f = a_A \Big(\frac{1}{|B|} \int_B f(t)\, dt \Big) = \frac{1}{|A|} \int_A \Big(\frac{1}{|B|} \int_B f(t)\, dt \Big) dx = \frac{1}{|B|} \int_B f = a_B f$$

for any f in $L_2(\mathbb{R}, \frac{dt}{1+t^2})$. □

Proposition 4.2.2. *Let A, B be two measurable sets. Then, the identity*

$$\|R_B a_A f\|_{\mathcal{M}\overline{\otimes}B(L_2(\mathbb{R},\frac{dt}{1+t^2}))} = \|f_A\|_{\mathcal{M}}$$

holds for any operator f in $L_\infty(\mathcal{M}; L_2^c(\mathbb{R}^n, \frac{dt}{1+t^2}))$.

Proof. The claim follows from the formula (4.2), that is,

$$\begin{aligned}\|R_B a_A f\|_{\mathcal{M}\overline{\otimes}B(L_2(\mathbb{R},\frac{dt}{1+t^2}))} &= \|R_B(f_A \otimes (1 \otimes \mathbb{1}))\|_{\mathcal{M}\overline{\otimes}B(L_2(\mathbb{R},\frac{dt}{1+t^2}))} \\ &= \|f_A^* f_A \Big(\int_B \frac{1+t^2}{|B|}\frac{dt}{1+t^2}\Big) \otimes (\mathbb{1}\otimes\mathbb{1})\|^{1/2}_{\mathcal{M}\overline{\otimes}B(L_2(\mathbb{R},\frac{dt}{1+t^2}))} \\ &= \|f_A^* f_A\|^{1/2}_{\mathcal{M}} = \|f_A\|_{\mathcal{M}}.\end{aligned}$$

□

Some other Hilbert spaces over the real line will be considered through this section. In particular, when the Lebesgue measure is considered, the function

$$t \mapsto \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{1+t^2}}$$

belongs to the space $L_2(\mathbb{R}, dt)$ and has L_2 norm equal to 1. Moreover, $\frac{\mathbb{1}}{\sqrt{1+t^2}}$ will be chosen as the distinguished element of this Hilbert space which appears in the definition of the column and row spaces $L_\infty(\mathcal{M}; L_2^c(\mathbb{R}, dt))$ and $L_\infty(\mathcal{M}; L_2^r(\mathbb{R}, dt))$.

Lemma 4.2.3. *Let A be a measurable set. Then, the pre-adjoints maps for R_A and a_A on $L_1(\mathcal{M}; L_2^r(\mathbb{R}, dt/(1+t^2)))$ act as follows on any operator $m \otimes (\mathbb{1} \otimes f) \in L_1(\mathcal{M}) \otimes L_2(\mathbb{R}, dt/(1+t^2))$*

$$\begin{aligned}(R_A)_* &: m \otimes (\mathbb{1} \otimes f) \longmapsto m \otimes \Big(\mathbb{1} \otimes \frac{\sqrt{1+t^2}}{\sqrt{|A|}}\chi_I f\Big) \\ (a_A)_* &: m \otimes (\mathbb{1} \otimes f) \longmapsto m \otimes \Big(\mathbb{1} \otimes \sqrt{\pi}\frac{1+t^2}{|A|}\chi_A \langle f, \mathbb{1}\rangle 1\Big).\end{aligned}$$

Moreover, the operator

$$\begin{array}{rccc} V & : L_\infty(\mathcal{M}; L_2^r(\mathbb{R}, (1+t^2)dt)) & \longmapsto & L_\infty(\mathcal{M}; L_2^r(\mathbb{R}, \frac{dt}{1+t^2})) \\ & m \otimes (\frac{\mathbb{1}}{1+t^2} \otimes f) & \longmapsto & m \otimes (\mathbb{1} \otimes f), \end{array}$$

admits a pre-adjoint

$$\begin{array}{rccc} V_* & : L_1(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dt}{1+t^2})) & \longmapsto & L_1(\mathcal{M}; L_2^c(\mathbb{R}, (1+t^2)dt)) \\ & m \otimes (f \otimes \mathbb{1}) & \longmapsto & m \otimes (\frac{f}{1+t^2} \otimes \frac{\mathbb{1}}{1+t^2}). \end{array}$$

Proof. The first claim follows from the identity

$$(\tau \otimes \mathrm{Tr})(T(m' \otimes (f' \otimes \mathbb{1}))\, m \otimes (\mathbb{1} \otimes f)) = (\tau \otimes \mathrm{Tr})(m' \otimes (f' \otimes \mathbb{1})\, T_*(m \otimes (\mathbb{1} \otimes f)))$$

for any $m' \otimes (f' \otimes \mathbb{1}) \in \mathcal{M} \otimes L_2(\mathbb{R}, \frac{dt}{1+t^2})$ and $m \otimes (\mathbb{1} \otimes f) \in L_1(\mathcal{M}) \otimes L_2(\mathbb{R}, \frac{dt}{1+t^2})$. Computing the pre-adjoint of R_A is a straightforward task, so we will only compute the pre-adjoint of a_A:

$$\begin{aligned}
&\langle m \otimes (\mathbb{1} \otimes f), a_I(m' \otimes (f' \otimes \mathbb{1}))\rangle \\
&\quad = \langle m \otimes (\mathbb{1} \otimes f), m' \sqrt{\pi}\, \mathrm{Tr}\big(\frac{1+t^2}{|A|}\chi_A f' \otimes \mathbb{1}\big)\big(1 \otimes \mathbb{1})\rangle \\
&\quad = \tau_{\mathcal{M}}(mm')\, \mathrm{Tr}((\mathbb{1} \otimes f)(1 \otimes \mathbb{1}))\, \langle \frac{1+t^2}{|A|}\chi_A f', 1\rangle_{L_2(\mathbb{R}, \frac{dt}{1+t^2})} \\
&\quad = \tau_{\mathcal{M}}(mm')\, \langle 1, f\rangle_{L_2(\mathbb{R}, \frac{dt}{1+t^2})}\, \langle \frac{1+t^2}{|A|}\chi_A f', 1\rangle_{L_2(\mathbb{R}, \frac{dt}{1+t^2})} \\
&\quad = \tau_{\mathcal{M}}(mm')\, \langle \frac{1+t^2}{|A|}\chi_A f', \langle f, 1\rangle 1\rangle_{L_2(\mathbb{R}, \frac{dt}{1+t^2})} \\
&\quad = \tau_{\mathcal{M}}(mm')\, \langle f', \langle f, 1\rangle \frac{1+t^2}{|A|}\chi_A\rangle_{L_2(\mathbb{R}, \frac{dt}{1+t^2})} \\
&\quad = \tau_{\mathcal{M}}(mm')\, \mathrm{Tr}\big(\mathbb{1} \otimes \sqrt{\pi}\frac{1+t^2}{|A|}\chi_A \langle f, \mathbb{1}\rangle 1 \cdot f' \otimes \mathbb{1}).
\end{aligned}$$

On the other hand, the expression for the pre-adjoint of V follows analogously from the duality expression

$$(\tau \otimes \mathrm{Tr})\big(V\big(m' \otimes (\frac{\mathbb{1}}{1+t^2} \otimes f')\big)\big(m \otimes (f \otimes \mathbb{1})\big) = (\tau \otimes \mathrm{Tr})\big(m' \otimes (\frac{\mathbb{1}}{1+t^2} \otimes f')\, V_*\big(m \otimes (f \otimes \mathbb{1})\big)\big)$$

for any $m' \otimes (\frac{\mathbb{1}}{1+t^2} \otimes f') \in \mathcal{M} \otimes L_2(\mathbb{R}, (1+t^2)dt)$ and $m \otimes (f \otimes \mathbb{1}) \in L_1(\mathcal{M}) \otimes L_2(\mathbb{R}, \frac{dt}{1+t^2})$. □

Now, we are ready to define the column and row BMO spaces.

Definition 4.2.4. *Given a von Neumann algebra $\mathcal{M}$ with n.s.f. trace τ, set the* column BMO space, *denoted as $\mathsf{BMO}_c(\mathbb{R}, \mathcal{M})$, to be the subspace of operators f in $L_\infty(\mathcal{M}; L_2^c(\mathbb{R}^n; \frac{dt}{1+t^2}))$ satisfying*

$$\|f\|_{\mathsf{BMO}_c} := \sup_{I \subseteq \mathbb{R}} \Big\| R_I(\mathrm{Id}_{\mathcal{M}\overline{\otimes}B(L_2(\Omega,\mu))} - a_I)f \Big\|_{\mathcal{M}\,\overline{\otimes}\, B\left(L_2(\mathbb{R}^n, \frac{dt}{1+t^2})\right)} < \infty, \tag{4.3}$$

where the supremum is considered over finite intervals I of $\mathbb{R}$. Likewise, the row BMO *space, $\mathsf{BMO}_r(\mathbb{R}, \mathcal{M})$, is the subspace of elements in $L_\infty(\mathcal{M}; L_2^r(\mathbb{R}, \frac{dt}{1+t^2}))$ for which the norm $\|f\|_{\mathsf{BMO}_r} := \|f^*\|_{\mathsf{BMO}_c}$ is finite.*

It is clear that $\|\cdot\|_{\mathsf{BMO}_c}$ is a norm modulo $\mathcal{M}$. We claim that this expression admits an easier-to-manage form for operators in $\mathcal{M}\otimes L_2(\mathbb{R},\frac{dt}{1+t^2})$, which recovers the expression which determines the definition for the BMO_c norm introduced in [54].

Lemma 4.2.5. *For any operator f in $\mathcal{M}\otimes L_2(\mathbb{R},\frac{dt}{1+t^2})$ it holds*

$$\|f\|_{\mathsf{BMO}_c} = \sup_{I\subseteq\mathbb{R}} \left\| \left(\frac{1}{|I|}\int_I |f-f_I|^2\right)^{\frac{1}{2}} \right\|_{\mathcal{M}}.$$

Proof. Let n be a natural number such that $f=\sum_{i=1}^n m_i\otimes f_i$ for some $m_i\in\mathcal{M}$ and $f_i\in L_2(\mathbb{R}^n,\frac{dt}{1+t^2})$, for $i=1,\dots,n$. Then, the right hand side from the statement above can be expressed as

$$\sup_{I\subseteq\mathbb{R}} \left\| \sum_{i,j=1}^n m_i^* m_j \,\frac{1}{|I|}\int_I \overline{(f_i-(f_i)_I)}(f_j-(f_j)_I) \right\|_{\mathcal{M}}^{1/2}.$$

On the other hand, it is clear that the expression inside the norm in (4.3) coincides with

$$\sum_{i=1}^n \left(m_i \otimes \left(\frac{\omega^{1/2}\chi_I}{\sqrt{|I|}}(f_i-(f_i)_I)\otimes \mathbb{1}\right)\right) =: F$$

so that it holds

$$\begin{aligned}
&\||F|^2\|_{\mathcal{M}\,\overline{\otimes}\,B\left(L_2(\mathbb{R}^n,\frac{dt}{1+t^2})\right)}\\
&= \left\| \sum_{i,j=1}^n m_i^* m_j \otimes \left(\frac{\omega^{1/2}\chi_I}{|I|^{1/2}}(f_i-(f_i)_I)\otimes\mathbb{1}\right)^* \left(\frac{\omega^{1/2}\chi_I}{|I|^{1/2}}(f_j-(f_j)_I)\otimes\mathbb{1}\right)\right\|_{\mathcal{M}\,\overline{\otimes} B(L_2)}\\
&= \left\| \sum_{i,j=1}^n m_i^* m_j \otimes \left(\mathbb{1}\otimes \frac{\omega^{1/2}\chi_I}{|I|^{1/2}}(f_i-(f_i)_I)\right)\left(\frac{\omega^{1/2}\chi_I}{|I|^{1/2}}(f_j-(f_j)_I)\otimes\mathbb{1}\right)\right\|_{\mathcal{M}\,\overline{\otimes} B(L_2)}\\
&= \left\| \sum_{i,j=1}^n m_i^* m_j \left(\frac{1}{|I|}\int_I \overline{(f_i-(f_i)_I)}(f_j-(f_j)_I)\right)\otimes(\mathbb{1}\otimes\mathbb{1})\right\|_{\mathcal{M}\overline{\otimes} B(L_2)}\\
&= \left\| \sum_{i,j=1}^n m_i^* m_j \,\frac{1}{|I|}\int_I \overline{(f_i-(f_i)_I)}(f_j-(f_j)_I)\right\|_{\mathcal{M}}
\end{aligned}$$

as a consequence of Lemma 4.1.1 (1)-(3). □

Proposition 4.2.6. *Let $f\in\mathsf{BMO}_c(\mathbb{R},\mathcal{M})$. Then,*

$$\|f\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))} \lesssim \|f\|_{\mathsf{BMO}_c} + \|f_{I_0}\|_{\mathcal{M}}$$

where I_0 is the interval $[-1,1)$.

Proof. This argument is an extension of Mei's proof of this result for the subspace $\mathcal{M} \otimes L_2(\mathbb{R}; \frac{dt}{1+t^2})$ [54]. Let I_0 be an arbitrary interval. Then any operator f in $\mathsf{BMO}_c(\mathbb{R}, \mathcal{M})$ satisfies the identity

$$
\begin{aligned}
\|f\|_{\mathcal{M}\overline{\otimes}B(L_2(\mathbb{R},\frac{dt}{1+t^2}))} &= \Big\| \widetilde{P}_{I_0} f + \sum_{j=0}^{\infty} \widetilde{P}_{2^{j+1}I_0 \setminus 2^j I_0} f \Big\|_{\mathcal{M}\overline{\otimes}B(L_2(\mathbb{R},\frac{dt}{1+t^2}))} \\
&\leq \big\| \widetilde{P}_{I_0} f \big\|_{\mathcal{M}\overline{\otimes}B(L_2(\mathbb{R},\frac{dt}{1+t^2}))} + \sum_{j=0}^{\infty} \Big\| \widetilde{P}_{2^{j+1}I_0 \setminus 2^j I_0} f \Big\|_{\mathcal{M}\overline{\otimes}B(L_2(\mathbb{R},\frac{dt}{1+t^2}))} \qquad (4.4)
\end{aligned}
$$

as a consequence of Lemmas 4.1.4 and 4.1.9. This latter result easily implies a bound for the first term at (4.4). Indeed,

$$
\begin{aligned}
\|\widetilde{P}_{I_0} f\|_{\mathcal{M}\overline{\otimes}B(L_2(\mathbb{R},\frac{dt}{1+t^2}))} &\leq \|\widetilde{P}_{I_0} a_{I_0} f\|_{\mathcal{M}\overline{\otimes}B(L_2(\mathbb{R},\frac{dt}{1+t^2}))} \\
&+ \|\widetilde{P}_{I_0}(\mathrm{Id} - a_{I_0}) f\|_{\mathcal{M}\overline{\otimes}B(L_2(\mathbb{R},\frac{dt}{1+t^2}))} \\
&\leq 2^{1/2} \, \|\widetilde{T}_{|I_0|^{-1}} \widetilde{T}_\omega \widetilde{P}_{I_0} a_{I_0} f\|_{\mathcal{M}\overline{\otimes}B(L_2(\mathbb{R},\frac{dt}{1+t^2}))} \\
&+ 2^{1/2} \, \|\widetilde{T}_\omega \widetilde{T}_{|I_0|^{-1}} \widetilde{P}_{I_0}(\mathrm{Id} - a_{I_0}) f\|_{\mathcal{M}\overline{\otimes}B(L_2(\mathbb{R},\frac{dt}{1+t^2}))} \\
&= 2^{1/2} \, \|R_{I_0} a_{I_0} f\|_{\mathcal{M}\overline{\otimes}B(L_2(\mathbb{R},\frac{dt}{1+t^2}))} \\
&+ 2^{1/2} \|R_{I_0}(\mathrm{Id} - a_{I_0}) f\|_{\mathcal{M}\overline{\otimes}B(L_2(\mathbb{R},\frac{dt}{1+t^2}))} \\
&\leq 2^{1/2} (\|f_{I_0}\|_{\mathcal{M}} + \|f\|_{\mathsf{BMO}_c})
\end{aligned}
$$

as a consequence of Proposition 4.2.2 and the definition of the BMO_c norm. On the other hand, given $j \geq 0$, define the interval $I_j = 2^j I_0$. Then, there follows

$$
\begin{aligned}
\|\widetilde{P}_{I_{j+1} \setminus I_j} f\|_{\mathcal{M}\overline{\otimes}B(L_2(\mathbb{R},\frac{dt}{1+t^2}))} &\leq \sup_{t \in I_{j+1} \setminus I_j} (1 + t^2)^{-1/2} \, \|\widetilde{T}_\omega \widetilde{P}_{I_{j+1} \setminus I_j} f\|_{\mathcal{M}\overline{\otimes}B(L_2(\mathbb{R},\frac{dt}{1+t^2}))} \\
&\leq \frac{1}{2^j} \|\widetilde{T}_\omega \widetilde{P}_{I_{j+1} \setminus I_j} f\|_{\mathcal{M}\overline{\otimes}B(L_2(\mathbb{R},\frac{dt}{1+t^2}))} \\
&= \frac{(2^{j+2})^{1/2}}{2^j} \|\widetilde{T}_{|I_{j+1}|^{-1}} \widetilde{T}_\omega \widetilde{P}_{I_{j+1}} (f - a_{I_{j+1}} f)\|_{\mathcal{M}\overline{\otimes}B(L_2(\mathbb{R},\frac{dt}{1+t^2}))} \\
&+ \frac{(2^{j+2})^{1/2}}{2^j} \, \|\widetilde{T}_{|I_{j+1}|^{-1}} \widetilde{T}_\omega \widetilde{P}_{I_{j+1}} (a_{I_{j+1}} f - a_{I_0} f)\|_{\mathcal{M}\overline{\otimes}B(L_2(\mathbb{R},\frac{dt}{1+t^2}))} \\
&+ \frac{(2^{j+2})^{1/2}}{2^j} \|\widetilde{T}_{|I_{j+1}|^{-1}} \widetilde{T}_\omega \widetilde{P}_{I_{j+1}} a_{I_0} f\|_{\mathcal{M}\overline{\otimes}B(L_2(\mathbb{R},\frac{dt}{1+t^2}))} \\
&= \frac{1}{2^{j/2-1}} \Big[\|R_{I_{j+1}} (f - a_{I_{j+1}} f)\|_{\mathcal{M}\overline{\otimes}B(L_2(\mathbb{R},\frac{dt}{1+t^2}))} \\
&+ \|R_{I_{j+1}} (a_{I_{j+1}} f - a_{I_0} f)\|_{\mathcal{M}\overline{\otimes}B(L_2(\mathbb{R},\frac{dt}{1+t^2}))} \\
&+ \|R_{I_{j+1}} a_{I_0} f\|_{\mathcal{M}\overline{\otimes}B(L_2(\mathbb{R},\frac{dt}{1+t^2}))} \Big] \\
&\leq \frac{1}{2^{j/2-1}} \Big[\|f\|_{\mathsf{BMO}_c} + \|a_{I_{j+1}} f - a_{I_0} f\|_{\mathcal{M}\overline{\otimes}B(L_2(\mathbb{R},\frac{dt}{1+t^2}))} + \|f_{I_0}\|_{\mathcal{M}} \Big].
\end{aligned}
$$

Therefore, it only remains to bound the second term in the latter sum, but it easily follows from Lemma 4.2.1. More clearly,

$$\begin{aligned}
\|a_{I_{j+1}}f - a_{I_0}f\|_{\mathcal{M}\overline{\otimes}B(L_2(\Omega,\mu))} &\leq \sum_{k=0}^{j} \|a_{I_k}f - a_{I_{k+1}}f\|_{\mathcal{M}\overline{\otimes}B(L_2(\mathbb{R},\frac{dt}{1+t^2}))} \\
&= \sum_{k=0}^{j} \|a_{I_k}(f - a_{I_{k+1}}f)\|_{\mathcal{M}\overline{\otimes}B(L_2(\mathbb{R},\frac{dt}{1+t^2}))} \\
&\leq \pi \sum_{k=0}^{j} \|\widetilde{T}_{\omega^2}\widetilde{T}_{|I_k|^{-2}}\widetilde{P}_{I_k}(f - a_{I_{k+1}}f)\|_{\mathcal{M}\overline{\otimes}B(L_2(\mathbb{R},\frac{dt}{1+t^2}))} \\
&\lesssim \sum_{k=0}^{j} \frac{2^{k/2}}{2^{k+1}} \|\widetilde{P}_{I_{k+1}}\widetilde{T}_{\omega}(f - a_{I_{k+1}})\|_{\mathcal{M}\overline{\otimes}B(L_2(\mathbb{R},\frac{dt}{1+t^2}))} \\
&\leq \sum_{k=0}^{j} \frac{2^{k+1/2}}{2^{k+1}} \|\widetilde{P}_{I_{k+1}}\widetilde{T}_{\omega}\widetilde{T}_{|I_{k+1}|^{-1}}(f - a_{I_{k+1}}f)\|_{\mathcal{M}\overline{\otimes}B(L_2(\mathbb{R},\frac{dt}{1+t^2}))} \\
&= \frac{1}{2}\sum_{k=0}^{j} \|R_{I_{k+1}}(f - a_{I_{k+1}}f)\|_{\mathcal{M}\overline{\otimes}B(L_2(\mathbb{R},\frac{dt}{1+t^2}))}.
\end{aligned}$$

In conclusion, for any operator f in $L_\infty(\mathcal{M}, L_2^c(\mathbb{R}, \frac{dt}{1+t^2}))$, we recover the estimate

$$\begin{aligned}
\|f\|_{L_\infty(\mathcal{M};L_2^c(\Omega,\mu))} &\lesssim (1 + \sum_{j=0}^{\infty} \frac{1}{2^{j/2-1}}) \|f_{I_0}\|_{\mathcal{M}} + (1 + \sum_{j=0}^{\infty} \frac{j+1}{2^{j/2-1}}) \|f\|_{\mathsf{BMO}_c} \\
&\lesssim \|f\|_{\mathsf{BMO}_c} + \|f_{I_0}\|_{\mathcal{M}}.
\end{aligned}$$

□

Remark 4.2.7. *Given $f \in \mathsf{BMO}_r(\mathbb{R}, \mathcal{M})$, and applying Theorem 4.2.6 to f^*, an analogous inequality holds*

$$\|f\|_{L_\infty(\mathcal{M};L_2^r(\mathbb{R},\frac{dt}{1+t^2}))} \lesssim \|f\|_{\mathsf{BMO}_r} + \|(f^*)_{I_0}\|_{\mathcal{M}}.$$

Along the next section, the study of the boundedness of Calderón-Zygmund operators on operator-valued Hardy spaces will require a concrete formulation in terms of atomic decompositions. In order to justify introducing these spaces, we will check that the dual of this new description of the column (resp. row) Hardy space coincides with $\mathsf{BMO}_r(\mathbb{R}, \mathcal{M})$ (resp. $\mathsf{BMO}_c(\mathbb{R}, \mathcal{M})$).

Definition 4.2.8. *Let $\mathcal{M}$ be a von Neumann algebra with n.s.f. trace. A* c-atom *is a function a belonging to $L_1(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})$ which admits a factorization of the form $a = bh$ for some function $b : \mathbb{R} \to L_2(\mathcal{M})$ and some $h \in L_2(\mathcal{M})$ with norm $\|h\|_{L_2(\mathcal{M})} \leq 1$, satisfying*

1. $\mathrm{supp}_{\mathbb{R}}(b) \subseteq I$ *for some interval* I,
2. $\|b\|_{L_2(\mathbb{R};L_2(\mathcal{M}))} \leq \frac{1}{\sqrt{|I|}}$,
3. $\int_I b = 0$.

Then, the column Hardy space $H_1^c(\mathcal{A})$ *is defined to be the subspace of elements in* $L_1(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})$ *of the form*

$$\sum_{i=0}^{\infty} \lambda_i a_i \text{ where } (\lambda_i)_i \in \ell_1 \text{ and } (a_i)_i \text{ } c\text{-atoms} \tag{4.5}$$

with respect to the norm

$$\|f\| = \inf\{\sum_{i=0}^{\infty} |\lambda_i| \ : \ f = \sum_{i=0}^{\infty} \lambda_i a_i \text{ in } L_1(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M}), \ (\lambda_i)_i \in \ell_1, \ (a_i)_i \text{ } c\text{-atoms}\}.$$

Under the above definition, any c-atom satisfies

$$\|a\|_{L_1(\mathbb{R};L_1(\mathcal{M}))} \leq 1$$

since, by the Hölder inequality,

$$\|a\|_{L_1(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})} \leq \|b\|_{L_2(\mathbb{R};L_2(\mathcal{M}))} \ \|h \ \chi_B\|_{L_2(\mathbb{R};L_2(\mathcal{M}))} \leq |B|^{-1/2} \ |B|^{1/2} = 1.$$

Therefore, $H_1^c(\mathcal{A})$ is contractively contained into $L_1(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})$.

Proposition 4.2.9. *The column Hardy space* $(H_1^c(\mathcal{A}), \|\cdot\|_{H_1^c})$ *is a Banach space.*

Proof. Consider a sequence $(f_n)_{n\geq 1} \subseteq H_1^c(\mathcal{A})$ such that $\sum_n \|f_n\|_{H_1^c} < \infty$. Then, it follows that $\sum_n \|f_n\|_{L_1}$ is finite, and since $L_1(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})$ is a Banach space, there exists some $f \in L_1$ such that $f = \sum_n f_n$. Moreover, given $\varepsilon > 0$, any f_n admits an atomic decomposition satisfying

$$f_n = \sum_i \lambda_i^n b_i^n h_i^n \text{ such that } \sum_i |\lambda_i^n| \leq \|f_n\|_{H_1^c} + \frac{\varepsilon}{2^n}.$$

Therefore, the identity $f = \sum_n \sum_i \lambda_i^n b_i^n h_i^n$ holds in L_1, but

$$\sum_n \sum_i |\lambda_i^n| \leq \sum_n \|f_n\|_{H_1^c} + \frac{\varepsilon}{2^n} = \sum_n \|f_n\|_{H_1^c} + \varepsilon,$$

so $f \in H_1^c(\mathcal{A})$ too. It only remains to proof that $\|f - \sum_{n=1}^N f_n\|_{H_1^c}$ tends to as 0 as N goes to infinity, but

$$\lim_{N\to\infty} \ f - \sum_{n=1}^{N} f_n\|_{H_1^c} \leq \lim_{N\to\infty} \sum_{n=N+1}^{\infty} \sum_{i=0}^{\infty} |\lambda_i^n| = 0,$$

since $\sum_n \sum_i |\lambda_i^n|$ is finite. □

Lemma 4.2.10. *The inclusion map $i : H_1^c(\mathcal{A}) \to L_1(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})$ is an injective contraction. Therefore, the adjoint map $i^* : L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M} \to H_1^c(\mathcal{A})^*$ is a weak*-continuous contractive operator with weak*-dense range, and $i^*(L_\infty(\mathbb{R}) \otimes \mathcal{M})$ is weak*-dense in $H_1^c(\mathcal{A})^*$.*

Proof. The injectivity of i implies that i^* has dense range. Moreover, it holds

$$i^*(\overline{L_\infty(\mathbb{R}) \otimes \mathcal{M}}^{w^*}) \subseteq \overline{i^*(L_\infty(\mathbb{R}) \otimes \mathcal{M})}^{w^*}$$

so the result follows. □

Given a Banach space $\mathbb{X}$, let $L_2^\circ(\mathbb{R}, (1+t^2)\, dt; \mathbb{X})$ denote the subspace of functions f in $L_2(\mathbb{R}, (1+t^2)\, dt; \mathbb{X})$ satisfying

$$\int_{\mathbb{R}} f(t)\, dt = 0.$$

Then, the classical argument by Meyer [56, Chapter 5, Proposition 1] extends to the Banach-valued setting yielding the inclusion as a subspace of $L_2^\circ(\mathbb{R}, (1+t^2)\, dt, \mathbb{X})$ into the vector-valued Hardy space $H_1(\mathbb{R}; \mathbb{X})$ [32]. More clearly, given $f \in L_2^\circ(\mathbb{R}, (1+t^2)dt; L_2(\mathcal{M}))$, there exists a sequence of atoms $(b_i)_i \subseteq H_1(\mathbb{R}; L_2(\mathcal{M}))$ and $(\lambda_i)_i \in \ell_1$ such that

$$f = \sum_{i=0}^{\infty} \lambda_i b_i \text{ in } L_1(\mathbb{R}; L_2(\mathcal{M})) \text{ and } \sum_{i=0}^{\infty} |\lambda_i| \lesssim \|f\|_{L_2^\circ(\mathbb{R},(1+t^2)dt; L_2(\mathcal{M}))}.$$

Since any c-atom $a = bh$ is the product of an L_2-atom b in $H_1(\mathbb{R}; L_2(\mathcal{M}))$ and an element h in $L_2(\mathcal{M})$, the argument by Meyer still works in the semicommutative case.

Proposition 4.2.11. *Let Q be the bilinear map given by*

$$\begin{array}{rccc} Q: & L_2^\circ(\mathbb{R}, (1+t^2)dt; L_2(\mathcal{M})) \times L_2(\mathcal{M}) & \longrightarrow & H_1^c(\mathcal{A}) \\ & (f,h) & \longmapsto & \sum_{i=0}^{\infty} \lambda_i b_i\, h \end{array}$$

where $\sum_{i=0}^{\infty} \lambda_i b_i$ denotes the atomic decomposition for f in $H_1(\mathbb{R}; L_2(\mathcal{M}))$ obtained via the $L_2(\mathcal{M})-$valued extension of the argument in [56, Chapter 5, Proposition 1]. Then Q extends to a bounded linear operator with dense range from $L_2^\circ(\mathbb{R}, (1+t^2)dt; L_2(\mathcal{M}))\widehat{\otimes}_\pi L_2(\mathcal{M})$ to $H_1^c(\mathcal{A})$. Therefore, the adjoint map

$$Q^* : (H_1^c(\mathcal{A}))^* \longrightarrow B(L_2(\mathcal{M}), L_2^\circ(\mathbb{R}, (1+t^2)dt)^* \otimes_2 L_2(\mathcal{M})^*)$$

is a weak-continuous injective bounded operator.*

Proof. Given an arbitrary element (f,h) in $L_2^\circ(\mathbb{R}, (1+t^2)dt; L_2(\mathcal{M})) \times L_2(\mathcal{M})$, there holds

$$\|Q(f,h)\|_{H_1^c(\mathcal{A})} = \Big\| \sum_{i=0}^{\infty} \lambda_i b_i\, h \Big\|_{H_1^c(\mathcal{A})} \leq \sum_{i=0}^{\infty} |\lambda_i|\, \|h\|_{L_2(\mathcal{M})} \Big\| b_i \frac{h}{\|h\|_{L_2(\mathcal{M})}} \Big\|_{H_1^c(\mathcal{A})}$$

$$= \sum_{i=0}^{\infty} |\lambda_i| \, \|h\|_{L_2(\mathcal{M})} \lesssim \|f\|_{L_2^\circ(\mathbb{R},(1+t^2)dt;L_2(\mathcal{M}))} \, \|h\|_{L_2(\mathcal{M})}.$$

Therefore Q is a bilinear map which extends to a bounded operator on the projective tensor product $L_2^\circ(\mathbb{R},(1+t^2)dt;L_2(\mathcal{M}))\widehat{\otimes}_\pi L_2(\mathcal{M})$ into $H_1^c(\mathcal{A})$ [76, Theorem 2.9]. Since c-atoms are elements of $L_2^\circ(\mathbb{R},(1+t^2)dt;L_2(\mathcal{M}))\widehat{\otimes}_\pi L_2(\mathcal{M})$, the range of Q is dense in $H_1^c(\mathcal{A})$, so the adjoint map Q^* is a weak*-continuous injection from the dual space $(H_1^c(\mathcal{A}))^*$ into

$$B(L_2(\mathcal{M}), L_2^\circ(\mathbb{R},(1+t^2)dt)^* \otimes_2 L_2(\mathcal{M})^*).$$

□

Lemma 4.2.12. *Given a semifinite von Neumann algebra $\mathcal{M}$ with n.s.f. trace, there holds*

$$\overline{Q^*(H_1^c(\mathcal{A})^*)}^{w^*} = \overline{Q^* i^*(L_\infty(\mathbb{R}) \otimes \mathcal{M})}^{w^*} = U(L_\infty(\mathcal{M}; L_2^{\circ,r}(\mathbb{R},(1+t^2)dt)))$$

where Q^* and U denote the maps from Propositions 4.2.11 and 4.1.5.

Proof. Let $g = \sum_{j=1}^n g_j \otimes m_j \in L_\infty(\mathbb{R}) \otimes \mathcal{M}$. We claim that there exists some $G \in L_\infty(\mathcal{M}; L_2^{\circ,r}(\mathbb{R},(1+t^2)dt))$ such that $Q^* i^*(g) = U(G)$. First, it holds

$$\begin{aligned}
\langle Q^* i^*(g), f \otimes h\rangle &= \langle i^*(g), \sum_{j=1}^{\infty} \lambda_j b_j h\rangle_{(H_1^c(\mathcal{A}))^*, H_1^c(\mathcal{A})} \\
&= \langle g, \sum_{j=1}^{\infty} \lambda_j b_j h\rangle_{L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M}, L_1(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})} \\
&= \langle g, fh\rangle_{L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M}, L_1(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})} \\
&= \tau_{\mathcal{M}} \int_{\mathbb{R}} hg\, f
\end{aligned}$$

for any $f \otimes h$ in $(L_2^\circ(\mathbb{R},(1+t^2)dt) \otimes_2 L_2(\mathcal{M}))\widehat{\otimes}_\pi L_2(\mathcal{M})$. On the other hand, taking

$$G = \sum_{j=1}^{n} \left(\frac{\mathbb{1}}{1+t^2} \otimes \frac{g_j - \frac{1}{\pi}\int_{\mathbb{R}} g_j \frac{dt}{1+t^2}}{1+t^2} \right) \otimes m_j \in (L_\infty \cap L_2^\circ)(\mathbb{R},(1+t^2)dt) \otimes \mathcal{M}$$

yields $\overline{Q^*(i^*(L_\infty(\mathbb{R}) \otimes \mathcal{M}))}^{w^*} \subseteq U(L_\infty(\mathcal{M}; L_2^{\circ,r}(\mathbb{R},(1+t^2)dt)))$ since

$$\begin{aligned}
\langle f, U(G) \otimes h\rangle &= \langle f, \sum_{j=1}^{n} \overline{\frac{g_j - \frac{1}{\pi}\int_{\mathbb{R}} g_j \frac{dt}{1+t^2}}{1+t^2}} \otimes \overline{m_j^* h^*}\rangle_{L_2^\circ(\mathbb{R},(1+t^2)dt;L_2(\mathcal{M}))} \\
&= \sum_{j=1}^{n} \tau_{\mathcal{M}} \int_{\mathbb{R}} (hm_j \otimes g_j) f = \tau_{\mathcal{M}} \int_{\mathbb{R}} hg\, f.
\end{aligned}$$

The map $Q^*i^*(g) \mapsto U(G)$ is injective since g_1 and g_2 induce the same functional if and only $g_1 - g_2 \in \mathcal{M}$. For the reverse inclusion, it only remains to check that, given

$$G = \sum_{j=1}^{n} \left(\frac{\mathbb{1}}{1+t^2} \otimes g_j \right) \otimes m_j \in (L_\infty \cap L_2^\circ)(\mathbb{R}, (1+t^2)dt) \otimes \mathcal{M},$$

if $g = \sum_{j=1}^{n} g_j(1+t^2) \otimes m_j$, then $U(G)$ and $Q^*i^*(g)$ coincide. Therefore, there exists and isomorphism between $Q^*i^*(L_\infty(\mathbb{R}) \otimes \mathcal{M})$ and the image under U of $(L_\infty \cap L_2^\circ)(\mathbb{R}, (1+t^2)dt) \otimes \mathcal{M}$, so their weak* closures coincide, yielding the statement. □

The previous statements enable us to represent any functional on $H_1^c(\mathcal{A})$ as an operator in $\mathsf{BMO}_r(\mathbb{R}, \mathcal{M})$.

Theorem 4.2.13. *Given a semifinite von Neumann algebra $\mathcal{M}$, it holds*

$$H_1^c(\mathcal{A})^* \subseteq \mathsf{BMO}_r(\mathbb{R}, \mathcal{M}).$$

Proof. By Lemma 4.2.12, the image of $H_1^c(\mathcal{A})^*$ under Q^* is a normed subspace of $U(L_\infty(\mathcal{M}; L_2^{\circ,r}(\mathbb{R}, (1+t^2)dt)))$. Indeed, by Corollary 4.1.6, $U^{-1}Q^*(H_1^c(\mathcal{A})^*)$ is a subspace of $L_\infty(\mathcal{M}; L_2^{\circ,r}(\mathbb{R}, (1+t^2)dt))$, so $VU^{-1}Q^*(H_1^c(\mathcal{A})^*)$, where V denotes the map from Lemma 4.2.3, is a subspace of $L_\infty(\mathcal{M}, L_2^r(\mathbb{R}, dt/(1+t^2)))$. Let $g \in H_1^c(\mathcal{A})^*$. Then,

$$\begin{aligned} \|VU^{-1}Q^*g\|_{\mathsf{BMO}_r} &= \sup_{I \subseteq \mathbb{R}} \|R_I(\mathrm{Id} - a_I)VU^{-1}Q^*g\|_{L_\infty(\mathcal{M}; L_2^r(\mathbb{R}, \frac{dt}{1+t^2}))} \\ &= \sup_I \sup_f |\langle f, R_I(\mathrm{Id} - a_I)VU^{-1}Q^*g\rangle| \end{aligned}$$

where the supremum is taken over $f \in L_1(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dt}{1+t^2}))$ such that

$$\|f\|_{L_1(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dt}{1+t^2}))} \leq 1.$$

As a consequence of Corollary 4.1.6, there exists some F in the chosen predual for

$$U(L_\infty(\mathcal{M}; L_2^c(\mathbb{R}, (1+t^2)dt)))$$

such that $f = U_*(F)$. Therefore,

$$\begin{aligned} \|VU^{-1}Q^*g\|_{\mathsf{BMO}_r} &= \sup_I \sup_{\|U_*(F)\| \leq 1} |\langle U_*(F), R_I(\mathrm{Id} - a_I)VU^{-1}Q^*g\rangle| \\ &= \sup_I \sup_{\|U_*(F)\| \leq 1} |\langle F, UR_I(\mathrm{Id} - a_I)VU^{-1}Q^*g\rangle| \\ &\leq \sup_I \sup_{\|F\| \leq \|U_*^{-1}\|} |\langle F, UR_I(\mathrm{Id} - a_I)VU^{-1}Q^*g\rangle| \\ &= \|U_*^{-1}\| \sup_I \sup_{\|F\| \leq 1} |\langle U_*(F), R_I(\mathrm{Id} - a_I)VU^{-1}Q^*g\rangle|. \end{aligned}$$

Moreover, fixed F and $\varepsilon > 0$, there exists $h_{\varepsilon,F} \in U(L_\infty(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dt}{1+t^2})))_\perp$ such that $U_*(F) = U_*(F + h_{\varepsilon,F})$ and $\|F + h_{\varepsilon,F}\| \leq 1 + \varepsilon$. Therefore

$$\begin{aligned}\|VU^{-1}Q^*g\|_{\mathsf{BMO}_r} &\leq \|U_*^{-1}\| \sup_I \sup_{\|F+h_{\varepsilon,F}\|\leq 1+\varepsilon} |\langle U_*(F + h_{\varepsilon,F}), R_I(\mathrm{Id} - a_I)VU^{-1}Q^*g\rangle| \\ &\leq \|U_*^{-1}\| \sup_I \sup_{\|F'\|\leq 1+\varepsilon} |\langle U_*(F'), R_I(\mathrm{Id} - a_I)VU^{-1}Q^*g\rangle| \\ &= (1+\varepsilon)\|U_*^{-1}\| \sup_I \sup_{\|F'\|\leq 1} |\langle U_*(F'), R_I(\mathrm{Id} - a_I)VU^{-1}Q^*g\rangle|\end{aligned}$$

where the supremum is taken over $F' \in L_2(\mathbb{R}, \frac{dt}{1+t^2}; L_2(\mathcal{M}))\widehat{\otimes}_\pi L_2(\mathcal{M})$. Indeed, we can replace the supremum on F' by tensors

$$f \otimes h = \sum_{j=1}^n \big(f_j \otimes m_j\big) \otimes h$$

such that $\|f\|_{L_2(\mathbb{R}, \frac{dt}{1+t^2})\otimes_2 L_2(\mathcal{M})} \leq 1$ and $\|h\|_{L_2(\mathcal{M})} \leq 1$. In other words, Remark 4.1.7 implies that

$$\begin{aligned}\|VU^{-1}Q^*g\|_{\mathsf{BMO}_r} &\lesssim \sup_I \sup_{\|f\|,\|h\|\leq 1} |\langle U_*(f \otimes h), R_I(\mathrm{Id} - a_I)VU^{-1}Q^*g\rangle| \\ &\leq \sup_I \sup_{\|f\|,\|h\|\leq 1} |\langle \sum_{j=1}^n \big(f_j \otimes \mathbb{1}\big) \otimes m_j h, R_I(\mathrm{Id} - a_I)VU^{-1}Q^*g\rangle|.\end{aligned}$$

Then, by Lemma 4.2.3, it holds

$$\begin{aligned}\|VU^{-1}Q^*g\|_{\mathsf{BMO}_r} &\lesssim \sup_I \sup_{f,h} |\langle \sum_{j=1}^n \big(f_j \otimes \mathbb{1}\big) \otimes m_j h, R_I(\mathrm{Id} - a_I)VU^{-1}Q^*g\rangle| \\ &= \sup_I \sup_{f,h} |\langle \sum_{j=1}^n \Big(\frac{\sqrt{1+t^2}}{\sqrt{|I|}}\chi_I f_j \otimes \mathbb{1}\Big) \otimes m_j h, (\mathrm{Id} - a_I)VU^{-1}Q^*g\rangle| \\ &= \sup_{I,f,h} |\langle \sum_{j=1}^n \Big(\Big(\frac{\sqrt{1+t^2}}{\sqrt{|I|}}\chi_I f_j - \frac{1+t^2}{|I|}\chi_I \,\langle \frac{\sqrt{1+t^2}}{\sqrt{|I|}}\chi_I f_j, 1\rangle\, 1\Big) \otimes \mathbb{1}\Big) \otimes m_j h,\ VU^{-1}Q^*g\rangle| \\ &= \sup_{I,f,h} |\langle \sum_{j=1}^n \Big(\Big(\frac{\chi_I}{\sqrt{|I|}\sqrt{1+t^2}} f_j - \frac{\chi_I}{|I|}\,\langle \frac{\sqrt{1+t^2}}{\sqrt{|I|}}\chi_I f_j, 1\rangle\, 1\Big) \otimes \frac{\mathbb{1}}{1+t^2}\Big) \otimes m_j h,\ U^{-1}Q^*g\rangle|.\end{aligned}$$

Now, recall that for any $j = 1, \ldots, n$, by denoting

$$F_j = \frac{\chi_I}{\sqrt{|I|}\sqrt{1+t^2}} f_j - \frac{\chi_I}{|I|}\langle \frac{\sqrt{1+t^2}}{\sqrt{|I|}}\chi_I f_j, 1\rangle 1,$$

there holds

$$U_*\big(\big(F_j \otimes m_j\big) \otimes h\big) = \Big(F_j \otimes \frac{\mathbb{1}}{1+t^2}\Big) \otimes m_j h.$$

Therefore,

$$\begin{aligned}\|VU^{-1}Q^*g\|_{\mathsf{BMO}_r} &\lesssim \sup_{I,f,h} |\langle \sum_{j=1}^n (F_j \otimes m_j) \otimes h, Q^* g\rangle| \\ &= \sup_{I,f,h} |\langle Q\Big(\sum_{j=1}^n (F_j \otimes m_j) \otimes h\Big), g\rangle_{H_1^c,(H_1^c)^*}| \\ &= \sup_{I,f,h} |\langle Q\Big(\Big(\frac{\chi_I}{\sqrt{|I|}\sqrt{1+t^2}} f - \frac{\chi_I}{|I|}\int_I \frac{1}{\sqrt{|I|}\sqrt{1+t^2}} f\Big) \otimes h\Big), g\rangle_{H_1^c,(H_1^c)^*}|.\end{aligned}$$

Moreover, notice that, fixed an interval I, the operators

$$f^I = \frac{\chi_I}{\sqrt{|I|}\sqrt{1+t^2}} f - \frac{\chi_I}{|I|}\int_I \frac{1}{\sqrt{|I|}\sqrt{1+t^2}} f$$

satisfy $\mathrm{supp}(f^I) \subseteq I$, $\int_{\mathbb{R}} f^I = 0$ and

$$\Big(\int_{\mathbb{R}} \Big| \frac{\chi_I}{\sqrt{|I|}\sqrt{1+t^2}} f - \frac{\chi_I}{|I|}\int_I \frac{1}{\sqrt{|I|}\sqrt{1+t^2}} f \Big|^2 dt\Big)^{1/2} \leq \frac{2}{\sqrt{|I|}} \|f\|_{L_2(\mathbb{R}, \frac{dt}{1+t^2})} \leq \frac{2}{\sqrt{|I|}}.$$

In other words, $f^I h$ is a c-atom, so

$$\begin{aligned}\|VU^{-1}Q^*g\|_{\mathsf{BMO}_r} &\lesssim \sup_{I,f,h} |\langle Q(f^I \otimes h), g\rangle_{H_1^c,(H_1^c)^*}| \\ &= \sup_{I,f,h} |\langle f^I h, g\rangle_{H_1^c,(H_1^c)^*}| \\ &\leq \sup_{a\ c-\text{atom}} |\langle a, g\rangle_{H_1^c,(H_1^c)^*}| \\ &= \|g\|_{H_1^c(\mathcal{A})^*}.\end{aligned}$$

Since ε is arbitrary, the best inequality to obtain is

$$\|VU^{-1}Q^*g\|_{\mathsf{BMO}_r} \leq \|U_*^{-1}\| \, \|g\|_{H_1^c(\mathcal{A})^*}.$$

□

In order to prove the reverse inclusion $\mathsf{BMO}_r(\mathbb{R}, \mathcal{M}) \subseteq H_1^c(\mathcal{A})^*$, we only need to check that every operator in $\mathsf{BMO}_r(\mathbb{R}, \mathcal{M})$ induces a continuous functional on $H_1^c(\mathcal{A})$. For this purpose, introduce some auxiliar maps.

Lemma 4.2.14. *Let J be a finite interval of $\mathbb{R}$, and let ϕ be a non-negative smooth function supported on $B(0,1)$ such that $\int \phi = 1$ and $\phi(-x) = \phi(x)$. Consider the bounded maps*

$$\begin{aligned} A^J : \quad L_1(\mathcal{M}; L_2^c(\mathbb{R}, dt)) &\longrightarrow L_1(\mathcal{M}; L_2^c(\mathbb{R}, \tfrac{dt}{1+t^2})) \\ m \otimes (f \otimes \tfrac{\mathbb{1}}{\sqrt{1+t^2}}) &\longmapsto m \otimes ((1+t^2)\chi_J f \otimes \mathbb{1}),\end{aligned}$$

$$\begin{array}{rccc} R_\phi : & L_1(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M}) & \longrightarrow & L_1(\mathcal{M}; L_2^c(\mathbb{R}, dt)) \\ & m \otimes f' & \longmapsto & m \otimes ((\phi * f') \otimes \frac{\mathbb{1}}{\sqrt{1+t^2}}), \end{array}$$

$$\begin{array}{rccc} B : & L_1(\mathcal{M}; L_2^c(\mathbb{R}, dt)) & \longrightarrow & L_1(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dt}{1+t^2})) \\ & m \otimes (f \otimes \frac{\mathbb{1}}{\sqrt{1+t^2}}) & \longmapsto & m \otimes (\sqrt{1+t^2} f \otimes \mathbb{1}) \end{array}$$

where $m \in L_1(\mathcal{M})$, $f \in L_2(\mathbb{R})$ *and* $f' \in L_1(\mathbb{R})$. *Then, there holds*

$$\|A^J\| \leq \sup_{t\in J}(1+t^2)^{1/2}, \ \|R_\phi\| \leq \|\phi\|_{L_2(\mathbb{R})}, \ \|B\| = 1$$

and

$$\begin{array}{rccc} (A^J)^* : & L_\infty(\mathcal{M}; L_2^r(\mathbb{R}, \frac{dt}{1+t^2})) & \longrightarrow & L_\infty(\mathcal{M}; L_2^r(\mathbb{R}, dt)) \\ & m \otimes (\mathbb{1} \otimes g) & \longmapsto & m \otimes (\frac{\mathbb{1}}{\sqrt{1+t^2}} \otimes \chi_J g), \end{array}$$

$$\begin{array}{rccc} R_\phi^* : & L_\infty(\mathcal{M}; L_2^r(\mathbb{R}, dt)) & \longrightarrow & L_\infty(\mathcal{M}; L_2^r(\mathbb{R}, dt)) \\ & m \otimes (\frac{\mathbb{1}}{\sqrt{1+t^2}} \otimes g) & \longmapsto & m \otimes (\frac{\mathbb{1}}{\sqrt{1+t^2}} \otimes \phi * g), \end{array}$$

$$\begin{array}{rccc} B^* : & L_\infty(\mathcal{M}; L_2^r(\mathbb{R}, \frac{dt}{1+t^2})) & \longrightarrow & L_\infty(\mathcal{M}; L_2^r(\mathbb{R}, dt)) \\ & m \otimes (\mathbb{1} \otimes g) & \longmapsto & m \otimes (\frac{\mathbb{1}}{\sqrt{1+t^2}} \otimes \frac{1}{\sqrt{1+t^2}} g), \end{array}$$

where $g \in L_2(\mathbb{R})$ *and* $m \in \mathcal{M}$.

Proof. The claims related to the maps A^J and B are straightforward computations in the spirit of Lemma 4.2.3. On the other hand, the universal property of projective tensor product assures that

$$\|R_\phi\| = \sup_{\substack{\|m\|_{L_1(\mathcal{M})}\leq 1, \\ \|f'\|_{L_1(\mathbb{R})}\leq 1}} \|R_n(m \otimes f')\|_{L_1(\mathcal{M};L_2^c(\mathbb{R},dt))},$$

but there holds

$$\begin{aligned} \|R_\phi(m \otimes f')\|_{L_1(\mathcal{M};L_2^c(\mathbb{R},dt))} &= \Big\| m \otimes (\phi * f' \otimes \frac{\mathbb{1}}{\sqrt{1+t^2}}) \Big\|_{L_1(\mathcal{M};L_2^c(\mathbb{R},dt))} \\ &= \Big\| \Big(m^*m \int |\phi * f'|^2\Big)^{1/2} \Big\|_{L_1(\mathcal{M})} \\ &= \|m\|_{L_1(\mathcal{M})} \, \|\phi * f'\|_{L_2(\mathbb{R})} \\ &\leq \|\phi\|_{L_2(\mathbb{R})} \, \|m\|_{L_1(\mathcal{M})} \, \|f'\|_{L_1(\mathbb{R})} \end{aligned}$$

by Young's inequality. Then, the adjoint R_ϕ^* admits the expression of the statement since

$$\begin{aligned} &\big(\tau_\mathcal{M}\otimes \mathrm{Tr}_{B(L_2(\mathbb{R},\frac{dt}{1+t^2}))}\big)\Big(m_1 \otimes (\phi * f' \otimes \frac{\mathbb{1}}{\sqrt{1+t^2}}) \cdot m_2 \otimes (\frac{\mathbb{1}}{\sqrt{1+t^2}} \otimes g)\Big) \\ &= \tau(m_1 m_2) \int \overline{g}(x) \int \phi(x-y) f'(y) \, dx \, dy \\ &= \tau(m_1 m_2) \int f'(y) \int \overline{g(x)} \phi(x-y) \, dy \, dx \end{aligned}$$

$$= \Big(\tau_{\mathcal{M}} \otimes \mathrm{Tr}_{B(L_2(\mathbb{R},\frac{dt}{1+t^2}))}\Big)\Big(m_1 \otimes f' \cdot m_2 \otimes \phi * g\Big)$$

for any $m_1 \in L_1(\mathcal{M})$ and $m_2 \in \mathcal{M}$. □

The starting point of our argument is that every operator $\varphi \in \mathsf{BMO}_r(\mathbb{R},\mathcal{M})$ induces a functional on the vector subspace generated by the $c-$atoms. For that purpose, any c-atom must be showed to admit a representation as an operator belonging to $L_1(\mathcal{M}; L_2^c(\mathbb{R},dt))$. Indeed, the map $\Gamma : \mathrm{span}\{c - \mathrm{atoms}\} \longrightarrow L_1(\mathcal{M}; L_2^c(\mathbb{R},dt))$ sending bh to $b(h \otimes (\frac{\mathbb{1}}{\sqrt{1+t^2}} \otimes \frac{\mathbb{1}}{\sqrt{1+t^2}}))$ is well-defined since

$$\Big\| b(h \otimes (\frac{\mathbb{1}}{\sqrt{1+t^2}} \otimes \frac{\mathbb{1}}{\sqrt{1+t^2}}))\Big\|_{L_1(\mathcal{M};L_2^c(\mathbb{R},dt))} = \Big\| \Big(\int |bh|^2\Big)^{1/2}\Big\|_{L_1(\mathcal{M})}$$
$$= \Big\| \int h^*|b|^2 h\Big\|^{1/2}_{L_{1/2}(\mathcal{M})} \leq \|h\|_{L_2(\mathcal{M})}\|b\|_{L_2(\mathbb{R};L_2(\mathcal{M}))}$$

whenever $b \in L_2(\mathcal{M}) \otimes L_2(\mathbb{R})$, and the estimate extends to general c-atoms by approximation.

Lemma 4.2.15. *Let $\varphi \in \mathsf{BMO}_r(\mathbb{R},\mathcal{M})$. If $a = bh$ is a c-atom in $H_1^c(\mathcal{A})$ supported on I, there holds*

$$|\langle \varphi, A^I(\Gamma(bh))\rangle_{L_\infty(\mathcal{M};L_2^r(\mathbb{R},\frac{dt}{1+t^2})),L_1(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}| \leq C\ \|\varphi\|_{\mathsf{BMO}_r}.$$

Proof. Let I be the interval for which $a = bh$ satisfies the definition of c-atom. Let a'_I denote the map from $L_\infty(\mathcal{M}; L_2^c(\mathbb{R},dt))$ into itself which sends $m \otimes (\frac{\mathbb{1}}{\sqrt{1+t^2}} \otimes g)$ into

$$\sqrt{\pi}\ \mathrm{Tr}_{B(L_2(\mathbb{R},dt))}\Big(\frac{\mathbb{1}}{\sqrt{1+t^2}} \otimes \frac{\sqrt{1+t^2}g\chi_I}{|I|}\Big)\ m' \otimes (\frac{\mathbb{1}}{\sqrt{1+t^2}} \otimes 1).$$

Then,

$$\begin{aligned}\langle \varphi, A^I(\Gamma(bh))\rangle &= \langle (A^I)^*\varphi, \Gamma(bh)\rangle_{L_\infty(\mathcal{M};L_2^r(\mathbb{R},dt)),L_1(\mathcal{M};L_2^c(\mathbb{R},dt))}\\ &= \langle (A^I)^*\varphi - a'_I(A^I)^*\varphi, \Gamma(bh)\rangle_{L_\infty(\mathcal{M};L_2^r(\mathbb{R},dt)),L_1(\mathcal{M};L_2^c(\mathbb{R},dt))}\\ &= \langle (B^{-1})^*[(A^I)^*\varphi - a'_I(A^I)^*\varphi], B\Gamma(bh)\rangle_{L_\infty(\mathcal{M};L_2^r(\mathbb{R},\frac{dt}{1+t^2})),L_1(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\ &= \langle \widetilde{P}_I\widetilde{T}_{|I|^{-1}}(B^{-1})^*[(A^I)^*(\varphi) - a'_I(A^I)^*(\varphi)], \widetilde{P}_I\widetilde{T}_{|I|}B\Gamma(bh)\rangle.\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}\langle \varphi, A^I(\Gamma(bh))\rangle &= \langle (A^I)^*\varphi - a'_I(A^I)^*\varphi, \Gamma(bh)\rangle_{L_\infty(\mathcal{M};L_2^r(\mathbb{R},dt)),L_1(\mathcal{M};L_2^c(\mathbb{R},dt))}\\ &= \langle (B^{-1})^*[(A^I)^*\varphi - a'_I(A^I)^*\varphi], B\Gamma(bh)\rangle_{L_\infty(\mathcal{M};L_2^r(\mathbb{R},\frac{dt}{1+t^2})),L_1(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\ &= \langle \widetilde{P}_I\widetilde{T}_{|I|^{-1}}(B^{-1})^*[(A^I)^*(\varphi) - a'_I(A^I)^*(\varphi)], \widetilde{P}_I\widetilde{T}_{|I|}B\Gamma(bh)\rangle.\end{aligned}$$

Recalling that $(B^{-1})^*(A^I)^*(\varphi) = \widetilde{T}_\omega \widetilde{T}_I$ and $\widetilde{P}_I(B^{-1})^* a'_I (A^I)^*\varphi = \widetilde{P}_I \widetilde{T}_\omega a_I \varphi$, then the duality product can be bounded as follows

$$\begin{aligned} |\langle \varphi, A^I(\Gamma(bh))\rangle| &\le \big\|\widetilde{P}_{I\varphi}\widetilde{T}_{|I|^{-1}}(B^{-1})^*[(A^I)^*(\varphi) - a'_{I\varphi}(A^I)^*\varphi]\big\| \, \big\|\widetilde{P}_I\widetilde{T}_{|I|}B(b_ih_i)\big\| \\ &= \big\|\widetilde{P}_I\widetilde{T}_{|I|^{-1}}\widetilde{T}_\omega(\varphi - a_I\varphi)\big\| \, \big\|\widetilde{P}_I\widetilde{T}_{|I|}B(b_ih_i)\big\| \\ &\le \|\varphi\|_{\mathsf{BMO}_r} \, \big\|\widetilde{P}_I\widetilde{T}_{|I|}B(\Gamma(bh))\big\|_{L_1(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}, \end{aligned}$$

so the statement is a consequence of the estimate

$$\begin{aligned} \|\widetilde{P}_I\widetilde{T}_{|I|}B\Gamma(bh)\|_{L_1(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))} &\le \sqrt{|I|}\,\|\Gamma(bh)\|_{L_1(\mathcal{M};L_2^c(\mathbb{R},dt))} \\ &= \sqrt{|I|}\,\Big\|\Big(\int_I |bh|^2\Big)^{1/2}\Big\|_{L_1(\mathcal{M})} = \sqrt{|I|}\,\Big\|\int_I h^*|b|^2h\Big\|^{1/2}_{L_{1/2}(\mathcal{M})} \\ &\le \sqrt{|I|}\,\|h\|_{L_2(\mathcal{M})}\,\Big\|\Big(\int_I |b|^2\Big)^{1/2}\Big\|_{L_2(\mathcal{M})} \le 1. \end{aligned} \tag{4.6}$$

□

Given $m \ge 1$, and given a function ϕ as in the statement of Lemma 4.2.14, define

$$R_n = R_{\phi_n} \text{ where } \phi_n(x) = n\phi(nx).$$

Then, given a finite interval J and $\varphi \in \mathsf{BMO}_r(\mathbb{R},\mathcal{M})$, there holds

$$\begin{aligned} &\Big\langle \varphi, A^J R_n\Big(\sum_i \lambda_i b_i h_i\Big)\Big\rangle_{L_\infty(\mathcal{M};L_2^r(\mathbb{R},\frac{dt}{1+t^2})),L_1(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))} \\ &= \Big\langle R_n^*(A^J)^*\varphi, \sum_i \lambda_i b_i h_i\Big\rangle_{L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M},L_1(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})} = 0 \end{aligned}$$

whenever $\sum_i \lambda_i b_i h_i = 0$ in $L_1(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})$. This is the starting point which ensures that any $\psi \in \mathsf{BMO}_r(\mathbb{R},\mathcal{M})$ with compact support induces a functional in $H_1^c(\mathcal{A})$.

Lemma 4.2.16. *Let ϕ be a function as in the statement of Lemma 4.2.14. Given a c-atom $a = bh$ supported on I, the following holds:*

1. $\|\phi_n * b - b\|_{L_2(\mathbb{R};L_2(\mathcal{M}))} \longrightarrow 0$ *as n goes to infinity,*
2. *the operator*

$$\frac{\sqrt{|I|}}{2\sqrt{|I| + \frac{2}{n}}}(\phi_m * b - b)h$$

is a c-atom supported on $I + \frac{1}{n}(-1,1)$.

Let $\{K_j\}_{j=1}^N$ be a family of finite intevals such that $(-1,1) = \bigcup_{j=1}^N K_j$. Then, *(3) the operator*

$$(\phi\chi_{K_j})_n * bh - \int (\phi\chi_{K_j})_n(y)\, dy \cdot bh$$

is a multiple of a c-atom supported on $I + \frac{1}{n}K_j$ *satisfying*

$$\|(\phi\chi_{K_j})_n * b - \int (\phi\chi_{K_j})_n(y)\, dy \cdot b\|_{L_2(\mathbb{R};L_2(\mathcal{M}))} \leq \frac{2}{\sqrt{|I|}} \int_{K_j} \varphi.$$

Proof. The claim (1) is a consequence of the fact that translation operators are continuous on $L_2(\mathbb{R};L_2(\mathcal{M}))$. On the other hand, (2) follows from the fact that $\phi_n * b - b$ is supported on $I + \frac{1}{n}(-1,1)$, it has integral zero, that is,

$$\int \phi_n * b(x)\, dx = \int \int \phi_n(y) b(x-y)\, dy\ dx = \int \phi_n(y) \int b(x-y)\, dx\ dy = 0,$$

and

$$\|\phi_n * b - b\|_{L_2(\mathbb{R};L_2(\mathcal{M}))} \leq \int \phi(y) \|b(\cdot - y) - b\|_{L_2(\mathbb{R};L_2(\mathcal{M}))} \leq \frac{2}{\sqrt{|I|}}.$$

Analogously, it is straightforward to check that the support of $(\phi\chi_{K_j})_m * b - b$ is supported on $I + \frac{1}{m}K_j$ and $\int (\phi\chi_{I_j})_n * b - b = 0$. Moreover,

$$\begin{aligned}
\|(\phi\chi_{K_j})_n * b - \int (\phi\chi_{K_j})_n(y)\, dy \cdot b\|_{L_2(\mathbb{R};L_2(\mathcal{M}))} \\
&\leq \int (\phi\chi_{K_j})_n(y)\ [b(\cdot - y) - b(\cdot)]\, dy\|_{L_2(\mathbb{R};L_2(\mathcal{M}))} \\
&\leq \frac{2}{\sqrt{|I|}} \int (\phi\chi_{K_j})_n(y)\, dy \\
&= \frac{2}{\sqrt{|I|}} \int_{K_j} \phi.
\end{aligned}$$

□

Given a c-atom $a = bh$, Lemma 4.2.16 justifies that the identity

$$R_n(bh) = \Gamma(\phi_n * bh)$$

holds.

Proposition 4.2.17. *Let* $\psi \in \mathsf{BMO}_r(\mathbb{R},\mathcal{M})$ *satisfying* $\widetilde{P}_J\psi = \psi$ *for some finite interval* J*. Then, there holds*

$$\sum_i \lambda_i\ \langle \psi, A^J\Gamma(b_i h_i)\rangle_{L_\infty(\mathcal{M};L_2^r(\mathbb{R},\frac{dt}{1+t^2})),L_1(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))} = 0$$

whenever $\sum_i \lambda_i b_i h_i = 0$ *in* $L_1(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})$.

Proof. As a consequence of the discussion preceding Lemma 4.2.16, it is sufficient to show that

$$\sum_i \lambda_i \ \langle \psi, A^J(\Gamma - R_n)(b_i h_i)\rangle_{L_\infty(\mathcal{M};L_2^r(\mathbb{R},\frac{dt}{1+t^2})),L_1(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))} = 0.$$

Arguing as in the proof of Lemma 4.2.15, it follows that, if $I_i^{(n)}$ denotes $I_i + \frac{1}{n}(-1,1)$, there holds

$$\begin{aligned}|\langle \psi, A^J(\Gamma - R_n)(b_i h_i)\rangle &= |\langle (A^J)^*\psi, (\Gamma - R_n)(b_i h_i)\rangle_{L_\infty(\mathcal{M};L_2^r(\mathbb{R},dt)),L_\infty(\mathcal{M};L_2^c(\mathbb{R},dt))}| \\ &= |\langle \widetilde{P}_{I_i^{(n)}} \widetilde{T}_{|I_i^{(n)}|^{-1}} (B^{-1})^*[(A^J)^*(\psi) - a'_{I_i^{(n)}}(A^J)^*(\psi)], \widetilde{P}_{I_i^{(n)}} \widetilde{T}_{|I_i^{(n)}|} B(\Gamma - R_n)(bh)\rangle|\end{aligned}$$

Since ψ has compact support contained in J, it follows that

$$\widetilde{P}_{I_i^{(n)}}(B^{-1})^*(A^J)^*\psi = \widetilde{P}_{I_i^{(n)}}\widetilde{T}_\omega \widetilde{P}_J\psi = \widetilde{P}_{I_i^{(n)}}\widetilde{T}_\omega\psi$$

and

$$\widetilde{P}_{I_i^{(n)}}(B^{-1})^* \ a'_{I_i^{(n)}} \ (A^J)^*\psi = \widetilde{P}_{I_i^{(n)}}\widetilde{T}_\omega \ a_{I_i^{(n)}} \ \widetilde{P}_J\psi = \widetilde{P}_{I_i^{(n)}}\widetilde{T}_\omega \ a_{I_i^{(n)}}\psi.$$

This implies the estimate

$$\begin{aligned}&|\langle \psi, A^J(\Gamma - R_n)(b_i h_i)\rangle| \\ &\quad\le \|\widetilde{P}_{I_i^{(n)}}\widetilde{T}_\omega\widetilde{T}_{I_i^{(n)}}[\psi - a_{I_i^{(n)}}\psi]\| \ \|\widetilde{P}_{I_i^{(n)}}\widetilde{T}_{|I_i^{(n)}|}B(\Gamma - R_n)(b_i h_i)\|_{L_1(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))} \\ &\quad\le \|\psi\|_{\mathsf{BMO}_r} \ \|\widetilde{P}_{I_i^{(n)}}\widetilde{T}_{|I_i^{(n)}|}B(\Gamma - R_n)(b_i h_i)\|_{L_1(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))} \\ &\quad\le \|\psi\|_{\mathsf{BMO}_r} \ \sqrt{|I_i| + 2/n} \ \|(\Gamma - R_n)(b_i h_i)\|_{L_1(\mathcal{M};L_2^c(\mathbb{R},dt))} \\ &\quad\le \|\psi\|_{\mathsf{BMO}_r} \ \sqrt{|I_i| + 2/n} \ \|\phi_n * b_i - b_i\|_{L_2(\mathbb{R};L_2(\mathcal{M}))}.\end{aligned}$$

Thus, it only remains to justify that dominated convergence theorem applies so that

$$\lim_{n\to\infty}\sum_i |\lambda_i| \ |\langle \psi, A^J(\Gamma - R_n)(b_i h_i)\rangle| = \sum_i |\lambda_i| \ \lim_{n\to\infty} |\langle \psi, A^J(\Gamma - R_n)(b_i h_i)\rangle|,$$

and

$$\begin{aligned}&\lim_{n\to\infty}\sum_i |\lambda_i| \ |\langle \psi, A^J(\Gamma - R_n)(b_i h_i)\rangle| \\ &\qquad\qquad \le \sum_i |\lambda_i| \|\psi\|_{\mathsf{BMO}_r} \ \lim_{n\to\infty} \sqrt{|I_i| + 2/n} \ \|\phi_n * b_i - b_i\|_{L_2(\mathbb{R};L_2(\mathcal{M}))} = 0\end{aligned}$$

by Lemma 4.2.16. Nevertheless, the estimate

$$\|\psi\|_{\mathsf{BMO}_r}\sqrt{|I_i| + 2/n} \ \|\phi_n * b_i - b_i\|_{L_2(\mathbb{R};L_2(\mathcal{M}))} \le \|\psi\|_{\mathsf{BMO}_r}\frac{\sqrt{|I_i| + 2/n}}{\sqrt{|I_i|}}$$

does not provide a satisfactory bound since the series

$$\sum_i |\lambda_i| \, \|\psi\|_{\mathsf{BMO}_r} \frac{\sqrt{|I_i| + 2/n}}{\sqrt{|I_i|}}$$

is not convergent in general. Therefore, a finer estimate is needed.

Fix $n \geq 1$. Whenever $|I_i| \geq 2/n$, there follows

$$|\langle \psi, A^J(\Gamma - R_n)(b_i h_i)\rangle| \leq \|\psi\|_{\mathsf{BMO}_r} \frac{\sqrt{|I_i| + 2/n}}{|I_i|} \leq \sqrt{2}\|\psi\|_{\mathsf{BMO}_r}. \tag{4.7}$$

Otherwise, consider the family of intervals $\{K^i_j\}_j$ defined as

$$K^i_j = (-1 + (j-1) n|I_i|, -1 + j \; n|I_i|) \text{ for } j = 1, \ldots, \lfloor 2/(n|I_i|) \rfloor,$$

and $K^i_j = (-1,1) \setminus \bigcup_{\ell=1}^{\lfloor 2/(n|I_i|)\rfloor} K^i_\ell$ for $j = \lfloor 2/(nI_i) \rfloor + 1$. In that case, recalling that

$$\phi_n * b_i - b_i = \sum_j (\phi \chi_{K^i_j})_n * b_i - \int (\phi \chi_{K_j})_n(y) \, dy \cdot b_i,$$

and setting $I_i^{(n,j)} = I + \frac{1}{n} K^i_j$ and $b_i^{(n,j)} = (\phi\chi_{K^i_j})_n * b_i - \int (\phi\chi_{K_j})_n(y)\,dy \cdot b_i$, it follows that

$$\begin{aligned}
|\langle \psi, A^J(\Gamma - R_m) b_i h_i\rangle| &\leq \sum_j |\langle (A^J)^* \psi, \Gamma(b_i^{(n,j)} h_i)\rangle| \\
&\leq \sum_j |\langle \widetilde{T}_{|I_i^{(n,j)}|^{-1}} \widetilde{P}_{I_i^{(n,j)}} \widetilde{T}_\omega [\psi - a_{I_i^{(n,j)}} \psi], \widetilde{T}_{|I_i^{(n,j)}|} \widetilde{P}_{I_i^{(n,j)}} B\Gamma(b_i^{(n,j)} h_i)\rangle| \\
&\leq \sum_j \|\psi\|_{\mathsf{BMO}_r} \sqrt{|I_i^{(n,j)}|} \, \|b_i^{(n,j)}\|_{L_2(\mathbb{R}; L_2(\mathcal{M}))}.
\end{aligned}$$

Therefore, point (3) from Lemma 4.2.16 implies that

$$\begin{aligned}
|\langle \psi, A^J(\Gamma - R_m) b_i h_i\rangle| &\leq 2\|\psi\|_{\mathsf{BMO}_r} \sum_j \frac{\sqrt{|I_i| + |K^i_j|/n}}{\sqrt{|I_i|}} \\
&\leq 2\|\psi\|_{\mathsf{BMO}_r} \frac{\sqrt{2|I_i|}}{\sqrt{|I_i|}} \sum_j \int_{K^i_j} \phi = 2\|\psi\|_{\mathsf{BMO}_r} \sqrt{2}.
\end{aligned}$$

In conclusion, this estimate along with (4.7) implies that

$$\sum_i |\lambda_i| \, |\langle \psi, A^J(\Gamma - R_n)(b_i h_i)\rangle| \leq 2\sqrt{2}\|\psi\|_{\mathsf{BMO}_r},$$

so the application of the dominated convergence theorem applies, yielding the statement.
□

In order to conclude the argument, it remains to show that any operator $\varphi \in \mathsf{BMO}_r(\mathbb{R},\mathcal{M})$ induces a well-defined functional on $H_1^c(\mathcal{A})$. The estimates obtained for compactly supported $\psi \in \mathsf{BMO}_r(\mathbb{R},\mathcal{M})$ will yield a general result under a suitable approximation argument.

Lemma 4.2.18. *Suppose $\varphi \in \mathsf{BMO}_c(\mathbb{R},\mathcal{M})$ and suppose J is an interval such that $\varphi_J = 0$. Let $3J$ be the interval concentric with J having length $3|J|$. Then there exists $\psi \in \mathsf{BMO}_c(\mathbb{R},\mathcal{M})$ such that*

- $\widetilde{P}_{3J}\psi = \psi$,
- $\widetilde{P}_J(\psi - \varphi) = 0$,
- *there exists some universal constant $C > 0$ such that*

$$\|\psi\|_{\mathsf{BMO}_c} \leq C\ \|\varphi\|_{\mathsf{BMO}_c}$$

Theorem 4.2.19. *Given a semifinite von Neumann algebra $\mathcal{M}$, there holds*

$$\mathsf{BMO}_r(\mathbb{R},\mathcal{M}) \subseteq H_1^c(\mathcal{A})^*.$$

Proof. Let $\varphi \in \mathsf{BMO}_r(\mathbb{R},\mathcal{M})$ and let $f \in H_1^c(\mathcal{A})$ admit an atomic decomposition $\sum_i \lambda_i b_i h_i$ which is equal to zero in $L_1(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})$. Given $\varepsilon > 0$, let $N \geq 1$ such that $\sum_{i>N} |\lambda_i| < \varepsilon$ and let J be a finite interval satisfying

$$\mathrm{supp}_{\mathbb{R}}\Big(\sum_{i=1}^N \lambda_i a_i\Big) \subseteq J.$$

Without loss of generality, we can assume that $\varphi_J = 0$, so by Lemma 4.2.18, there exists some $\psi \in \mathsf{BMO}_r(\mathbb{R},\mathcal{M})$ satisfying $\widetilde{P}_{3J}\psi = \psi$, $\widetilde{P}_J(\varphi - \psi) = 0$ and

$$\|\psi\|_{\mathsf{BMO}_r} \leq C\|\varphi\|_{\mathsf{BMO}_r}$$

for some universal constant $C > 0$. Therefore, as a consequence of Proposition 4.2.17,

$$\begin{aligned}
\sum_i \lambda_i \langle \varphi, A^{I_i}\Gamma(b_i h_i)\rangle &= \sum_i \lambda_i \langle \varphi, A^{I_i}\Gamma(b_i h_i)\rangle - \sum_i \langle \psi, A^{3J}\Gamma(b_i h_i)\rangle \\
&= \sum_{i=1}^N \lambda_i\ \langle \varphi - \psi, A^{I_i}\Gamma(b_i h_i)\rangle \\
&+ \sum_{i=N+1}^\infty \lambda_i\ (\langle \varphi, A^{I_i}\Gamma(b_i h_i)\rangle - \langle \psi, A^{3J}\Gamma(b_i h_i)\rangle) \\
&= \sum_{i=N+1}^\infty \lambda_i\ (\langle \varphi, A^{I_i}\Gamma(b_i h_i)\rangle - \langle \psi, A^{3J}\Gamma(b_i h_i)\rangle),
\end{aligned}$$

so

$$|\sum_i \lambda_i \langle \varphi, A^{I_i}\Gamma(b_i h_i)\rangle| \le \sum_{i=N+1}^{\infty} |\lambda_i|(1+C)\|\varphi\|_{\mathsf{BMO}_r} < (1+C)\|\varphi\|_{\mathsf{BMO}_r}\varepsilon.$$

In conclusion, the statement follows since ε is arbitrarily small. □

This last theorem completes the proof of the identity

$$(H_1^c(\mathcal{A}))^* \simeq \mathsf{BMO}_r(\mathbb{R},\mathcal{M}).$$

Therefore, a new description for a predual of $\mathsf{BMO}_r(\mathbb{R},\mathcal{M})$ has been obtained just in terms of a new atomic decomposition, which will be crucial for the study of the boundedness of Calderón-Zygmund operators from $H_1^c(\mathcal{A})$ to $L_1(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})$.

4.3 Proof of a result by Garnett

The justification for Lemma 4.2.18 is included in this section. This result was stated in the commutative setting by Garnett [21] and by Mei in the semicommutative one too, both without including a complete proof [54]. For that reason, a general version of the argument is developed here. Before giving the explicit construction, some preliminar results are showed.

Lemma 4.3.1. *Suppose $g \in L_\infty(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dt}{1+t^2}))$ and let A be a measurable set such that $|A| \neq 0$. Then, there holds*

$$\begin{aligned}\|\sqrt{\pi}(\mathrm{Id}\otimes \mathrm{Tr}_{B(L_2(\mathbb{R},\frac{dt}{1+t^2}))})(\widetilde{T}_\omega \widetilde{P}_A \widetilde{T}_{|A|^{-1}} g)\otimes (1\otimes \mathbb{1})\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))} \\ \le \|g\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}.\end{aligned}$$

Proof. Given $g \in L_2(\mathbb{R}, \frac{dt}{1+t^2})$, there holds

$$\begin{aligned}\int g\chi_A \frac{\sqrt{1+t^2}}{\sqrt{|A|}}\frac{dt}{1+t^2}\ {}_{L_2(\mathbb{R},\frac{dt}{1+t^2})} &= \int g\frac{1}{\sqrt{1+t^2}}\frac{\chi_A}{\sqrt{|A|}}\,dt \\ &\le \Big(\int |g|^2\frac{dt}{1+t^2}\Big)^{1/2}\Big(\int \frac{\chi_A}{|A|}\Big)^{1/2} = \|g\|_{L_2(\mathbb{R},\frac{dt}{1+t^2})}.\end{aligned}$$

Then, the statement follows by Corollary 4.1.3. □

Lemma 4.3.2. *Let $\varphi \in \mathsf{BMO}_c(\mathbb{R},\mathcal{M})$ and let Q and Q' be two bounded intervals such that $|Q| \sim |Q'|$ with distance $d(Q,Q') \lesssim |Q|$. Then, there holds*

$$\|a_Q\varphi - a_{Q'}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}$$

$$\leq \Big(\frac{|R|^{1/2}}{|Q|^{1/2}} + \frac{|R|^{1/2}}{|Q'|^{1/2}}\Big) \, \|\varphi\|_{\mathsf{BMO}_c}$$

where R is the smallest interval concentric to Q containing Q' such that $|R| \sim |Q|$.

Proof. Let R denote an interval as in the statement. Then, it follows that

$$\begin{aligned}\|a_Q\varphi - a_{Q'}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))} &\leq \|a_Q\varphi - a_R\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))} \\ &+ \|a_Q\varphi - a_R\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}.\end{aligned}$$

As a consequence of Lemma 4.3.1, there holds

$$\begin{aligned}\|a_Q\varphi - a_R\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))} &= \|a_Q(\varphi - a_R\varphi)\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))} \\ &\leq \|\widetilde{T}_\omega \widetilde{T}_{|Q|^{-1}} \widetilde{P}_Q(\varphi - a_R\varphi)\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))} \\ &= \frac{|R|^{1/2}}{|Q|^{1/2}} \, \|\widetilde{T}_\omega \widetilde{T}_{|R|^{-1}} \widetilde{P}_R(\varphi - a_R\varphi)\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))} \\ &\leq \frac{|R|^{1/2}}{|Q|^{1/2}} \|\varphi\|_{\mathsf{BMO}_c},\end{aligned}$$

as well as an analogous estimate for $\|a_{Q'}\varphi - a_R\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}$, yielding the claim from the statement. □

The previous estimate is the key tool for proving Lemma 4.2.18. Indeed, it encodes the *almost-characterization* of elements from BMO spaces: even when $f \in \mathsf{BMO}$ is not bounded, the differences between averages can be controlled somehow by the norm of f.

Lemma 4.3.3. *Let A, B and C be some measurable sets such that $A \subseteq B$ and $|B|, |C| \neq 0$. Then, it holds*

$$\frac{|A|}{|B|} \, a_C g = a_B \widetilde{P}_A a_C g$$

for any $g \in L_\infty(\mathcal{M}; L_2^c(\mathbb{R}, \frac{dt}{1+t^2}))$.

Proof. This is a straightforward verification. □

Let $J = (a, b)$ be a finite interval and let $c_J = \frac{a+b}{2}$ denote the center of J. Write

$$J = J_0 \cup \bigcup_{n=1}^{\infty} J_n \cup \bigcup_{n=1}^{\infty} J'_n$$

where $d(J_n, \partial J) = |J_n|$ for $n \geq 0$ and $d(J'_n, \partial J) = |J'_n|$ for $n \geq 1$. Then J_0 coincides with the middle third of J, that is,

$$J_0 = \frac{1}{3}J = \Big(c_J - \frac{1}{2}\frac{1}{3}|J|, c_J + \frac{1}{2}\frac{1}{3}|J|\Big),$$

while for any $n \geq 1$,

$$J_n = \Big(c_J + \frac{|J|}{3}\sum_{k=0}^{n-1}\frac{1}{2^k}, c_J + \frac{|J|}{3}\sum_{k=0}^{n}\frac{1}{2^k}\Big)$$
$$J_n' = \Big(c_J - \frac{|J|}{3}\sum_{k=0}^{n}\frac{1}{2^k}, c_J - \frac{|J|}{3}\sum_{k=0}^{n-1}\frac{1}{2^k}\Big).$$

Thus, $|J_n| = |J_n'| = \frac{|J|}{3}\frac{1}{2^n}$. Set K_n (respectively K_n') as the reflection of J_n (respectively J_n') across b (respectively a). Then,

$$K_n = \Big(b + \frac{|J|}{3} - \frac{|J|}{3}\sum_{k=1}^{n}\frac{1}{2^k}, b + \frac{|J|}{3} - \frac{|J|}{3}\sum_{k=1}^{n-1}\frac{1}{2^k}\Big),$$
$$K_n' = \Big(a - \frac{|J|}{3} + \frac{|J|}{3}\sum_{k=1}^{n-1}\frac{1}{2^k}, a - \frac{|J|}{3} + \frac{|J|}{3}\sum_{k=1}^{n}\frac{1}{2^k}\Big),$$

so $|K_n| = |K_n'| = |J_n| = |J_n'|$. Finally, define $L = (b + \frac{|J|}{3}, \infty)$ and $L' = (-\infty, a - \frac{|J|}{3})$. Assuming $\varphi \in \mathsf{BMO}_c$ and $\varphi_J = 0$, this construction yields the desired operator ψ, which is given by the following expression:

$$\begin{aligned}\psi &= \widetilde{P}_J\varphi + \sum_{n\geq 1}\varphi_{J_n}\otimes(\chi_{K_n}\otimes \mathbb{1}) + \sum_{n\geq 1}\varphi_{J_n'}\otimes(\chi_{K_n'}\otimes\mathbb{1})\\ &= \widetilde{P}_J\varphi + \sum_{n\geq 1}\widetilde{P}_{K_n}a_{J_n}\varphi + \sum_{n\geq 1}\widetilde{P}_{K_n'}a_{J_n'}\varphi.\end{aligned}$$

Therefore, the average of ψ on some finite interval $I \subseteq \mathbb{R}$ coincides with

$$\begin{aligned}a_I\psi &= a_I\widetilde{P}_{I\cap J}\varphi + \sum_{n\geq 1}\frac{|K_n\cap I|}{|I|}a_{J_n}\varphi + \sum_{n\geq 1}\frac{|K_n'\cap I|}{|I|}a_{J_n'}\varphi\\ &= a_I\widetilde{P}_{I\cap J}\varphi + \sum_{n\geq 1}\frac{|K_n\cap I|}{|I|}\varphi_{J_n}\otimes(1\otimes\mathbb{1}) + \sum_{n\geq 1}\frac{|K_n'\cap I|}{|I|}\varphi_{J_n'}\otimes(1\otimes\mathbb{1}).\end{aligned}$$

From the definition, it is clear that $\widetilde{P}_{3J}\psi = \psi$ and $\widetilde{P}_J(\varphi-\psi) = 0$. On the other hand, setting $K = \bigcup_{n\geq 1}K_n$ and $K' = \bigcup_{n\geq 1}K_n'$, it follows that

(4.8)

$$\begin{aligned}&\|\psi\|_{\mathsf{BMO}_c} = \sup_I \|R_I(\mathrm{Id} - a_I)\psi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\ &\leq \sup_I\Big[\|\widetilde{P}_K R_I(\mathrm{Id}-a_I)\psi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))} + \|\widetilde{P}_{K'}R_I(\mathrm{Id}-a_I)\psi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\ &+\|\widetilde{P}_J R_I(\mathrm{Id}-a_I)\psi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\end{aligned}$$

$$+ + \|\widetilde{P}_L R_I(\mathrm{Id} - a_I)\psi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))} + \|\widetilde{P}_{L'} R_I(\mathrm{Id} - a_I)\psi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\Big],$$

so it is sufficient to bound each one of these summands when I is fixed. Although some of them may become zero (for instance, the last norm is zero whenever $I \cap L = \emptyset$), every case reduces to bound these terms when they do not vanish. The structure of the decomposition and Lemma 4.3.1 ensure that

- $\|a_{J_n}\varphi - a_{J_{n+1}}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))} \leq \big(\frac{2|J_{n+1}|}{|J_{n+1}|} + \frac{2|J_{n+1}|}{|J_n|}\big) \|\varphi\|_{\mathsf{BMO}_c} \leq 6 \|\varphi\|_{\mathsf{BMO}_c}$,
- $\|a_{J_{n_0}}\varphi - a_{\widetilde{K}}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))} \leq \big(\frac{|\widetilde{K}|}{|J_{n_0}|} + 1\big) \|\varphi\|_{\mathsf{BMO}_c} = (1 + \sum_{j \geq n_0} 2^{-j})\|\varphi\|_{\mathsf{BMO}_c}$ where $\widetilde{K} = \cup_{n=1}^{\infty} J_n$,
- $\|a_{J_1}\varphi - a_J\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))} \leq \big(1 + \frac{|J|}{|J_1|}\big) \|\varphi\|_{\mathsf{BMO}_c} \leq 10 \|\varphi\|_{\mathsf{BMO}_c}$.

These and finer estimates will be used along the proof of the lemma.

Given $A \subseteq J$ or $A \subseteq K, K'$, define $\widetilde{A}$ as the reflection across the closer point of the border ∂J to A. Then,

$$\begin{aligned}
&\|\widetilde{P}_{K\cap I} R_I(\mathrm{Id} - a_I)\psi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))} \\
&\quad \leq \|\widetilde{P}_{K\cap I}\widetilde{T}_\omega \widetilde{T}_{|I|^{-1}}(\psi - a_{\widetilde{K\cap I}}\psi)\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))} \\
&\quad + \|\widetilde{P}_{K\cap I}\widetilde{T}_\omega \widetilde{T}_{|I|^{-1}}(a_I \widetilde{P}_{J\cap I}\psi - \frac{|J\cap I|}{|I|} a_{\widetilde{K\cap I}}\varphi)\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))} \\
&\quad + \|\widetilde{P}_{K\cap I}\widetilde{T}_\omega \widetilde{T}_{|I|^{-1}}\Big(\sum_n \frac{|K_n\cap I|}{|I|} a_{J_n}\varphi - \frac{|K\cap I|}{|I|} a_{\widetilde{K\cap I}}\varphi\Big)\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))} \\
&\quad + \|\widetilde{P}_{K\cap I}\widetilde{T}_\omega \widetilde{T}_{|I|^{-1}}\Big(\sum_n \frac{|K_n'\cap I|}{|I|} a_{J_n}\varphi - \frac{|K'\cap I|}{|I|} a_{\widetilde{K\cap I}}\varphi\Big)\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))} \\
&\quad + \|\widetilde{P}_{K\cap I}\widetilde{T}_\omega \widetilde{T}_{|I|^{-1}}\Big(\frac{|L\cap I|}{|I|} a_{\widetilde{K\cap I}}\varphi\Big)\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))} \\
&\quad + \|\widetilde{P}_{K\cap I}\widetilde{T}_\omega \widetilde{T}_{|I|^{-1}}\Big(\frac{|L'\cap I|}{|I|} a_{\widetilde{K\cap I}}\varphi\Big)\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))} \\
&\quad = A_1 + A_2 + A_3 + A_4 + A_5 + A_6.
\end{aligned}$$

Now, we estimate each of these terms. First, assuming that n_0 and N_0 respectively denote the smallest and largest natural number n such that K_n intersects I, there holds

$$\begin{aligned}
A_1 &\leq \ \widetilde{P}_{K\cap I}\widetilde{T}_\omega \widetilde{T}_{|I|^{-1}}\big(\sum_n \widetilde{P}_{K_n\cap I}(a_{J_n}\varphi - a_{\widetilde{K\cap I}}\varphi)\big)\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))} \\
&\leq \frac{|K\cap I|^{1/2}}{|I|^{1/2}} \sum_n \frac{|K_n\cap I|}{|K\cap I|} \|\widetilde{P}_{K_n\cap I}\widetilde{T}_\omega \widetilde{T}_{|I\cap K|^{-1}}(a_{J_n}\varphi - a_{\widetilde{K\cap I}}\varphi)\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))} \\
&\leq \frac{|K\cap I|^{1/2}}{|I|^{1/2}} \sum_n \frac{|K_n\cap I|}{|K\cap I|} \|\widetilde{P}_{K_n\cap I}\widetilde{T}_\omega \widetilde{T}_{|I\cap K|^{-1}}(\sum_{k=n_0}^{n-1} a_{J_{k+1}}\varphi - a_{J_k}\varphi)\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{|K\cap I|^{1/2}}{|I|^{1/2}}\sum_n \frac{|K_n\cap I|}{|K\cap I|}\|\widetilde{P}_{K_n\cap I}\widetilde{T}_\omega\widetilde{T}_{|I\cap K|^{-1}}(a_{J_{n_0}}\varphi-a_{\widetilde{K\cap I}}\varphi)\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&\le \frac{1}{\sqrt{\pi}}\frac{|K\cap I|^{1/2}}{|I|^{1/2}}\sum_n \frac{|K_n\cap I|}{|K\cap I|}(n-n_0)\cdot 3\cdot\sum_{k=n_0}^{N_0-1}\|a_{J_{k+1}}\varphi-a_{J_k}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&+\frac{1}{\sqrt{\pi}}\frac{|K\cap I|^{1/2}}{|I|^{1/2}}\sum_n \frac{|K_n\cap I|}{|K\cap I|}\Big(1+\frac{|J_{n_0}|\sum_j 2^{-j}}{|J_{n_0}|}\Big)\|a_{J_{n_0}}\varphi-a_{\widetilde{K\cap I}}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&\lesssim \frac{1}{\sqrt{\pi}}\frac{|K\cap I|^{1/2}}{|I|^{1/2}}\sum_n \frac{|K_n\cap I|}{|K\cap I|}[(n-n_0)+1]\,\|\varphi\|_{\mathsf{BMO}_c}\lesssim \frac{|K\cap I|^{1/2}}{|I|^{1/2}}\,\|\varphi\|_{\mathsf{BMO}_c}.
\end{aligned}
$$

On the other hand, Lemma 4.3.3 implies that

$$
\begin{aligned}
A_2 &= \frac{|K\cap I|^{1/2}}{|I|^{1/2}}\frac{1}{\sqrt{\pi}}\,\|a_I\widetilde{P}_{J\cap I}\psi-\frac{|J\cap I|}{|I|}a_{\widetilde{K\cap I}}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&= \frac{|K\cap I|^{1/2}}{|I|^{1/2}}\frac{1}{\sqrt{\pi}}\,\|a_I\widetilde{P}_{J\cap I}\psi-a_I\widetilde{P}_{J\cap I}a_{\widetilde{K\cap I}}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&\le \frac{|K\cap I|^{1/2}}{|I|^{1/2}}\|\widetilde{T}_\omega\widetilde{P}_{J\cap I}\widetilde{T}_{|I|^{-1}}(\varphi-a_{\widetilde{K\cap I}}\varphi)\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&\le \frac{|K\cap I|^{1/2}|J\cap I|^{1/2}}{|I|}\|\widetilde{T}_\omega\widetilde{P}_{J\cap I}\widetilde{T}_{|J\cap I|^{-1}}(\varphi-a_{\widetilde{K\cap I}}\varphi)\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&\le \frac{|K\cap I|^{1/2}|J\cap I|^{1/2}}{|I|}\,\|\widetilde{T}_\omega\widetilde{P}_{J\cap I}\widetilde{T}_{|J\cap I|^{-1}}(\varphi-a_{J\cap I}\varphi)\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&+\frac{|K\cap I|^{1/2}|J\cap I|^{1/2}}{|I|}\,\|a_{J\cap I}\varphi-a_{\widetilde{K\cap J}}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&\le \frac{|K\cap I|^{1/2}|J\cap I|^{1/2}}{|I|}\|\varphi\|_{\mathsf{BMO}_c}[1+\max\{\frac{|K\cap I|^{1/2}}{|J\cap I|^{1/2}},\frac{|J\cap I|^{1/2}}{|K\cap I|^{1/2}}\}],
\end{aligned}
$$

$$
\begin{aligned}
A_3 &\le \frac{|K\cap I|^{1/2}}{|I|^{1/2}}\sum_n \frac{|K_n\cap I|}{|I|}\|a_{J_n}\varphi-a_{\widetilde{K\cap I}}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&\le \frac{|K\cap I|^{3/2}}{|I|^{3/2}}\sum_n \frac{|K_n\cap I|}{|K\cap I|}[(n-n_0)+1]\,\|\varphi\|_{\mathsf{BMO}_c}\lesssim \frac{|K\cap I|^{3/2}}{|I|^{3/2}}\,\|\varphi\|_{\mathsf{BMO}_c}.
\end{aligned}
$$

Whenever A_4 appears in the above estimate, there holds $J\cap I=J$, so

$$
\begin{aligned}
A_4 &\le \frac{|K\cap I|^{1/2}|K'\cap I|}{|I|^{3/2}}\sum_n \frac{|K_n'\cap I|}{|K'\cap I|}\|a_{J_n'}\varphi-a_{\widetilde{K\cap J}}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&\le \frac{|K\cap I|^{1/2}|K'\cap I|}{|I|^{3/2}}\sum_n \frac{|K_n'\cap I|}{|K'\cap I|}\|a_{J_n'}\varphi-a_{\widetilde{K'\cap I}}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}
\end{aligned}
$$

$$+\frac{|K\cap I|^{1/2}|K'\cap I|}{|I|^{3/2}}\sum_n \frac{|K'_n\cap I|}{|K'\cap I|}\|a_{\widetilde{K'\cap I}}\varphi - a_J\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}$$
$$+\frac{|K\cap I|^{1/2}|K'\cap I|}{|I|^{3/2}}\sum_n \frac{|K'_n\cap I|}{|K'\cap I|}\|a_J\varphi - a_{\widetilde{K\cap I}}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}$$
$$\lesssim \frac{|K\cap I|^{1/2}|K'\cap I|}{|I|^{3/2}}\Big[1+\frac{|J|^{1/2}}{|K'\cap I|^{1/2}}+\frac{|J|^{1/2}}{|K\cap I|^{1/2}}\Big]\|\varphi\|_{\mathsf{BMO}_c}.$$

Finally,

$$\begin{aligned}
A_5 &\le \frac{|K\cap I|^{1/2}|L\cap I|}{|I|^{3/2}}\|a_{\widetilde{K\cap I}}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&\le \frac{|K\cap I|^{1/2}|L\cap I|}{|I|^{3/2}}\ [\|a_{\widetilde{K\cap I}}-a_{J_1}\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&+\|a_{J_1}\varphi - a_{\widetilde{K\cap I}}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}]\\
&\le \frac{|K\cap I|^{1/2}|L\cap I|}{|I|^{3/2}}\ [1+\frac{|J_1|\cdot 2}{|J_1|}+1+9]\,\|\varphi\|_{\mathsf{BMO}_c}\\
&\lesssim \frac{|K\cap I|^{1/2}|L\cap I|}{|I|^{3/2}}\ \|\varphi\|_{\mathsf{BMO}_c},
\end{aligned}$$

and

$$\begin{aligned}
A_6 &\le \frac{|K\cap I|^{1/2}|L'\cap I|}{|I|^{3/2}}\ \|a_{\widetilde{K\cap I}}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&= \frac{|K\cap I|^{1/2}|L'\cap I|}{|I|^{3/2}}\|a_{\widetilde{K\cap I}}\varphi - a_J\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&\lesssim \frac{|K\cap I|^{1/2}|L'\cap I|}{|I|^{3/2}}\ \|\varphi\|_{\mathsf{BMO}_c}+\frac{|J|^{1/2}|L'\cap I|}{|I|^{3/2}}\ \|\varphi\|_{\mathsf{BMO}_c}.
\end{aligned}$$

Analogous estimates can be computed for the norm

$$\|\widetilde{P}_{K'\cap I}R_I(\mathrm{Id}-a_I)\psi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\le A'_1+\ldots+A'_6$$

by interchanging K and K' as well as L and L'. Moreover,

$$\begin{aligned}
&\|\widetilde{P}_JR_I(\mathrm{Id}-a_I)\psi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&\le \|\widetilde{P}_{J\cap I}\widetilde{T}_\omega\widetilde{T}_{|I|^{-1}}(\psi - a_{J\cap I}\psi)\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&+\|\widetilde{P}_{J\cap I}\widetilde{T}_\omega\widetilde{T}_{|I|^{-1}}(a_I\widetilde{P}_{J\cap I}\psi-\frac{|J\cap I|}{|I|}a_{J\cap I}\varphi)\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&+\|\widetilde{P}_{J\cap I}\widetilde{T}_\omega\widetilde{T}_{|I|^{-1}}\Big(\sum_n\frac{|K_n\cap I|}{|I|}a_{J_n}\varphi-\frac{|K\cap I|}{|I|}a_{J\cap I}\varphi\Big)\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}
\end{aligned}$$

$$
\begin{aligned}
&+\|\widetilde{P}_{J\cap I}\widetilde{T}_\omega\widetilde{T}_{|I|^{-1}}\Big(\sum_n \frac{|K_n'\cap I|}{|I|}a_{J_n}\varphi - \frac{|K\cap I|}{|I|}a_{J\cap I}\varphi\Big)\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&+\|\widetilde{P}_{J\cap I}\widetilde{T}_\omega\widetilde{T}_{|I|^{-1}}\Big(\frac{|L\cap I|}{|I|}a_{J\cap I}\varphi\Big)\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&+\|\widetilde{P}_{J\cap I}\widetilde{T}_\omega\widetilde{T}_{|I|^{-1}}\Big(\frac{|L'\cap I|}{|I|}a_{J\cap I}\varphi\Big)\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&= B_1 + B_2 + B_3 + B_4 + B_5 + B_6.
\end{aligned}
$$

Then, it is clear that $B_1 \leq \|\varphi\|_{\mathsf{BMO}_c}$ follows from the identity $\widetilde{P}_{I\cap J}(\varphi-\psi)=0$. On the other hand,

$$
\begin{aligned}
B_2 &= \frac{1}{\sqrt{\pi}}\,\frac{|J\cap I|^{1/2}}{|I|^{1/2}}\,\|a_I\widetilde{P}_{J\cap I}\varphi - a_I\widetilde{P}_{J\cap I}a_{J\cap I}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&\leq \frac{|J\cap I|}{|I|}\,\|\widetilde{P}_{J\cap I}\widetilde{T}_\omega\widetilde{T}_{|J\cap I|^{-1}}(\varphi - a_{J\cap I}\varphi)\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&\leq \frac{|J\cap I|}{|I|}\,\|\varphi\|_{\mathsf{BMO}_c}.
\end{aligned}
$$

$$
\begin{aligned}
B_3 &\leq \frac{|I\cap J|^{1/2}}{|I|^{1/2}}\sum_n \frac{|K_n\cap I|}{|I|}\|a_{J_n'}\varphi - a_{J\cap I}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&= \frac{|I\cap J|^{1/2}|K\cap I|}{|I|^{3/2}}\sum_n \frac{|K_n\cap I|}{|K\cap I|}\|a_{J_n}\varphi - a_{J\cap I}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&\leq \frac{|I\cap J|^{1/2}}{|I|^{1/2}}\sum_n \frac{|K_n\cap I|}{|I|}\|a_{J_n'}\varphi - a_{J\cap I}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&\leq \frac{|I\cap J|^{1/2}|K\cap I|}{|I|^{3/2}}\sum_n \frac{|K_n\cap I|}{|K\cap I|}\|a_{J_n}\varphi - a_{\widetilde{K\cap J}}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&+ \frac{|I\cap J|^{1/2}|K\cap I|}{|I|^{3/2}}\sum_n \frac{|K_n\cap I|}{|K\cap I|} + \|a_{\widetilde{K\cap J}}\varphi - a_{J\cap I}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&\leq \frac{|I\cap J|^{1/2}|K\cap I|}{|I|^{3/2}}[1+\max\{\frac{|K\cap I|^{1/2}}{|I\cap J|^{1/2}},\frac{|J\cap I|^{1/2}}{|K\cap I|^{1/2}}\}],
\end{aligned}
$$

$$
\begin{aligned}
B_4 &\leq \frac{|J\cap I|^{1/2}|K'\cap I|}{|I|^{3/2}}\sum_n \frac{|K_n'\cap I|}{|K'\cap I|}\,\|a_{J_n'}\varphi - a_{J\cap I}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&\leq \frac{|J\cap I|^{1/2}|K'\cap I|}{|I|^{3/2}}\sum_n \frac{|K_n'\cap I|}{|K'\cap I|}\,\|a_{J_n'}\varphi - a_{\widetilde{K'\cap I}}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&+ \frac{|J\cap I|^{1/2}|K'\cap I|}{|I|^{3/2}}\sum_n \frac{|K_n'\cap I|}{|K'\cap I|}\,\|a_{J\cap I}\varphi - a_{\widetilde{K'\cap I}}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}
\end{aligned}
$$

$$\lesssim \frac{|J\cap I|^{1/2}|K'\cap I|}{|I|^{3/2}}[1+\max\{\frac{|K'\cap I|^{1/2}}{|J\cap I|^{1/2}},\frac{|J\cap I|^{1/2}}{|K'\cap I|^{1/2}}\}]\|\varphi\|_{\mathsf{BMO}_c},$$

$$\begin{aligned}B_5 &\le \frac{|I\cap J|^{1/2}|L\cap I|}{|I|^{3/2}}\|a_{J\cap I}\varphi - a_J\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\ &\le \frac{|I\cap J|^{1/2}|L\cap I|}{|I|^{3/2}}\|a_{J\cap I}\varphi - a_{\widetilde{K}}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\ &+\frac{|I\cap J|^{1/2}|L\cap I|}{|I|^{3/2}}\|a_{J_1}\varphi - a_{\widetilde{K}}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\ &+\frac{|I\cap J|^{1/2}|L\cap I|}{|I|^{3/2}}\|a_{J_1}\varphi - a_J\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\ &\lesssim \frac{|I\cap J|^{1/2}|L\cap I|}{|I|^{3/2}}(1+\frac{|K|^{1/2}}{|J\cap I|^{1/2}})\|\varphi\|_{\mathsf{BMO}_c}\\ &\le \frac{|I\cap J|^{1/2}|L\cap I|}{|I|^{3/2}}\|\varphi\|_{\mathsf{BMO}_c}+\frac{|K|^{1/2}|L\cap I|}{|I|^{3/2}}\ \|\varphi\|_{\mathsf{BMO}_c},\end{aligned}$$

$$\begin{aligned}B_6 &\le \frac{|J\cap I|^{1/2}|L'\cap I|}{|I|^{3/2}}\|a_{J\cap I}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\ &\lesssim \frac{|J\cap I|^{1/2}|L'\cap I|}{|I|^{3/2}}\|\varphi\|_{\mathsf{BMO}_c}+\frac{|K'|^{1/2}|L'\cap I|}{|I|^{3/2}}\ \|\varphi\|_{\mathsf{BMO}_c}.\end{aligned}$$

Now, it only remains to prove that $\|\widetilde{P}_{L\cap J}R_I(\mathrm{Id}-a_I)\psi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\lesssim\|\varphi\|_{\mathsf{BMO}_c}$ since, by symmetry, an analogous estimate follows for $\|\widetilde{P}_{L'\cap J}R_I(\mathrm{Id}-a_I)\psi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}$. Recall that

$$\begin{aligned}&\|\widetilde{P}_{L\cap I}R_I(\mathrm{Id}-a_I)\psi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\ &\quad\le \|\widetilde{P}_{L\cap I}\widetilde{T}_\omega\widetilde{T}_{|I|^{-1}}a_I\widetilde{P}_{I\cap J}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\ &\quad+\|\sum_{n\ge1}\frac{|K_n\cap I|}{|I|}\widetilde{P}_{L\cap I}\widetilde{T}_\omega\widetilde{T}_{|I|^{-1}}a_{J_n}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\ &\quad+\|\sum_{n\ge1}\frac{|K_n'\cap I|}{|I|}\widetilde{P}_{L\cap I}\widetilde{T}_\omega\widetilde{T}_{|I|^{-1}}a_{J_n'}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\ &\quad= C_1+C_2+C_3.\end{aligned}$$

Then, if $\widetilde{K}$ denotes the union $\bigcup_{n=1}^\infty J_n$, there holds

$$C_1\le\frac{|L\cap I|^{1/2}}{|I|^{1/2}}\|a_I\widetilde{P}_{I\cap J}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}$$

$$
\begin{aligned}
&\leq \frac{|L\cap I|^{1/2}}{|I|^{1/2}} \|\widetilde{P}_{I\cap J}\widetilde{T}_{\omega}\widetilde{T}_{|I|^{-1}}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&\leq \frac{|L\cap I|^{1/2}|I\cap J|^{1/2}}{|I|}\Big[\|\widetilde{P}_{I\cap J}\widetilde{T}_{\omega}\widetilde{T}_{|I\cap J|^{-1}}(\varphi - a_{I\cap J}\varphi)\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&+ \frac{|L\cap I|^{1/2}|I\cap J|^{1/2}}{|I|}\|\widetilde{P}_{I\cap J}\widetilde{T}_{\omega}\widetilde{T}_{|I\cap J|^{-1}}(a_{I\cap J}\varphi - a_{\widetilde{K}}\varphi)\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&+ \frac{|L\cap I|^{1/2}|I\cap J|^{1/2}}{|I|}\|\widetilde{P}_{I\cap J}\widetilde{T}_{\omega}\widetilde{T}_{|I\cap J|^{-1}}(a_{\widetilde{K}}\varphi - a_{J_1}\varphi)\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&+ \frac{|L\cap I|^{1/2}|I\cap J|^{1/2}}{|I|}\|\widetilde{P}_{I\cap J}\widetilde{T}_{\omega}\widetilde{T}_{|I\cap J|^{-1}}(a_{J_1}\varphi - a_{J}\varphi)\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\Big]\\
&\lesssim \frac{|L\cap I|^{1/2}|I\cap J|^{1/2}}{|I|}[1 + \frac{|K|^{1/2}}{|I\cap J|^{1/2}}] \|\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&\leq \Big[\frac{|L\cap I|^{1/2}|I\cap J|^{1/2}}{|I|} + \frac{|L\cap I|^{1/2}|K|^{1/2}}{|I|}\Big] \|\varphi\|_{\mathsf{BMO}_c}.
\end{aligned}
$$

On the other hand, it follows that

$$
\begin{aligned}
C_2 &\leq \frac{|L\cap I|^{1/2}}{|I|^{1/2}}\sum_n \frac{|K_n\cap I|}{|I|}\|a_{J_n}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&= \frac{|L\cap I|^{1/2}|K\cap I|}{|I|^{3/2}}\sum_n \frac{|K_n\cap I|}{|K\cap I|}\|a_{J_n}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&\leq \frac{|L\cap I|^{1/2}|K\cap I|}{|I|^{3/2}}\sum_n \frac{|K_n\cap I|}{|K\cap I|}\|a_{J_n}\varphi - a_{J_1}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&+ \frac{|L\cap I|^{1/2}|K\cap I|}{|I|^{3/2}}\sum_n \frac{|K_n\cap I|}{|K\cap I|}\|a_{J_1}\varphi - a_{J}\varphi\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R},\frac{dt}{1+t^2}))}\\
&\lesssim \frac{|L\cap I|^{1/2}|K\cap I|}{|I|^{3/2}} \|\varphi\|_{\mathsf{BMO}_c},
\end{aligned}
$$

and analogous computations show that

$$
C_3 \lesssim \frac{|L\cap I|^{1/2}|K'\cap I|}{|I|} \|\varphi\|_{\mathsf{BMO}_c}.
$$

From the expression (4.8) for the BMO_c norm of ψ and the computations above, it follows that

$$
\|\psi\|_{\mathsf{BMO}_c} \lesssim \|\varphi\|_{\mathsf{BMO}_c},
$$

so that the constant does not depends on J or φ. An analogous argument shows a row version of Lemma 4.2.18.

4.4 Calderón-Zygmund operators with operator-valued kernel

Let $\mathcal{M}$ be a von Neumann algebra over a separable Hilbert space. Through this section we establish the conditions under which a kernel K, defined outside the diagonal with values in $\mathcal{M}$, will induce a Calderón-Zygmund operator from $H_1^c(\mathcal{A})$ into $L_1(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})$. As one could expect, the resulting operator will be defined up to a left pointwise multiplication operator. Before giving a suitable definition, some facts about vector-valued functions and von Neumann algebras should be tackled (see [14, Chapter 2] and [77, Section 1.22]).

Let $\mathbb{X}$ be a Banach space, and let (Ω,μ) be a $\sigma-$finite measure space (or more generally, a localizable measure space [79]). Then, a function $f:\Omega\longrightarrow\mathbb{X}$ is said to be $\mu-$*measurable* whenever there exists a sequence of simple functions $(f_n)_{n\geq 1}$, $f_n=\sum_i x_i^n\ \chi_{A_i^n}$ for some $x_i^n\in\mathbb{X}$ and $\mu-$measurable sets A_i^n, such that

$$\lim_{n\to\infty}\|f_n-f\|_{\mathbb{X}}=0\ \mu-\text{almost everywhere}.$$

On the other hand, we will say that a function $f:\Omega\longrightarrow\mathbb{X}^*$ is *weak* μ-measurable* if $J_x\circ f$ is measurable for each $x\in\mathbb{X}$, where J_x denotes the continuous functional on $\mathbb{X}^*$ given by $J_x(x^*)=x^*(x)$ for every x^* in $\mathbb{X}^*$.

Define $L_\infty(\Omega,\mu;\mathcal{M})$ as the Banach space of all $\mathcal{M}-$valued weak* $\mu-$measurable functions which are essentially bounded, that is,

$$\operatorname{ess\,sup}_{t\in\Omega}\|f(t)\|_{\mathcal{M}}<\infty.$$

Then, $L_\infty(\Omega,\mu;\mathcal{M})$ is a von Neumann algebra under the pointwise multiplication. Moreover, the map

$$f\otimes m\longmapsto f(t)m,\quad f\in L_\infty(\Omega),\ m\in\mathcal{M},$$

can be extended to a isomorphism of $L_\infty(\Omega)\overline{\otimes}\mathcal{M}$ onto $L_\infty(\Omega,\mu;\mathcal{M})$.

Definition 4.4.1. *Assume that T is a bounded operator on $L_2(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})$. We say that T is a* Calderón-Zygmund operator *if there exists some function*

$$K:\mathbb{R}\times\mathbb{R}\ \setminus\ \{x=y\}\longrightarrow\mathcal{M}$$

such that for every pair of intervals I,J satisfying $d(I,J)>0$, there exist

- $K_{I,J}\in L_\infty(I\times J)\overline{\otimes}\mathcal{M}$,
- $\widehat{K}_{I,J}\in L_\infty(I\times J;\mathcal{M})$ *such that*

$$\widehat{K}_{I,J}(t)=K(t)\text{ for almost every }t\in I\times J,$$

and

$$\left(\tau \circ \int\right)(g\, Tf) = \langle f(y)g(x), K_{I,J}\rangle_{L_1(L_\infty(I\times J)\overline{\otimes}\mathcal{M}),L_\infty(I\times J)\overline{\otimes}\mathcal{M}}$$

for any $f, g \in (L_\infty \cap L_2)(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})$ *satisfying*

$$\mathrm{supp}_{\mathbb{R}}\|f(y)\|_{L_2(\mathcal{M})} \subseteq J \text{ and } \mathrm{supp}_{\mathbb{R}}\|g(x)\|_{L_2(\mathcal{M})} \subseteq I.$$

In addition, T *will be required to satisfy*

$$T(fh) = T(f)h$$

for any compactly supported $f \in L_2(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})$ *and* $h \in \mathcal{M}$.

When studying the boundedness of Calderón-Zygmund operators, the kernel K *is usually supposed to satisfy some smoothness conditions. In particular,* K *will be said to satisfy the* Hörmander condition *whenever*

$$\int_{|x-y|\geq 2|y'-y|} \|K(x,y) - K(x,y')\|_{\mathcal{M}}\, dx \leq C \tag{4.9}$$

for some constant $C > 0$.

Lemma 4.4.2. *Let* $\mathcal{M}$ *be a von Neumann algebra. Let* T *be a Calderón-Zygmund operator bounded on* $L_2(\mathbb{R}; L_2(\mathcal{M}))$ *with associated kernel*

$$K : \mathbb{R} \times \mathbb{R} \ \setminus \ \{x = y\} \longrightarrow \mathcal{M}.$$

If K *satisfies the condition*

$$\int_{|x-y|\geq \lambda|y'-y|} \|K(x,y) - K(x,y')\|_{\mathcal{M}}\, dx \leq C$$

for some constants $\lambda > 1$ *and* $C > 0$, *and*

$$T(a) = T(b)h \text{ for any } c-\text{atom},$$

then there holds

$$\|T(a)\|_{L_1(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})} \leq \max\{C, \lambda^{1/2}\|T\|\}.$$

Proof. Let $a = bh$ be a c−atom in $H_1^c(\mathcal{A})$ with support contained in the interval $I = B(y_0, d)$ and let $\lambda I = B(y_0, \lambda d)$. Then, the norm

$$\|T(a)\|_{L_1(\mathbb{R};L_1(\mathcal{M}))} = \int_{\lambda I} \|T(a)\|_{L_1(\mathcal{M})} + \int_{(\lambda I)^c} \|T(a)\|_{L_1(\mathcal{M})}$$

can be bounded in two steps. First, the continuity of T on $L_2(\mathbb{R}; L_2(\mathcal{M}))$ implies that

$$\begin{aligned}\int_{\lambda I} \|T(a)\|_{L_1(\mathcal{M})} &= \int_{\lambda I} \|T(b)h\|_{L_1(\mathcal{M})} \le \Big(\int_{\lambda I} \|T(b)\|^2_{L_2(\mathcal{M})}\Big)^{1/2} \Big(\int_{\lambda I} \|h\|^2_{L_2(\mathcal{M})}\Big)^{1/2} \\ &\le \|T(b)\|_{L_2(\mathbb{R};L_2(\mathcal{M}))} \ |\lambda I|^{1/2} \|h\|_{L_2(\mathcal{M})} \\ &\le \|T : L_2(\mathbb{R}; L_2(\mathcal{M})) \to L_2(\mathbb{R}; L_2(\mathcal{M}))\| \ \|b\|_{L_2(\mathbb{R};L_2(\mathcal{M}))} \ |\lambda I|^{1/2} \ \|h\|_{L_2(\mathcal{M})} \\ &\le \lambda^{1/2} \ \|T : L_2(\mathbb{R}; L_2(\mathcal{M})) \to L_2(\mathbb{R}; L_2(\mathcal{M}))\|.\end{aligned}$$

On the other hand, since I and $(\lambda I)^c$ are disjoint measurable sets and b has integral zero, it follows

$$\begin{aligned}\|T(a)\chi_{(\lambda I)^c}\|_{L_1(\mathbb{R};L_1(\mathcal{M}))} &= \|T(b)h\chi_{(\lambda I)^c}\|_{L_1(\mathbb{R};L_1(\mathcal{M}))} = \sup_{\substack{g\in L_\infty((\lambda I)^c)\overline{\otimes}\mathcal{M} \\ \|g\|_{L_\infty\overline{\otimes}\mathcal{M}}\le 1}} \tau \int T(b)hg \\ &= \sup_g \langle b(y)hg(x), K_{(\lambda I)^c,I}\rangle| = \sup_g \tau \int_{(\lambda I)^c} \int_I \widehat{K}_{(\lambda I)^c,I}(x,y)\, b(y)hg(x)\, dxdy \\ &= \sup_g \tau \int_{(\lambda I)^c} \int_I (\widehat{K}_{(\lambda I)^c,I}(x,y) - \widehat{K}_{(\lambda I)^c,I}(x,y_0))\, b(y)hg(x)\, dx\, dy \\ &= \Big\| \int_I (\widehat{K}_{(\lambda I)^c,I}(x,y) - \widehat{K}_{(\lambda I)^c,I}(x,y_0))\, b(y)h\, dy \Big\|_{L_1((\lambda I)^c;L_1(\mathcal{M}))} \\ &\le \int_{(\lambda I)^c} \int_I \|(\widehat{K}_{(\lambda I)^c,I}(x,y) - \widehat{K}_{(\lambda I)^c,I}(x,y_0))\, b(y)h\|_{L_1(\mathcal{M})}\, dx\, dy \\ &\le \int_{(\lambda I)^c} \int_I \|(\widehat{K}_{(\lambda I)^c,I}(x,y) - \widehat{K}_{(\lambda I)^c,I}(x,y_0))\|_{\mathcal{M}} \ \|b(y)\|_{L_2(\mathcal{M})} \ \|h\|_{L_2(\mathcal{M})}\, dx\, dy \\ &\le C \int_I \|b(y)\|_{L_2(\mathcal{M})} \ \|h\|_{L_2(\mathcal{M})}\, dy \le C\|b\|_{L_2(\mathbb{R};L_2(\mathcal{M}))} \ \|h\chi_I\|_{L_2(\mathbb{R};L_2(\mathcal{M}))} \le C.\end{aligned}$$

□

In order to extend a Calderón-Zygmund operator to the whole Hardy space $H_1^c(\mathcal{A})$, we will proceed as in [57]. A family of bounded kernels will be constructed, yielding a bounded family of Calderón-Zygmund operators which will approximate the original operator.

Lemma 4.4.3. *Let T be a Calderón-Zygmund operator whose associated kernel K satisfies the Hörmander condition (4.9). Then, there exists a sequence of kernels $(K_m)_{m\ge 1} \subseteq L_\infty(\mathbb{R}\times\mathbb{R})\overline{\otimes}\mathcal{M}$ such that*

$$\int_{|x-y|\ge 4|y-y'|} \|K_m(x,y) - K_m(x,y')\|_{\mathcal{M}}\, dx \le C, \tag{4.10}$$

so that the constant C is uniform in m. Moreover, there holds

$$\lim_{m\to\infty} \|T_m(f) - T(f)\|_{L_2(\mathbb{R};L_2(\mathcal{M}))} = 0$$

for every $f \in L_2(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})$.

Proof. Let ϕ be a non-negative smooth function supported on $B(0,1) = (-1,1)$ with integral $\int \phi = 1$ and satisfying $\phi(-t) = \phi(t)$, and let R_m be the operator defined on $L_2(\mathbb{R}; L_2(\mathcal{M}))$ to itself as $R_m(f) = f * \phi_m$ where $\phi_m(x) = m\phi(mx)$. Then, for any $f, g \in L_2(\mathbb{R}; L_2(\mathcal{M}))$ it follows that

$$\begin{aligned}
\langle R_m T R_m f, g\rangle &= \tau \int R_m T R_m f(x)\, g(x)\, dx = \tau \int \int T R_m f(t)\, \phi_m(x-t)\, dt\; g(x)\, dx \\
&= \tau \int \int T\Big(\int \phi_m(\cdot - y)\, f(y)\, dy \Big)(t)\, \phi_m(x-t)\, dt\; g(x)\, dx \\
&= \tau \int \int \int T\phi_m(\cdot - y)(t)\, f(y)\, dy\; \phi_m(x-t)\, dt\; g(x)\, dx,
\end{aligned}$$

where the last identity holds since the integral in y can be approximated by a particular chosen sequence of Riemann sums whenever ϕ_m and f belong to the Schwartz class $\mathcal{S}(\mathbb{R}; L_2(\mathcal{M}))$ (see [26, p. 200] and [26, Theorem 2.3.20]). Then, the argument follows for any $f \in L_2(\mathbb{R}; L_2(\mathcal{M}))$ by density. In conclusion,

$$\langle R_m T R_m f, g\rangle = \tau \int \int \langle T\tau_y \phi_m, \tau_x \phi_m\rangle\; f(y)\, g(x)\, dx\; dy,$$

where $\tau_x \phi(t) = \phi(t-x)$. Therefore,

$$K_m(x,y) = \langle T\tau_y \phi_m, \tau_x \phi_m\rangle$$

is the kernel for $R_m T R_m$ in the sense of Definition 4.4.1. Under this definition, $K_m \in L_\infty(\mathbb{R}\times\mathbb{R}) \overline{\otimes} \mathcal{M}$ for any $m \geq 1$. Indeed,

$$\begin{aligned}
\|K_m(x,y)\|_{\mathcal{M}} &= \sup_{\substack{g \in \mathcal{S}(\mathcal{M}) \\ \|g\|_{L_1(\mathcal{M})} \leq 1}} |\langle K_m(x,y), g\rangle| = \sup_g |\langle \int T(\tau_y \phi_m)\, \tau_x \phi_m, g\rangle| \\
&= \sup_g |(\int \circ \tau)\big(T(\tau_y \phi_m)\, \tau_x \phi_m g\big)|,
\end{aligned}$$

where the supremum can be taken over g belonging to $\mathcal{S}(\mathcal{M})$, the subspace of operators in $\mathcal{M}$ *supported by a τ-finite projection.* The operator g admits some partial isometry u such that $g = u|g|$, so it follows that

$$\begin{aligned}
\|K_m(x,y)\|_{\mathcal{M}} &= \sup_g |(\int \circ \tau)\big(T(\tau_y \phi_m) u |g|^{1/2}\; \tau_x \phi_m |g|^{1/2}\big)| \\
&= \sup_g |(\int \circ \tau)\big(T(\tau_y \phi_m u |g|^{1/2})\, \tau_x \phi_m |g|^{1/2}\big)| \\
&\leq \sup_g \|T(\tau_y \phi_m u |g|^{1/2})\|_{L_2(\mathbb{R}; L_2(\mathcal{M}))}\; \|\tau_x \phi_m |g|^{1/2}\|_{L_2(\mathbb{R}; L_2(\mathcal{M}))} \\
&\leq \sup_g \|\tau_y \phi_m u |g|^{1/2}\|_{L_2(\mathbb{R}; L_2(\mathcal{M}))}\; \|\tau_x \phi_m |g|^{1/2}\|_{L_2(\mathbb{R}; L_2(\mathcal{M}))} \\
&= \sup_g \|\tau_y \phi_m\|_{L_2(\mathbb{R})}\; \|g\|^{1/2}_{L_1(\mathcal{M})}\; \|\tau_x \phi\|_{L_2(\mathbb{R})}\; \|g\|^{1/2}_{L_1(\mathcal{M})}
\end{aligned}$$

$$\leq m.$$

On the other hand, the kernel K_m satisfies the Hörmander condition too. First, consider the case in which $|x-y|>4/m$. Then, there holds

$$\begin{aligned}\int_{|x-y|\geq 4|y'-y|} &\|K_m(x,y)-K_m(x,y')\|_{\mathcal{M}}\,dx\\ &= \int_{|x-y|\geq 4|y'-y|} \|\langle T(\tau_y\phi_m)-T(\tau_{y'}\phi_m),\tau_x\phi_m\rangle_{L_2(\mathbb{R})}\|_{\mathcal{M}}\,dx\\ &= \int_{|x-y|\geq 4|y'-y|}\sup_{\|g\|_{L_1(\mathcal{M})}\leq 1}\tau_{\mathcal{M}}\big(\langle T(\tau_y\phi_m)-T(\tau_{y'}\phi_m),\tau_x\phi_m\rangle_{L_2(\mathbb{R})}\,g\big)\,dx.\end{aligned}$$

The functions $\tau_y\phi_m-\tau_{y'}\phi_m$ and $\tau_x\phi_m$ have disjoint supports since, whenever $u,v\in B(0,1)$,

$$|x-\frac{u}{m}-(y-\frac{v}{m})|\geq |x-y|-\frac{2}{m}>\frac{2}{m}$$

and

$$|x-\frac{u}{m}-(y'-\frac{v}{m})|\geq |x-y'|-\frac{2}{m}\geq\frac{3}{4}|x-y|-\frac{2}{m}>\frac{|x-y|}{4}>0.$$

Let $I_{y,y'}$ and I_x denote some disjoint intervals containing the support of $\tau_y\phi_m-\tau_{y'}\phi_m$ and $\tau_x\phi_m$ respectively. Then, there holds

$$\begin{aligned}&\int_{|x-y|\geq 4|y'-y|}\|K_m(x,y)-K_m(x,y')\|_{\mathcal{M}}\,dx\\ &= \int_{|x-y|\geq 4|y'-y|}\sup_g|\langle(\tau_y\phi_m-\tau_{y'}\phi_m)\otimes\tau_x\phi_m\, g,K_{I_x,I_{y,y'}}\rangle|\,dx\\ &= \int_{|x-y|\geq 4|y'-y|}\sup_g\tau\Big(\int\int\widehat{K}_{I_x,I_{y,y'}}(u,v)\,T(\tau_y\phi_m(v)-\tau_{y'}\phi_m(v))\,\tau_x\phi_m(u)g\,du\,dv\Big)\,dx\\ &\lesssim \int_{|x-y|\geq 4|y'-y|}\int\int\Big\|\widehat{K}_{I_x,I_{y,y'}}(x-\frac{u}{m},y+\frac{v}{m})-\widehat{K}_{I_x,I_{y,y'}}(x-\frac{u}{m},y'+\frac{v}{m})\Big\|_{\mathcal{M}}\,du\,dv\,dx.\end{aligned}$$

Now, the Hörmander condition for the kernel K implies that K_m satisfies (4.10) since

$$|x-\frac{u}{m}-y-\frac{v}{m}|\geq|x-y|-\frac{2}{m}>\frac{1}{2}|x-y|\geq 2|y'-y|.$$

On the other hand, whenever $|x-y|\leq 4/m$ holds, it yields

$$\begin{aligned}&\int_{|x-y|\geq 4|y'-y|}\|K_m(x,y)-K_m(x,y')\|_{\mathcal{M}}\,dx\\ &=\int_{|x-y|\geq 4|y'-y|}\sup_{\substack{g\in L_2(\mathcal{M})\\ \|g\|_{L_2(\mathcal{M})}\leq 1}}\Big\|\int T(\tau_y\phi_m-\tau_{y'}\phi_m)(t)\,g\,\tau_x\phi_m(t)\,dt\Big\|_{L_2(\mathcal{M})}\,dx\end{aligned}$$

$$\leq \sup_g \int_{|x-y|\geq 4|y'-y|} \iint \|T(\tau_y\phi_m - \tau_{y'}\phi_m)(t)\, g\ \tau_x\phi_m(t)\|_{L_2(\mathcal{M})}\ dx\ dt$$
$$\leq \sup_g \int_{|x-y|\geq 4|y'-y|} \Big(\int \|T(\tau_y\phi_m - \tau_{y'}\phi_m)(t)\, g\|^2_{L_2(\mathcal{M})}\ dt\Big)^{1/2} \Big(\int |\tau_x\phi_m(t)|^2\ dt\Big)^{1/2}\ dx$$
$$\leq \|T\|\ \sup_g \int_{|x-y|\geq 4|y'-y|} m^{1/2}\ \|(\tau_y\phi_m - \tau_{y'}\phi_m)\, g\|_{L_2(\mathbb{R};L_2(\mathcal{M}))}\ dx.$$

This integral is finite since, by the mean value theorem for vector-valued functions,

$$\begin{aligned}\|(\tau_y\phi_m - \tau_{y'}\phi_m)\, g\|_{L_2(\mathbb{R};L_2(\mathcal{M}))} &= \Big(\int_{B(0,1/m)} \|(\phi_m(t-y) - \phi_m(t-y'))g\|^2_{L_2(\mathcal{M})}\ dt\Big)^{1/2} \\ &\leq \Big(\int_{B(0,1/m)} |y-y'|^2\ \sup_c\ \frac{\partial\phi_m}{\partial t}(c)\, g\|^2_{L_2(\mathcal{M})}\ dt\Big)^{1/2} \\ &= m^2\ \Big(\int_{B(0,1/m)} |y-y'|^2\ \sup_c\ \phi'(mc)\, g\|^2_{L_2(\mathcal{M})}\Big)^{1/2} \\ &\leq m^{3/2}\ |y'-y|\ \sup_c \|\phi'(mc)g\|_{L_2(\mathcal{M})}.\end{aligned}$$

Indeed, then it follows from the assumption $|x-y| \leq 4/m$ that

$$\begin{aligned}&\int_{|x-y|\geq 4|y'-y|} \|K_m(x,y) - K_m(x,y')\|_{\mathcal{M}}\ dx \\ &\leq \|T\|\ \sup_g \int_{|x-y|\geq 4|y'-y|} m^{1/2}\ \|(\tau_y\phi_m - \tau_{y'}\phi_m)\, g\|_{L_2(\mathbb{R};L_2(\mathcal{M}))}\ dx \\ &\leq \|T\| \int_{|x-y|\geq 4|y'-y|} m^2\ |y'-y|\ \sup_{g,c} \|\phi'(mc)g\|_{L_2(\mathcal{M})}\ dx \\ &= \|T\|\|\phi'\|_{L_\infty} \int_{|x-y|\geq 4|y'-y|} m^2\ |y'-y|\ dx \\ &\lesssim \|T\|\|\phi'\|_{L_\infty} \int_{|x-y|\geq 4|y'-y|} \frac{|y'-y|}{|x-y|^2}\ dx = 4\,\|T\|\|\phi'\|_{L_\infty}.\end{aligned}$$

Finally, given $f \in L_2(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})$ there holds

$$\begin{aligned}\|T_m(f) - T(f)\|_{L_2} &\leq \|R_m T(R_m(f) - f)\|_{L_2} + \|(R_m - \mathrm{Id})T(f)\|_{L_2} \\ &\leq \|T\|\ \|R_m(f) - f\|_{L_2} + \|(R_m - \mathrm{Id})T(f)\|_{L_2} \to 0\end{aligned}$$

as m tends to infinity. □

Before proving that a Calderón-Zygmund operator extends to a bounded operator from $H_1^c(\mathcal{A})$ into $L_1(\mathcal{A})$, a fundamental property of Calderón-Zygmund operators is included.

Proposition 4.4.4. *Let $\mathcal{M}$ be a von Neumann algebra and let T be Calderón-Zygmund operator which is bounded on $L_2(\mathbb{R};L_2(\mathcal{M}))$ whose kernel is zero. Then T is an operator of pointwise multiplication by some $F \in L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M}$.*

Proof. Let $f \in L_\infty(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M}) \cap L_2(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})$ such that $\mathrm{supp}_{\mathbb{R}}\|f\|$ is compact and contained in some interval J. Then, we claim that for any $g, g' \in L_2(\mathbb{R}; L_2(\mathcal{M}))$ it holds

$$\langle g', T(fg)\rangle_{L_2(\mathbb{R};L_2(\mathcal{M}))} = \langle g', T(f)g\rangle_{L_2(\mathbb{R};L_2(\mathcal{M}))}. \tag{4.11}$$

As a consequence of Definition 4.4.1, given $m \in \mathcal{M} \cap L_2(\mathcal{M})$ and I an arbitrary interval, it holds

$$\begin{aligned}\langle g'(\chi_{(\bar I)^c} \otimes 1), T(f(\chi_I \otimes m))\rangle &= \langle f(y)(\chi_I \otimes m)(y)g'(x)(\chi_{(\bar I)^c} \otimes 1)(x), K_{(\bar I)^c, I\cap J}\rangle = 0,\\ \langle g'(\chi_I \otimes 1), T(f(\chi_{(\bar I)^c} \otimes m))\rangle &= \langle f(y)(\chi_{(\bar I)^c} \otimes m)(y)g'(x)(\chi_I \otimes 1)(x), K_{I, J\cap(\bar I)^c}\rangle = 0.\end{aligned}$$

Moreover, it follows

$$\begin{aligned}\langle g'(\chi_I \otimes 1), T(f)(\chi_I \otimes m)\rangle &= \langle g'(\chi_I \otimes 1), T(f(\chi_I \otimes m))\rangle\\ &\quad + \langle g'(\chi_I \otimes 1), T(f(\chi_{(\bar I)^c} \otimes m))\rangle\\ &= \langle g'(\chi_I \otimes 1), T(f(\chi_I \otimes m))\rangle\end{aligned}$$

and

$$\langle g'(\chi_{(\bar I)^c} \otimes 1), T(f)(\chi_I \otimes m)\rangle = 0 = \langle g'(\chi_{(\bar I)^c} \otimes 1), T(f(\chi_I \otimes m))\rangle,$$

yielding $\langle g', T(f)(\chi_I \otimes m)\rangle = \langle g', T(f(\chi_I \otimes m))\rangle$. Indeed, this identity remains valid when replacing $\chi_I \otimes m$ by $h \otimes b \in L_2(\mathbb{R}) \otimes (\mathcal{M} \cap L_2(\mathcal{M}))$, so claim (4.11) follows.

On the other hand, there exists a increasing sequence of projections $(e_i)_{i=1}^\infty$ which have finite trace and strongly converges to 1 in $L_2(\mathcal{M})$. Therefore, taking $1 \le j \le j'$,

$$\begin{aligned}\langle g', T(\chi_{B(0,j)} \otimes e_j)\rangle &= \langle g', T(\chi_{B(0,j)}\chi_{B(0,j')} \otimes e_j e_{j'})\rangle\\ &= \langle g', T(\chi_{B(0,j')} \otimes e_{j'})(\chi_{B(0,j)} \otimes e_j)\rangle,\end{aligned}$$

so there holds $T(\chi_{B(0,j)} \otimes e_j)(\chi_{B(0,j)} \otimes e_j) = T(\chi_{B(0,j')} \otimes e'_j)(\chi_{B(0,j)} \otimes e_j)$, and there exists some operator $F \in L_2(\mathbb{R}; L_2(\mathcal{M}))$ such that

$$\langle g', F\ (\chi_{B(0,j)} \otimes e_j)\rangle = \langle g', T(\chi_{B(0,j)} \otimes e_j)(\chi_{B(0,j)} \otimes e_j)\rangle$$

for every $j \ge 1$. Then, given $g \in L_2(\mathbb{R}; L_2(\mathcal{M}))$ such that $(\chi_{B(0,j)} \otimes e_j)g = g$,

$$\begin{aligned}\langle g', T(g)\rangle &= \langle g', T((\chi_{B(0,j)} \otimes e_j)g)\rangle = \langle g', T((\chi_{B(0,j)} \otimes e_j))\ (\chi_{B(0,j)} \otimes e_j)g\rangle\\ &= \langle g', F\ (\chi_{B(0,j)} \otimes e_j)g\rangle = \langle g', Fg\rangle\end{aligned}$$

from which follows that $T(g) = Fg$ for every $g \in L_2(\mathbb{R}; L_2(\mathcal{M}))$. It only remains to check that F is a bounded operator. Otherwise, given $n \in \mathbb{N}$, there would hold

$$\Big(\int \Big(\otimes \tau\Big)\Big(F e_{(n,+\infty)}(|F|)\Big) > 0$$

where $e_{(n,+\infty)}(|F|)$ denotes the spectral projection for $|F|$ on the set $(n,+\infty)$. Since $\int\otimes\tau$ is semifinite, there would exist a projection $p_n \leq e_{(n,+\infty)}(|F|)$ with finite trace. Then, $p_n \in L_2(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})$ and $|F|p_n \geq np_n$ so

$$\|T(p_n)\|_{L_2} \geq n\|p_n\|_{L_2},$$

what contradicts the boundedness of T. □

Theorem 4.4.5. *Let $\mathcal{M}$ be a von Neumann algebra. Let T be a Calderón-Zygmund operator bounded on $L_2(\mathbb{R};L_2(\mathcal{M}))$ with associated kernel $K : \mathbb{R}\times\mathbb{R}\setminus\{x=y\}\to\mathcal{M}$. If K satisfies Hörmander condition* (4.9)*, then T extends to a bounded operator from $H_1^c(\mathcal{A})$ into $L_1(\mathbb{R};L_1(\mathcal{M}))$.*

Proof. Let $f = \sum_i \lambda_i b_i h_i$ in $H_1^c(\mathcal{A})$ such that $\sum_{i=1}^\infty \lambda_i b_i h_i = 0$ in $L_1(\mathcal{M}\overline{\otimes}L_\infty(\mathbb{R}))$. First, recall that since $K_m(x,\cdot)$ belongs to $L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M}$, there holds

$$T_m(\sum_i \lambda_i b_i h_i)(x) = \int K_m(x,y)\sum_i \lambda_i b_i(y) h_i \, dy = 0 \text{ in } L_1(\mathcal{M}) \text{ for almost every } x,$$

and

$$T_m(a)(x) = \int K_m(x,y)b(y)h\, dy = \int K_m(x,y)b(y)dt\cdot h.$$

Therefore, the operator T can be extended to c–atoms. Indeed, given a c–atom $a = bh = b'h'$, $T_m(b)h = T_m(b')h'$ in $L_1(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})$. Moreover, $T(a)$ can be defined as $T(b)h = T(b')h'$ since the norm of the difference is

$$\|T(b)h - T(b')h'\|_{L_1(L_\infty(\mathbb{R})\overline{\otimes}\mathcal{M})} \leq \|T(b)h - T_m(b)h\|_{L_1} + \|T_m(b')h' - T(b')h'\|_{L_1}$$

and

$$\begin{aligned} \|T(b)h - T_m(b)h\|_{L_1(\mathcal{A})} &\leq \|T(b)h - TR_m(b)h\|_{L_1} + \|(R_mTR_m(b) - TR_m(b))h\|_{L_1} \\ &\leq \|T(R_mb - b)h\|_{L_1} + \|(R_m - \mathrm{Id})[TR_m(b)h]\|_{L_1}. \end{aligned} \tag{4.12}$$

Regard that, given a c–atom $a = bh$ with support contained in the interval I, then $R_m a = \phi_m * a = (\phi_m * b)h$ is a multiple of a c–atom since

$$\mathrm{supp}_{\mathbb{R}}\|\phi_m * b(\cdot)\|_{L_2(\mathcal{M})} \subseteq I + B(0,1/m), \quad \int (\phi_m * b)(x)\,dx = 0,$$

and $\|\phi * b\|_{L_2(\mathbb{R};L_2(\mathcal{M}))} \leq \frac{1}{\sqrt{|I|}}$. Moreover, $\phi_m * a - a$ is then also a multiple of a c–atom and

$$\left\|\frac{\phi_m * a - a}{\|\phi_m * b - b\|_{L_2}\sqrt{|I| + |B(0,1/m)|}}\right\|_{H_1^c} \leq 1,$$

which implies that $\|\phi_m * a - a\|_{H_1^c} \leq \sqrt{|I| + |B(0,1/m)|}\ \|\phi_m b - b\|_{L_2}$ goes to 0 as m grows for a fixed c−atom a. Then, the first term from (4.12) goes to zero as m grows by a similar argument to the proof of Lemma 4.4.2, while the second norm goes to zero since R_m is given by convolution against an approximate identity. Therefore, it follows that $T(b)h = T(b')h'$ in $L_1(\mathcal{A})$.

Again, as a consequence of Lemma 4.4.2,

$$\|T(a) - T_m(a)\|_{L_1(\mathbb{R};L_1(\mathcal{M}))} \leq C \tag{4.13}$$

for some universal constant $C > 0$. Moreover,

$$\begin{aligned}\|T(a) - T_m(a)\|_{L_1} &\leq \|R_m T R_m(a) - R_m T(a)\|_{L_1} + \|R_m T(a) - T(a)\|_{L_1}\\ &\leq \|T(R_m a - a)\|_{L_1} + \|(R_m - \mathrm{Id})Ta\|_{L_1}\\ &\leq C\ \sqrt{|I| + 2/m}\ \|\phi_m b - b\|_{L_2} + \|(R_m - \mathrm{Id})Ta\|_{L_1}. \qquad (4.14)\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\Big\|\sum_{i=1}^{\infty} \lambda_i\ T(b_i h_i)\Big\|_{L_1(\mathbb{R};L_1(\mathcal{M}))} &\leq \Big\|\sum_{i=1}^{\infty} \lambda_i\ T_m(b_i h_i)\Big\|_{L_1(\mathbb{R};L_1(\mathcal{M}))}\\ &+ \Big\|\sum_{i=1}^{\infty} \lambda_i\ (T(b_i h_i) - T_m(b_i h_i))\Big\|_{L_1(\mathbb{R};L_1(\mathcal{M}))}\\ &\leq \sum_{i=1}^{\infty} |\lambda_i|\ \|T(b_i h_i) - T_m(b_i h_i)\|_{L_1(\mathbb{R};L_1(\mathcal{M}))}\\ &\leq \sum_{i=1}^{\infty} |\lambda_i|\ \|T(b_i h_i) - T_m(b_i h_i)\|_{L_1(\mathbb{R};L_1(\mathcal{M}))}.\end{aligned}$$

The former series tends to 0 as m grows to infinity as consequence of (4.13) and (4.14). This argument justifies the extension of the map T to the whole $H_1^c(\mathcal{A})$ as

$$T(f) = \sum_{i=1}^{\infty} \lambda_i T(b_i h_i)$$

regardlesss of the chosen atomic decomposition for f. □

Remark 4.4.6. *As we already mentioned in the introduction to this chapter, all of our results hold for functions in* $\mathbb{R}^n$*. They also readily extend to more general measure metric spaces so long as the underlying measure is doubling, that is, if the condition*

$$\mu(B(x,2r)) \leq C_\mu\ \mu(B(x,r),$$

holds for all $x \in \mathrm{supp}_{\mathbb{R}}(\mu)$ *and all* $r > 0$*. When* μ *is a measure on* $\mathbb{R}^n$ *which fails the doubling condition the definition of the appropriate* BMO*-type space and a predual is more involved and due to Tolsa in the classical case [80]. A semicommutative definition in that context can be found in [12]. We believe that all our arguments can be transferred to that setting, although the details can be repetitive and are omitted.*

Appendix A

Vector-valued Hardy spaces

This appendix contains an explicit argument for the construction of the map

$$Q : L_2^\circ(\mathbb{R}, (1+t^2), dt) \longrightarrow H_1(\mathbb{R}; L_2(\mathcal{M}))$$

as stated in the discussion preceding Proposition 4.2.11. Indeed, the map Q is considered whenever $L_2(\mathcal{M})$ is replaced by any other Banach space. Moreover, a brief study of *molecules*, which give an additional description of the vector-valued Hardy space, is included.

Given a Banach space $\mathbb{X}$ we say that a function a belonging to $L_1(\mathbb{R}^n;\mathbb{X})$ is a $L_2(\mathbb{R}^n;\mathbb{X})-$atom in $H_1(\mathbb{R}^n;\mathbb{X})$ whenever it satisfies the following conditions:

- $\mathrm{supp}(a) \subseteq B$ for some ball B,
- $\|a\|_{L_2(\mathbb{R}^n;\mathbb{X})} \leq \frac{1}{\sqrt{|B|}}$,
- $\int_B a = 0$.

Then, $H_1(\mathbb{R}^n;\mathbb{X})$ is defined as the subspace of those functions f in L_1 admiting a decomposition

$$f = \sum_{i=1}^{\infty} \lambda_i \, a_i \quad \text{in } L_1(\mathbb{R}^n;\mathbb{X}) \tag{A.1}$$

for some absolutely summable sequence $(\lambda_i)_{i=1}^{\infty}$ and some family of $L_2(\mathbb{R}^n;\mathbb{X})-$atoms $\{a_i\}_{i=1}^{\infty}$. According to [32], the definition of $H_1(\mathbb{R}^n;\mathbb{X})$ can be given via maximal functions the same way it is done in the scalar-valued case. This justifies which sort of convergence is considered in (A.1). It can be checked that the norm

$$\|f\|_{H_1(\mathbb{R}^n;\mathbb{X})} = \inf\{\sum_{i=1}^{\infty} |\lambda_i| \ : \ f = \sum_{i=1}^{\infty} \lambda_i a_i \text{ for some } (\lambda_i)_i \in \ell_1 \ \textit{and} \ L_2 - \text{atoms } (a_i)_i\}$$

is equivalent to that ones defined via maximal functions with values in arbitrary Banach spaces. On the other hand, we include below the characterization via *molecules* of $H_1(\mathbb{R}^n;\mathbb{X})$ which streamlines proof for the classical case [56, Ch. 5 Sec. 5].

Let $\mathbb{X}$ be a Banach space and consider the function $w_s(x) = (1+|x|)^s$ for some $s > n$. Then, the space

$$L_2(\mathbb{R}^n, w(x)\,dx;\mathbb{X}) = \left\{ f \in L_0(\mathbb{R}^n;\mathbb{X}) \ : \ \int_{\mathbb{R}^n} \|f(x)\|_{\mathbb{X}}^2 \, w_s(x)\,dx \ < \infty \right\}$$

is contained in $L_1(\mathbb{R}^n;\mathbb{X})$. We will consider the subspace

$$M^s(\mathbb{X}) = \left\{ f \in L_2(\mathbb{R}^n, w_s(x)dx;\mathbb{X}) \ : \ \int_{\mathbb{R}^n} f = 0 \right\}.$$

Lemma A.0.1. *$M^s(\mathbb{X})$ is a dense linear subspace of $H_1(\mathbb{R}^n;\mathbb{X})$.*

Proof. Let $f \in M^s$ and define

$$f_0 = f\ \chi_{\{|x|\le 1\}} \text{ and } f_j = f\ \chi_{\{2^{j-1}<|x|\le 2^j\}} \text{ for any } j > 0.$$

These functions satisfy

$$\begin{aligned}
\|f_j\|_{L_2(\mathbb{R}^n;\mathbb{X})} &= \left(\int_{2^{j-1}<|x|\le 2^j} \|f(x)\|_{\mathbb{X}}^2 \ dx \right)^{1/2} \\
&= \left(\int_{2^{j-1}<|x|\le 2^j} \|f(x)\|_{\mathbb{X}}^2 \ w_s(x)\,w_s(x)^{-1} \ dx \right)^{1/2} \\
&\le \left(\int_{2^{j-1}<|x|\le 2^j} \|f(x)\|_{\mathbb{X}}^2 \ w_s(x)\,dx \right)^{1/2} 2^{-js/2} \ 2^{s/2} =: R_j \ 2^{-js/2}
\end{aligned}$$

so that the sequence $(R_j)_{j\ge 0}$ belongs to ℓ_2. Let I_j be the integral $\int_{\mathbb{R}^n} f_j$. Then, by the Cauchy-Schwarz inequality and a similar computation, it follows that, for $j \ge 0$

$$\begin{aligned}
\|I_j\|_{\mathbb{X}} &\le \int_{2^{j-1}<|x|\le 2^j} \|f(x)\|_{\mathbb{X}} \ dx \\
&\le \left(\int_{2^{j-1}<|x|\le 2^j} \|f(x)\|_{\mathbb{X}}^2 w_s(x)\,dx \right)^{1/2} \left(\int_{2^{j-1}<|x|\le 2^j} w_s(x)^{-1} \ dx \right)^{1/2} \\
&\le c(n)\,R_j \ 2^{-j(s-n)/2}
\end{aligned}$$

where $c(n) = \left(\frac{\pi^{n/2}(1-2^{-n})}{\Gamma(n/2+1)}\right)^{1/2}$. Therefore, this yields some estimates for $S_j = \sum_{k\ge j} I_j$. Indeed,

$$\|S_j\|_{\mathbb{X}} \le c(n) \sum_{k\ge j} R_k \ 2^{-(s-n)k/2}.$$

Now, let us replace the functions f_j by some *perturbed* atoms a_j given by

$$a_j(x) = f_j(x) + S_{j+1}\ |B(0,2^{j+1})|^{-1}\ \chi_{|x|\le 2^{j+1}}(x) - S_j\ |B(0,2^j)|^{-1}\ \chi_{|x|\le 2^j}(x).$$

Then, the sequence $(a_j)_{j\ge 0}$ satisfies

$$\begin{aligned}\int_{\mathbb{R}^n} a_j(x)\,dx &= \int_{2^{j-1}<|x|\le 2^j} f(x)\,dx + S_{j+1} - S_j\\ &= \int_{2^{j-1}<|x|\le 2^j} f(x)\,dx - I_j = 0.\end{aligned}$$

Moreover, it is easy to check that, by hypothesis, the support of a_j is contained in $B(0,2^{j+1})$ and

$$\begin{aligned}\|a_j\|_{L_2(\mathbb{R}^n;\mathbb{X})} &\le \|f_j\|_2 + \|S_{j+1}\|_{\mathbb{X}}\ |B(0,2^{j+1})|^{-1/2}\ \|\chi_{|x|\le 2^{j+1}}\|_2\\ &+ \|S_j\|_{\mathbb{X}}\ |B(0,2^j)|^{-1/2}\ \|\chi_{|x|\le 2^j}\|_2\\ &\le R_j\ 2^{-js/2} + c(n) \sum_{k\ge j+1} R_k\ 2^{-(s-n)k/2}\ |B(0,2^{j+1})|^{-1/2}\\ &+ c(n) \sum_{k\ge j} R_k\ 2^{-(s-n)k/2}\ |B(0,2^j)|^{-1/2}\\ &= |B(0,2^{j+1})|^{-1/2}\ \Big[|B(0,2^{j+1})|^{1/2} R_j\ 2^{-js/2} + c(n) \sum_{k\ge j+1} R_k\ 2^{-(s-n)k/2}\\ &+ c(n) \sum_{k\ge j} R_k\ 2^{-(s-n)k/2}\ \frac{|B(0,2^{j+1})|^{1/2}}{|B(0,2^j)|^{1/2}}\Big]\\ &= |B(0,2^{j+1})|^{-1/2}\ \Big[c(n) 2^{-j(s-n)/2}\ R_j + c(n) \sum_{k\ge j+1} R_k\ 2^{-(s-n)k/2}\\ &+ c(n) \sum_{k\ge j} R_k\ 2^{-(s-n)k/2}\Big].\end{aligned}$$

Therefore, by taking

$$\lambda_i = c(n)\ 2^{-j(s-n)/2}\ R_j + c(n) \sum_{k\ge j+1} R_k\ 2^{-(s-n)k/2} + c(n) \sum_{k\ge j} R_k\ 2^{-(s-n)k/2}$$

then $(\lambda_i)_{i\in\mathbb{N}} \in \ell_1$ by previous computations and since

$$\begin{aligned}\sum_{j=0}^{\infty} \sum_{k\ge j} R_j\ 2^{-(s-n)k/2} &\le \sum_{j=0}^{\infty} \Big(\sum_{k\ge j} R_j^2\Big)^{1/2} \Big(\sum_{k\ge j} 2^{-(s-n)k}\Big)^{1/2}\\ &\lesssim \frac{1}{(1-2^{-(s-n)})^{1/2}} \sum_{j=0}^{\infty} 2^{-(s-n)j/2}\ \|f\|_{L_2(\mathbb{R},(1+t^2)dt;L_2(\mathcal{M}))}\\ &= \frac{1}{(1-2^{-(s-n)})^{1/2}} \frac{1}{1-2^{-(s-n)/2}}\ \|f\|_{L_2(\mathbb{R},(1+t^2)dt;L_2(\mathcal{M}))}.\end{aligned}$$

Moreover, redefining a_i as a_i/λ_i, we obtain the expression

$$f = \sum_{i\in\mathbb{N}} \lambda_i \ a_i$$

where each a_i is an atom with support $B(0, 2^{i+1})$ and convergence holds in the sense of $L_1(\mathbb{R}, (1+t^2)dt; \mathbb{X})$. In conclusion, f belongs to $H_1(\mathbb{R}^n; \mathbb{X})$.

□

Definition A.0.2. *Let $s > n$ be a real number. A* molecule $f \in M^s$*, centered on x_0 and of width $d > 0$, is defined to be a function belonging to M^s which is normalized by*

$$\Big(\int_{\mathbb{R}^n} \|f(x)\|_{\mathbb{X}}^2 \ \Big(1 + \frac{|x - x_0|}{d}\Big)^s \ dx\Big)^{1/2} \leq d^{-n/2}.$$

Remark A.0.3. *Given a molecule f with center x_0 and width d, some modifications in te proof of the previous lemma implies that the $H_1(\mathbb{R}^n; \mathbb{X})-$norm of the molecule does not depend on d and x_0 but on s and n. More clearly, by defining*

$$f_0 = f \ \chi_{\{|x-x_0|\leq d\}} \text{ and } f_j = f \ \chi_{\{2^{j-1}d|x-x_0|\leq 2^j d\}} \text{ for any } j > 0,$$

then it follows

$$\|S_j\|_{\mathbb{X}} \leq c(n) \, d^{n/2} \ \sum_{k\geq j} R_k 2^{-(s-n)k/2}.$$

Define the atoms a_j by the relation

$$a_j(x) \cdot \lambda_i = f_j(x) + S_{j+1} \ |B(0, 2^{j+1}d)|^{-1} \ \chi_{|x|\leq 2^{j+1}d}(x) - S_j \ |B(0, 2^j d)|^{-1} \ \chi_{|x|\leq 2^j d}(x),$$

where the coefficients λ_i are given by

$$\lambda_i = d^{n/2} \ c(n) \ 2^{-j(s-n)/2} \ R_j + d^{n/2} \ c(n) \sum_{k\geq j+1} R_k \ 2^{-(s-n)k/2}$$
$$+ d^{n/2} c(n) \sum_{k\geq j} R_k \ 2^{-(s-n)k/2},$$

so that their ℓ_1 sum is bounded by a constant which does not depend on x_0 or d by translation invariance and since R_j is bounded by $d^{-n/2}$ by hypothesis. Then, it is easy to check the identity $f = \sum_i \lambda_i a \ a_i$, that $\text{supp}(a_j) \subseteq B(x_0, 2^{j+1}d)$ *and* $\|a_j\|_{L_2(\mathbb{R}^n;\mathbb{X})} \leq |B(x_0, 2^{j+1}d)|^{-1/2}$.

Conclusiones

Para concluir, destacamos algunas ideas que extraemos de esta tesis doctoral.

Los resultados presentados en los capítulos 1 y 2 indican que el análisis armónico no conmutativo proporciona un enfoque natural para el estudio de desigualdades para funciones en el cubo de Hamming. Por ejemplo, la estructura de cociclo se revela como un ingrediente crucial para codificar las posibles geometrías de un grupo discreto dado. Estas técnicas han llevado a otras aplicaciones en el análisis de álgebras de von Neumann de grupo [39, 24, 40], y esperamos que se consoliden como una herramienta para el estudio de desigualdades con origen en la geometría métrica de espacios de Banach. Por otro lado, la *dimension-free Pisier's inequality* obtenida en el capítulo 3 refuerza las posibilidades del análisis no conmutativo en este contexto, y esperamos que las aplicaciones que sigan de este resultado lo confirmen.

El capítulo 4 constituye una formulación rigurosa del espacio BMO con valores en operadores introducido por Mei [54], un espacio que encuentra aplicaciones en varios trabajos recientes [30, 11]. Nuestro enfoque del espacio de Hardy con valores en operadores supone una novedad ya que su definición solo depende de una decomposición atómica apropiada. Además, la acotación de operadores de Calderón-Zygmund de H_1^c en L_1 con kernels con valores en operadores establece un marco prometedor para otros resultados sobre interpolación con espacios BMO.

Como posibles proyectos de trabajo futuro, nos concentramos en varios problemas abiertos. Primero, en cuanto al contenido de los capítulos 1 y 2, sería muy deseable obtener una versión puramente no conmutativa de la desigualdad X_p métrica. En particular, una basada en probabilidad libre arrojaría luz sobre el significado de las "traslaciones no conmutativas" en álgebras de von Neumann de grupo. En otras palabras, la principal dificultad de este enfoque es representar la función

$$(x, \varepsilon) \in \mathbb{Z}_{8m}^n \times \Omega_n \mapsto f(x + \varepsilon)$$

cuando se sustituye el par $(\mathbb{Z}_{8m}^n, \mathbb{Z}_2^n = \widehat{\Omega_n})$ por otros grupos no abelianos (G, H). Si $f = \lambda_{\mathrm{G}}(g) \in \mathcal{L}(\mathrm{G})$, entonces

$$\text{“}f(x + \varepsilon)\text{”} \simeq \lambda_{\mathrm{G}}(g) \otimes \Lambda_{\mathrm{H}}(\lambda_{\mathrm{G}}(g))$$

donde $\Lambda(\lambda_{\mathrm{G}}(g))$ denota la “restricción” del caracter $\lambda_{\mathrm{G}}(g)$ a $\mathcal{L}(\mathrm{H})$. Sin embargo, no hemos encontrado una formulación satisfactoria hasta el momento, aunque guardamos la sospecha de que Λ debería tomar valores en un álgebra semiconmutativa que represente a $\mathcal{L}(\mathrm{H})$ en algún sentido.

Otras cuestiones surgen de nuestro trabajo en el capítulo 2. Una forma adecuada de las desigualdades X_p o unas nuevas desigualdades métricas podrían proporcionar resultados de no-embedabilidad para subconjuntos de la clase de Schatten S_q^n en el espacio S_p cuando $2 < q < p$. Aparte de la discusión incluida al final del capítulo 2, hemos intentado otro enfoque. Una versión matricial de la desigualdad métrica X_p puede ser construida a partir de una desigualdad que implica sucesiones bisimétricas en espacios L_p no conmutativos [46, Theorem 7.1]. En realidad, este resultado se sigue de la iteración de la desigualdad de Johnson, Maurey, Schechtman y Tzafriri [34] para sucesiones simétricas en espacios L_p no conmutativos [46]. El único resultado que se sigue de estas desigualdades X_p matriciales es el siguiente: dados $2 < q < p$ y $m, n \in \mathbb{N}$, la distorsión bi-Lipschitz de $S_q^n([m])$ en $L_p(\mathcal{M})$ cumple

$$c_{L_p(\mathcal{M})}(S_q^n([m])) \gtrsim_{p,q} \min\{m^{\frac{1}{2}-\frac{1}{q}}, n^{\frac{(p-q)(q-2)}{q^2(p-2)}}\}.$$

Sin embargo, este resultado ya se sigue del Corolario 2.4.2 pues

$$\begin{aligned}\min\Big\{n^{\frac{(p-q)(q-2)}{q^2(p-2)}}, m^{1-\frac{2}{q}}\Big\} &\lesssim_{p,q} c_{L_p(\mathcal{M})}([m]_q^n)\\ &\leq c_{L_p(\mathcal{M})}(S_q^n([m]))\, c_{S_q^n([m])}([m]_q^n) = c_{L_p(\mathcal{M})}(S_q^n([m])).\end{aligned}$$

Esto puede sugerir que el orden correcto de la distorsión $c_{L_p(\mathcal{M})}(S_q^n([m]))$ coincide con $c_{L_p(\mathcal{M})}([m]_q^n)$, pero no tenemos ninguna pista sobre si alguna técnica en esta dirección daría una cota superior adecuada.

Entre los méritos del trabajo de Naor [60] sobre desigualdades métricas X_p óptimas, debemos destacar su relación profunda con el análisis de Fourier. A lo largo de esta tesis, hemos explorado la generalización de este argumento en el contexto de álgebras de von Neumann de grupo, pero nos planteamos si una estrategia opuesta funcionaría. En otras palabras, nuestros resultados hasta ahora tratan con las herramientas que hay en el “lado de Fourier”: pensamos en $f \in L_p(\Omega_n)$ como un operador que es combinación lineal de cuantizaciones de funciones de Walsh W_{A}, pero no como una función definida en un grupo cuyas representaciones irreducibles tienen dimensión uno. Por ejemplo, podríamos intentar generalizar reemplazando funciones dependientes de variables de Rademacher por funciones en el grupo unitario. Este enfoque se ve apoyado por la desigualdad alternativa a [46, Theorem 7.1] que está dada por una expressión que implica a la integral de Haar en $U(n)$, y que está basada en el trabajo de Marcus y Pisier en series de Fourier aleatorias [51].

En cuanto a la segunda parte de esta tesis, proponemos algunos problemas. Es bien conocido que varios espacios de funciones como los espacios L_p, el espacio de Hardy $H_1(\mathbb{R}^n)$ y $\mathsf{BMO}(\mathbb{R}^n)$ pueden ser descritos en términos de expansiones de ondículas. En realidad, los argumentos presentes en las Secciones 5.4 y 5.6 de [29] se pueden adaptar para demostrar que un sistema de ondículas suave es una base completamente incondicional para $L_p(\mathbb{R}^n; L_p(\mathcal{M}))$. En particular, estaríamos interesados en extender los resultados ya conocidos para $H_1(\mathbb{R}^n)$ [56, 32] a $H_1(\mathcal{A})$. Para ello, el Teorema 4.4.5 será una pieza clave (del mismo modo que en el argumento clásico), y puede que proporcione resultados de interpolación a partir de los métodos de Bourgain y Pisier [4].

Otra posibilidad para estudiar los espacios de Hardy $H_1(\mathcal{A})$ son las *moléculas*, introducidas en el contexto euclídeo por Meyer [56] (ver también el Apéndice A). Un análogo semiconmutativo sería especialmente relevante, ya que daría teoremas de acotación para operadores de Calderón-Zygmund de H_1^c en sí mismo. Además, el argumento anterior está basado en demostrar que los operadores de Calderón-Zygmund envían átomos en moleculas, por lo que esta nueva construcción sería complementaria a nuestro trabajo.

Conclusions

To conclude, we highlight some ideas extracted from this Ph.D. thesis.

The results presented in chapters 1 and 2 indicate that noncommutative harmonic analysis provides a natural approach for studying inequalities for functions on the Hamming cube. For instance, the cocycle structure turns out to be a crucial ingredient in order to encode the distinct possible geometries of a given discrete group. These techniques have found some other applications in analysis on group von Neumann algebras [39, 24, 40], and we expect it to consolidate as a tool for the study of inequalities with origin in the metric geometry of Banach spaces. On the other hand, the dimension-free Pisier's inequality obtained in chapter 3 reinforces the possibilities of noncommutative analysis in this context, and we expect that applications which follow from this result will confirm it.

Chapter 4 constitutes a rigorous formulation of the operator-valued BMO space introduced by Mei [54], a space which finds applications in several works from recent years [30, 11]. Our approach through the operator-valued Hardy space supposes a novelty since its definition only depends on an appropriate atomic decomposition. Moreover, the boundedness of Calderón-Zygmund operators from H_1^c to L_1 with operator-valued kernels establishes a promising framework for further results about interpolation with BMO spaces.

As further work, we focus on several open problems. First, regarding the content of chapters 1 and 2, a purely noncommutative version of the metric X_p inequality would be desirable. In particular, one based on free probability would shed light about the meaning of "noncommutative translations" in group von Neumann algebras. In other words, the main difficulty from this approach is representing the function

$$(x, \varepsilon) \in \mathbb{Z}_{8m}^n \times \Omega_n \mapsto f(x + \varepsilon)$$

when replacing the pair $(\mathbb{Z}_{8m}^n, \mathbb{Z}_2^n = \widehat{\Omega_n})$ by nonabelian groups (G, H). Whenever $f = \lambda_{\mathrm{G}}(g) \in \mathcal{L}(\mathrm{G})$, then

$$\text{"}f(x + \varepsilon)\text{"} \simeq \lambda_{\mathrm{G}}(g) \otimes \Lambda_{\mathrm{H}}(\lambda_{\mathrm{G}}(g))$$

where $\Lambda(\lambda_{\mathrm{G}}(g))$ denotes the "restriction" of the character $\lambda_{\mathrm{G}}(g)$ to $\mathcal{L}(\mathrm{H})$. However, we have not found a satisfactory formulation yet, although we have the suspicion that Λ would need to take values in a semicommutative algebra which represents $\mathcal{L}(\mathrm{H})$ in some sense.

Some other questions arise from our work from chapter 2. A suitable form of X_p inequalities or some new metric inequalities may provide nonembeddability results for subsets of the Schatten class S_q^n into the space S_p whenever $2 < q < p$. Apart from the discussion included at the end of chapter 2, we tried another approach. A matricial version of the metric X_p inequality can be constructed relying on an inequality which involves bisymmetric sequences in noncommutative L_p spaces [46, Theorem 7.1]. Actually, this result follows from the iteration of the inequality by Johnson, Maurey, Schechtman and Tzafriri [34] for symmetric sequences in noncommutative L_p spaces [46]. The only result that follows from these matricial X_p inequalities is the following: for every $2 < q < p$ and $m, n \in \mathbb{N}$, the bi-Lipschitz distorsion of $S_q^n([m])$ into $L_p(\mathcal{M})$ satisfies

$$c_{L_p(\mathcal{M})}(S_q^n([m])) \gtrsim_{p,q} \min\{m^{\frac{1}{2}-\frac{1}{q}}, n^{\frac{(p-q)(q-2)}{q^2(p-2)}}\}.$$

However, this result already follows from Corollary 2.4.2 since

$$\begin{aligned} \min\left\{n^{\frac{(p-q)(q-2)}{q^2(p-2)}}, m^{1-\frac{2}{q}}\right\} &\lesssim_{p,q} c_{L_p(\mathcal{M})}([m]_q^n) \\ &\leq c_{L_p(\mathcal{M})}(S_q^n([m]))\, c_{S_q^n([m])}([m]_q^n) = c_{L_p(\mathcal{M})}(S_q^n([m])). \end{aligned}$$

This may suggest that the correct order for the distortion $c_{L_p(\mathcal{M})}(S_q^n([m]))$ coincides with $c_{L_p(\mathcal{M})}([m]_q^n)$, but we have no clue about whether any technique in this direction would give a suitable upper bound.

Among the merits of the work by Naor [60] about sharp metric X_p inequalities, we must highlight its deep relationship with Fourier analysis. During this thesis, we have explored the extension of this argument to the context of group von Neumann algebras, but we ask ourselves if an opposite strategy would work. In other words, our results so far deal with the tools which are present in the "Fourier side": we think of $f \in L_p(\Omega_n)$ as an operator which is a linear combination of quantizations of the Walsh functions W_{A}, but not as a function on a group whose irreducible representations are one-dimensional. For instance, we could try to generalize replacing functions depending on Rademacher variables by functions on the unitary group $U(n)$. This approach is supported by an alternative inequality to [46, Theorem 7.1] which is given by an expression that involves the Haar integral on $U(n)$, and which is based on the work by Marcus and Pisier on random Fourier series [51].

Regarding the second part of this thesis, some further directions can be proposed. It is a well-known fact that function spaces such as L_p spaces, the Hardy space $H_1(\mathbb{R}^n)$ and $\mathsf{BMO}(\mathbb{R}^n)$ can be described in terms of wavelet expansions. Actually, the arguments from Sections 5.3 and 5.6 from [29] can be adapted in order to show that a smooth system of wavelets is a completely unconditional basis for $L_p(\mathbb{R}^n; \mathcal{A})$. In particular, we would be interested in extending the already known results for $H_1(\mathbb{R}^n)$ [56, 32] to $H_1(\mathcal{A})$. For that purpose, Theorem 4.4.5 will be a key tool (as it is in the classical argument), and it may yield interpolation results following the methods by Bourgain and Pisier [4].

Another possibility for studying operator-valued Hardy spaces $H_1^c(\mathcal{A})$ are *molecules*, introduced in the Euclidean context by Meyer [56] (see also Appendix A). A semicommutative analogue will be specially relevant, since it would provide boundedness theorems for Calderón-Zygmund operators from H_1^c to itself. Moreover, the former argument is based on showing that a Calderón-Zygmund operator sends atoms to molecules, so this new construction would be complementary to our work.

Bibliography

[1] S. Banach. *Théorie des opérations linéaires.* Éditions Jacques Gabay, Sceaux, 1993. Reprint of the 1932 original.

[2] P. Biane. Free hypercontractivity. *Comm. Math. Phys.*, 184(2):457–474, 1997.

[3] J. Bourgain. The metrical interpretation of superreflexivity in Banach spaces. *Israel J. Math.*, 56:222–230, 1986.

[4] J. Bourgain. Some consequences of Pisier's approach to interpolation. *Israel J. Math.*, 77(1-2):165–185, 1992.

[5] M. Bożejko, T. Januszkiewicz, and R.J. Spatzier. Infinite Coxeter groups do not have Kazhdan's property. *J. Operator Theory*, 19:63–67, 1988.

[6] L. Cadilhac. Weak boundedness of Calderón-Zygmund operators on noncommutative L_1-spaces. *J. Funct. Anal.*, 274(3):769–796, 2018.

[7] L. Cadilhac, J. M. Conde-Alonso, and J. Parcet. Spectral multipliers in group algebras and noncommutative Calderón-Zygmund theory. *J. Math. Pures Appl. (9)*, 163:450–472, 2022.

[8] A. I. Cano-Mármol, J. M. Conde-Alonso, and J. Parcet. Trigonometric chaos and X_p inequalities I: Balanced Fourier truncations over discrete groups. Preprint available at https://arxiv.org/abs/2209.05986.

[9] A. I. Cano-Mármol, J. M. Conde-Alonso, and J. Parcet. Trigonometric chaos and X_p inequalities II: X_p inequalities with sharp scaling parameter. Preprint available at https://arxiv.org/abs/2209.05991.

[10] E. A. Carlen and E. H. Lieb. Optimal hypercontractivity for Fermi fields and related noncommutative integration inequalities. *Comm. Math. Phys.*, 155(1):27–46, 1993.

[11] J. M. Conde-Alonso, A.M. González-Pérez, J. Parcet, and E. Tablate. Schur multipliers in Schatten-von Neumann classes. Ann. of Math., to appear, 2023.

[12] J. M. Conde-Alonso and J. Parcet. Nondoubling Calderón-Zygmund theory: a dyadic approach. *J. Fourier Anal. Appl.*, 25(4):1267–1292, 2019.

[13] K. de Leeuw. On L_p multipliers. *Ann. of Math.*, 81:364–379, 1965.

[14] J. Diestel and J. J. Uhl, Jr. *Vector measures.* Mathematical Surveys, No. 15. American Mathematical Society, Providence, R.I., 1977. With a foreword by B. J. Pettis.

[15] E. G. Effros and Z. Ruan. *Operator spaces*, volume 23 of *London Mathematical Society Monographs. New Series.* The Clarendon Press, Oxford University Press, New York, 2000.

[16] P. Enflo. On infinite-dimensional topological groups. In *Séminaire sur la Géométrie des Espaces de Banach (1977–1978)*, pages Exp. No. 10–11, 11. École Polytech., Palaiseau, 1978.

[17] C. Fefferman and E. M. Stein. H^p spaces of several variables. *Acta Math.*, 129(3-4):137–193, 1972.

[18] T. Figiel. Singular integral operators: a martingale approach. In *Geometry of Banach spaces (Strobl, 1989)*, volume 158 of *London Math. Soc. Lecture Note Ser.*, pages 95–110. Cambridge Univ. Press, Cambridge, 1990.

[19] G. B. Folland. *A course in abstract harmonic analysis.* Textbooks in Mathematics. CRC Press, Boca Raton, FL, second edition, 2016.

[20] J. García-Cuerva and J. L. Rubio de Francia. *Weighted norm inequalities and related topics*, volume 116 of *North-Holland Mathematics Studies.* North-Holland Publishing Co., Amsterdam, 1985. Notas de Matemática [Mathematical Notes], 104.

[21] J. B. Garnett. *Bounded analytic functions*, volume 236 of *Graduate Texts in Mathematics.* Springer, New York, first edition, 2007.

[22] E. D. Gluskin, A. Pietsch, and J. Puhl. A generalization of Khintchine's inequality and its application in the theory of operator ideals. *Studia Math.*, 67(2):149–155, 1980.

[23] S. Goldstein and L. Labuschagne. *Notes on noncommutative L^p and Orlicz spaces.* Wydawnictwo Uniwersytetu Lódzkiego, Lódź, 2020.

[24] A. M. González-Pérez, M. Junge, and J. Parcet. Smooth Fourier multipliers in group algebras via Sobolev dimension. *Ann. Sci. Éc. Norm. Supér. (4)*, 50:879–925, 2017.

[25] A. M. González-Pérez, M. Junge, and J. Parcet. Singular integrals in quantum Euclidean spaces. *Mem. Amer. Math. Soc.*, 272(1334):xiii+90, 2021.

[26] L. Grafakos. *Modern Fourier analysis*, volume 250 of *Graduate Texts in Mathematics.* Springer, New York, second edition, 2009.

[27] U. Haagerup. An example of a nonnuclear C^*-algebra, which has the metric approximation property. *Invent. Math.*, 50:279–293, 1978/79.

[28] U. Haagerup and G. Pisier. Bounded linear operators between C^*-algebras. *Duke Math. J.*, 71:889–925, 1993.

[29] E. Hernández and G. Weiss. *A first course on wavelets.* Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1996. With a foreword by Yves Meyer.

[30] G. Hong and Z. Yin. Wavelet approach to operator-valued Hardy spaces. *Rev. Mat. Iberoam.*, 29(1):293–313, 2013.

[31] L. Hörmander. Estimates for translation invariant operators in L^p spaces. *Acta Math.*, 104:93–140, 1960.

[32] T. Hytönen. Vector-valued wavelets and the Hardy space $H^1(\mathbb{R}^n, X)$. *Studia Math.*, 172(2):125–147, 2006.

[33] P. Ivanisvili, R. van Handel, and A. Volberg. Rademacher type and Enflo type coincide. *Ann. of Math. (2)*, 192(2):665–678, 2020.

[34] W. B. Johnson, B. Maurey, G. Schechtman, and L. Tzafriri. Symmetric structures in Banach spaces. *Mem. Amer. Math. Soc.*, 19(217), 1979.

[35] W. B. Johnson, G. Schechtman, and J. Zinn. Best constants in moment inequalities for linear combinations of independent and exchangeable random variables. *Ann. Probab.*, 13(1):234–253, 1985.

[36] M. Junge. Fubini's theorem for ultraproducts of noncommutative L_p-spaces. *Canad. J. Math.*, 56:983–1021, 2004.

[37] M. Junge, C. Le Merdy, and Q. Xu. H^∞ functional calculus and square functions on noncommutative L^p-spaces. *Astérisque*, (305):vi+138, 2006.

[38] M. Junge and T. Mei. Noncommutative Riesz transforms—a probabilistic approach. *Amer. J. Math.*, 132:611–680, 2010.

[39] M. Junge, T. Mei, and J. Parcet. Smooth Fourier multipliers on group von Neumann algebras. *Geom. Funct. Anal.*, 24:1913–1980, 2014.

[40] M. Junge, T. Mei, and J. Parcet. Noncommutative Riesz transforms—dimension free bounds and Fourier multipliers. *J. Eur. Math. Soc.*, 20:529–595, 2018.

[41] M. Junge, T. Mei, J. Parcet, and R. Xia. Algebraic Calderón-Zygmund theory. *Adv. Math.*, 376:Paper No. 107443, 72, 2021.

[42] M. Junge, C. Palazuelos, J. Parcet, and M. Perrin. Hypercontractivity in group von Neumann algebras. *Mem. Amer. Math. Soc.*, 249(1183), 2017.

[43] M. Junge and J. Parcet. The norm of sums of independent noncommutative random variables in $L_p(\ell_1)$. *J. Funct. Anal.*, 221(2):366–406, 2005.

[44] M. Junge, J. Parcet, and Q. Xu. Rosenthal type inequalities for free chaos. *Ann. Probab.*, 35:1374–1437, 2007.

[45] M. Junge and Q. Xu. Noncommutative Burkholder/Rosenthal inequalities. *Ann. Probab.*, 31:948–995, 2003.

[46] M. Junge and Q. Xu. Noncommutative Burkholder/Rosenthal inequalities. II. Applications. *Israel J. Math.*, 167:227–282, 2008.

[47] M. Ĭ. Kadec. Linear dimension of the spaces L_p and l_q. *Uspehi Mat. Nauk*, 13(6(84)):95–98, 1958.

[48] F. Lust-Piquard. Inégalités de Khintchine dans C_p $(1 < p < \infty)$. *C. R. Acad. Sci. Paris Sér. I Math.*, 303:289–292, 1986.

[49] F. Lust-Piquard. Riesz transforms associated with the number operator on the Walsh system and the fermions. *J. Funct. Anal.*, 155:263–285, 1998.

[50] P. Mankiewicz. On Lipschitz mappings between Fréchet spaces. *Studia Math.*, 41:225–241, 1972.

[51] M. B. Marcus and G. Pisier. *Random Fourier series with applications to harmonic analysis.* Annals of Mathematics Studies, No. 101. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1981.

[52] B. Maurey. Type, cotype and K-convexity. In *Handbook of the geometry of Banach spaces, Vol. 2*, pages 1299–1332. North-Holland, Amsterdam, 2003.

[53] R. E. Megginson. *An introduction to Banach space theory*, volume 183 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1998.

[54] T. Mei. Operator valued Hardy spaces. *Mem. Amer. Math. Soc.*, 188(881):vi+64, 2007.

[55] T. Mei and É. Ricard. Free Hilbert transforms. *Duke Math. J.*, 166:2153–2182, 2017.

[56] Y. Meyer. *Wavelets and operators*, volume 37 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1992. Translated from the 1990 French original by D. H. Salinger.

[57] Y. Meyer and R. Coifman. *Wavelets*, volume 48 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1997. Calderón-Zygmund and multilinear operators, Translated from the 1990 and 1991 French originals by David Salinger.

[58] S. G. Mikhlin. On the multipliers of Fourier integrals. *Dokl. Akad. Nauk SSSR*, 109:701–703, 1956.

[59] A. Naor. An introduction to the Ribe program. *Jpn. J. Math.*, 7(2):167–233, 2012.

[60] A. Naor. Discrete Riesz transforms and sharp metric X_p inequalities. *Ann. of Math.*, 184:991–1016, 2016.

[61] A. Naor, G. Pisier, and G. Schechtman. Impossibility of dimension reduction in the nuclear norm. *Discrete Comput. Geom.*, 63(2):319–345, 2020.

[62] A. Naor and G. Schechtman. Remarks on non linear type and Pisier's inequality. *J. Reine Angew. Math.*, 552:213–236, 2002.

[63] A. Naor and G. Schechtman. Metric X_p inequalities. *Forum Math.* Π, 4:e3, 81, 2016.

[64] R. O'Donnell. *Analysis of Boolean functions.* Cambridge University Press, New York, 2014.

[65] R. E. A. C. Paley. Some theorems on abstract spaces. *Bull. Amer. Math. Soc.*, 42(4):235–240, 1936.

[66] J. Parcet. Pseudo-localization of singular integrals and noncommutative Calderón-Zygmund theory. *J. Funct. Anal.*, 256(2):509–593, 2009.

[67] G. Pisier. Holomorphic semigroups and the geometry of Banach spaces. *Ann. of Math.*, 115:375–392, 1982.

[68] G. Pisier. Non-commutative vector valued L_p-spaces and completely p-summing maps. *Astérisque*, (247):vi+131, 1998.

[69] G. Pisier. *Introduction to operator space theory*, volume 294 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 2003.

[70] G. Pisier and Q. Xu. Non-commutative L^p-spaces. In *Handbook of the geometry of Banach spaces, Vol. 2*, pages 1459–1517. North-Holland, Amsterdam, 2003.

[71] M. Ribe. On uniformly homeomorphic normed spaces. *Ark. Mat.*, 14(2):237–244, 1976.

[72] É. Ricard. *Décomposition de H1, Multiplicateurs de Schur et Espaces d'Operateurs.* PhD thesis, Université Paris 6, 2001.

[73] H. P. Rosenthal. On the subspaces of L^p $(p > 2)$ spanned by sequences of independent random variables. *Israel J. Math.*, 8:273–303, 1970.

[74] W. Rudin. *Fourier analysis on groups.* Wiley Classics Library. John Wiley & Sons, Inc., New York, 1990. Reprint of the 1962 original, A Wiley-Interscience Publication.

[75] W. Rudin. *Functional analysis.* International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, second edition, 1991.

[76] R. A. Ryan. *Introduction to tensor products of Banach spaces.* Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, 2002.

[77] S. Sakai. *C^*-algebras and W^*-algebras.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 60. Springer-Verlag, New York-Heidelberg, 1971.

[78] I. J. Schoenberg. Metric spaces and completely monotone functions. *Ann. of Math.*, 39:811–841, 1938.

[79] I. E. Segal. Equivalences of measure spaces. *Amer. J. Math.*, 73:275–313, 1951.

[80] X. Tolsa. BMO, H^1, and Calderón-Zygmund operators for non doubling measures. *Math. Ann.*, 319(1):89–149, 2001.

[81] Q. Xu. Noncommutative L^p spaces. Book in preparation.