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Problems of Harmonic Analysis in high dimensions.

Memoria para optar al título de Doctor en Matemáticas presentada por:

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Contents

Introduction and summary of the main results

Many classical results of Harmonic Analysis are devoted to estimating the norms of certain operators on the spaces L^p. We say that a linear or sublinear operator T, defined a priori over regular functions, is bounded on $L^p(\mathbb{R}^n, d\mu)$ if there exists a constant C_p so that

$$\|Tf\|_{L^p(\mathbb{R}^n,d\mu)} \leq C_p \|f\|_{L^p(\mathbb{R}^n,d\mu)}, \tag{1}$$

for every f in the domain of T. One also says that T is weakly bounded on $L^p(d\mu)$ (or of weak type $L^p(d\mu)$) if there exists a constant c_p so that

$$\mu(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\})^{1/p} \leq \frac{c_p}{\lambda} \|f\|_{L^p(\mathbb{R}^n,d\mu)}, \tag{2}$$

for all f in the domain of T.

This kind of estimates are crucial in the development of many theories in analysis. Just to mention a few applications, they ensure, for instance, that the action of these operators can be defined over all L^p under some mild additional conditions, guarantee the pointwise convergence of certain sequences of operators and provide arguments for the existence, control and regularity of the solutions of partial differential equations.

For many of the most important operators in Harmonic Analysis like the Fourier transform, the Hardy-Littlewood and the spherical maximal operators, Riesz transforms, maximal operators associated to semi-groups, etc it is known for certain values of p that there exist C_p or c_p so that (1) or (2) hold independently of the dimension n considered.

These results led to the following question, can this uniformity in the dimension be related to an infinite dimensional phenomenon? That is, can one build a reasonable Harmonic Analysis over an infinite dimensional space showing this uniformity as a natural reflection? One may also wonder if the limits when

$n \to \infty$ of the aforementioned operators and their uniform bounds have some reasonable meaning.

As far as we are concerned, these questions are not solved yet. It seems that if we want to have a better idea of what is going on in this setting more work is needed to study the behaviour in high dimensions of these operators. This is the topic of this thesis. Although we will briefly comment some uniformity results for some other important operators, most part of our work is devoted to the Hardy-Littlewood maximal operator and its variants when changing the integration measure.

This thesis is divided into five chapters. In the first one we recall some uniform bounds with respect to the Lebesgue measure for important operators of Harmonic Analysis. We focus on three tools that have been quite successful in order to prove uniform bounds with respect to the dimension: Fourier transform, general theory of semi-groups and the method of rotations. For example the general semi-group theory applies to the Ornstein-Uhlenbeck maximal operator proving that it is uniformly bounded on L^p with $p > 1$, (the uniform weak boundedness on L^1 remains an open problem). As an example for the method of rotations we choose the proof by J. Duoandikoetxea and J.L. Rubio de Francia [28] that Riesz transforms are bounded in $L^p(\mathbb{R}^n)$ with a constant independent of n. Also, using this general principle, we show that for the universal maximal Kakeya operator it is possible to give a uniform weak type bound on $L^n(\mathbb{R}^n)$ when the action is restricted to radial functions (Theorem 1.4).

In this chapter too, we present some results concerning upper bounds for the operator norms of maximal functions in $\mathbb{R}^n$ for large n. First we treat the case of average means over Euclidean balls. In this case the maximal operator acting on radial functions has a weak type dimension-free bound (Theorem 1.9), this was proved in [54]. When the action is restricted to radially decreasing functions the constant for the weak type bound is 1 in every dimension (Theorem 1.10), this was proved by J.M. Aldaz and J. Pérez-Lázaro in [5]. Concerning the maximal operator applied to general functions in L^1, E.M. Stein and J.O. Strömberg proved in [74] that the constants in the weak bounds grow at most like $\mathcal{O}(n)$ as $n \to \infty$ (Theorem 1.8). We show the argument that E.M. Stein ([70], [71] and [74]) used to obtain L^p bounds uniform in the dimension for the maximal operator (Theorem 1.13).

The situation is more complicated when considering maximal functions associated to the balls given by general norms. Thus, for the constants in the L^1 weak type bounds, E.M. Stein and J.O. Strömberg [74] established that they cannot grow faster than $\mathcal{O}(n \log n)$ as $n \to \infty$ (Theorem 1.16). J. Bourgain [10] proved that the action on L^2 can be bounded independently of n and the norm (Theorem 1.17). Then, he (see [11], [12]) and, independently, A. Carbery [17] proved that this result could be extended to L^p for any $p > 3/2$. Dimension independent bounds on L^p for all $p > 1$ were proved by D. Müller in [61] for maximal functions associated to the balls given by the ℓ^q norms in $\mathbb{R}^n$ with $1 \leq q < \infty$ (Theorem 1.28), although in this case these constants depend on q. We show that the technique used in [17] gives a different proof for Theorem 1.17.

Chapter 2 is devoted to lower bounds for the weak type operator norm of maximal functions. The first step in this direction is due to M. de Guzmán [42], who showed that the weak boundedness of a maximal convolution operators on L^1 is equivalent to the weak boundedness over finite sums of Dirac deltas. M.T. Menárguez and F. Soria [53] pointed out that in both cases the operator norms must be the same (Theorem 2.1), and this discretisation technique provides a method to compute lower bounds for them. In this same article some lower bounds are given for the weak type operator norms of maximal function associated to cubes. This method was exploited by many researchers trying to compute the exact value of the weak type norm of the one-dimensional maximal operator. This was finally achieved by A. Melas in [48] and [49]. Turning back to the high-dimensional problems, the next important step was presented by J.M. Aldaz in [3]. He showed that the weak type operator norms of maximal functions associated to cubes grow to infinity with the dimension (Theorem 2.3). G. Aubrun proved in [7] that this growth is at least of the order of $C(\log n)^{1-\varepsilon}$ for any $\varepsilon > 0$. We present in Theorem 2.9, an extension of this last result to Orlicz spaces. More precisely, it is showed that the weak operator norms on some Orlicz spaces, of maximal operator associated to cubes, also grow to infinity with the dimension. We give a lower bound for this growth related to the Young function defining the Orlicz space.

In the rest of the chapters we look at a modification of the problem. It is also possible to define maximal functions with an underlying measure μ different of Lebesgue measure:

$$M_\mu f(x) = \sup_{\mu(B(x,r))>0} \frac{1}{\mu(B(x,R))} \int_{B(x,R)} |f(y)|\, d\mu(y).$$

Since we are considering centred operators, the boundedness of these operators is an easy consequence of Besicovitch covering Lemma. Then, we may ask ourselves if the operator norms for these bounds are uniformly bounded in dimension. It makes sense to focus on measures with a radial density, since they can be defined for every dimension. Some answers for this question are given in Chapters 3, 4 and 5.

In Chapter 3 we consider the case that μ is a finite measure. Using the discretisation method, J.M. Aldaz proved in [2] that if μ is a finite measure with a bounded radial density the weak $L^1(\mu)$ operator norm of M_μ grows exponentially to infinity, with a lower bound only depending on n and not on μ (Theorem 3.3). In Theorem 3.4 we give the following extension of this last result. If μ is a finite measure with a bounded and radially decreasing density, then the $L^p(\mu)$ operator norm of M_μ grows exponentially to infinity with a lower bound independent of μ for a small range of p near 1 ($1 \leq p \leq 1.005$). This unboundedness in dimension occurs no matter if restricting the action to radially decreasing functions. J.M. Aldaz and J. Pérez-Lázaro [4] proved independently the same for a slightly larger range of p near one ($1 \leq p \leq 1.0378$) and a wider class of measures in the sense that some unbounded densities and infinite measures are allowed. In both cases the method fails to produce general unboundedness results for significantly larger values of p, as we will see in some examples.

Chapter 4 gives a complete picture of the situation in two particular examples. These are the Lebesgue measure restricted to the unit ball and the Gaussian measure. Considering μ as the Lebesgue measure restricted to the unit ball, we show that for $1 \leq p < 2$ the $L^p(\mu)$ operator norm of M_μ grows exponentially to infinity as $n \to \infty$, even for radially decreasing functions (Theorem 4.1). However, if $p > 2$ we show that for these functions there exist uniform bounds in $L^p(\mu)$ (Theorem 4.3). With respect to the Gaussian density, in Theorem 4.11 we prove that no uniform bounds on L^p for any $p \geq 1$ can be given for its associated maximal operators (except when $p = \infty$). This result is also extended to families of measures given by densities with exponential decay or double exponential decay (Theorems 4.12 and 4.13).

Last, in Chapter 5 we study the case that μ is a measure with certain doubling properties. We show that some of the classical results given in Chapter 1 can be extended to a more general setting. Our model example will be the power weights, that is, the measures over $\mathbb{R}^n$ given by the densities $|\cdot|^\alpha$ with $\alpha > -n$. We will also deal with measures μ defined in general metric measure spaces. Briefly, we say that μ is n-micro-doubling if dilation by the factor $(1 + 1/n)$ of a ball gives a ball of comparable measure. We say that μ is weakly regular if the measure of two intersecting balls with the same radius is comparable. Finally, we say that a family of measures is uniformly weakly regular or n-micro-doubling if these constants can be bounded uniformly throughout all the family. We will see that power weights satisfy these properties. When μ belongs to a family of uniform n-micro-doubling and weakly regular measures, A. Naor and T. Tao showed in [63] that M_μ has a weak L^1 bound smaller than $\mathcal{O}(n \log n)$. This is reminiscent of what Theorem 1.16 by Stein and Strömberg says. The result is obtained through a localisation principle of which we present a different proof based on geometrical arguments, rather than on probabilistic methods (see Theorem 5.4).

The second main result in this chapter is given in Theorem 5.9. Here we show that there are families of measures (that include the power weights) whose associated maximal operators satisfy uniform L^p bounds for each $p > 1$.

Introducción y resumen de resultados y conclusiones

Muchos resultados clásicos del Análisis Armónico estiman la norma de ciertos operadores en $L^p(\mathbb{R}^n, d\mu)$. Se dice que un operador lineal o sublineal T, definido a priori sobre funciones regulares, está acotado en $L^p(\mathbb{R}^n, d\mu)$ si existe una constante positiva C_p de modo que la desigualdad

$$\|Tf\|_{L^p(\mathbb{R}^n,d\mu)} \leq C_p\|f\|_{L^p(\mathbb{R}^n,d\mu)}, \tag{3}$$

se cumple para toda f en el dominio de T. También decimos que el operador T está débilmente acotado en $L^p(d\mu)$ (o que es de tipo débil $L^p(d\mu)$) si existe una constante positiva c_p tal que

$$\mu(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\})^{1/p} \leq \frac{c_p}{\lambda}\|f\|_{L^p(\mathbb{R}^n,d\mu)}, \tag{4}$$

para toda f en el dominio de T.

Este tipo de desigualdades es crucial en el desarrollo de muchas teorías en Análisis. Por poner algunos ejemplos de su utilidad, estas acotaciones garantizan que la acción de estos operadores se pueda definir sobre todo L^p, aseguran la convergencia puntual de ciertas sucesiones de operadores y dan lugar a argumentos que implican la existencia, control o regularidad de soluciones de ecuaciones diferenciales.

De muchos de los operadores más importantes del Análisis Armónico como la Transformada de Fourier, el operador maximal de Hardy-Littlewood, el operador maximal esférico, las transformadas de Riesz o los operadores maximales asociados a semigrupos, se sabe que para algunos valores de p existen C_p o c_p de modo que (3) o (4) son válidas en todas las dimensiones n.

Estos resultados llevaron a que se planteasen las siguientes preguntas. ¿Este comportamiento uniforme respecto de la dimensión esta relacionado con un fenómeno en infinitas dimensiones? Es decir, ¿puede construirse un Análisis Armónico sobre un espacio de dimensión infinita con resultados en los que se vea reflejada esta uniformidad? También podemos preguntarnos si se les puede dar algún sentido razonable a los límites de dichos operadores y sus cotas cuando $n \to \infty$.

Que sepamos, estas cuestiones están muy lejos de responderse de manera satisfactoria. Parece que para intentar comprender mejor el sentido de estas preguntas se necesita todavía estudiar mejor el comportamiento de estos operadores en dimensiones altas. Este es el problema que tratamos en esta tesis. La mayor parte de ella esta dedicada al estudio del operador maximal de Hardy y Littlewood y algunas de sus variantes, aunque brevemente comentamos algunas acotaciones uniformes para otros operadores importantes.

Esta memoria se divide en cinco capítulos. En el primero recogemos algunas acotaciones independientes de la dimensión respecto de la medida de Lebesgue para distintos operadores habituales en Análisis Armónico. Nos centramos en ejemplificar tres herramientas que se han demostrado útiles para obtener acotaciones uniformes: la transformada de Fourier, la teoría general de semigrupos y el método de rotaciones. Entre otras cosas, veremos que la teoría general de semigrupos puede aplicarse al operador maximal de Ornstein-Uhlenbeck para probar acotaciones uniformes en L^p con $p > 1$ (la acotación de tipo débil L^1 con constantes independientes de la dimensión es un problema abierto). Como ejemplo del método de rotaciones, escogemos la prueba con la que J. Duoandikoetxea y J.L. Rubio de Francia [28] demostraron que las transformadas de Riesz están acotadas en $L^p(\mathbb{R}^n)$ con constantes independientes de la dimensión. También usando este principio general del método de rotaciones, probamos que el operador universal de Kakeya actuando sobre funciones radiales satisface una acotación de tipo débil restringido $L^n(\mathbb{R}^n)$ con constantes que no dependen de la dimensión (Teorema 1.4).

También en este capítulo, comenzamos nuestro trabajo sobre el operador maximal de Hardy-Littlewood. Comentaremos algunos de los resultados más importantes que intentan acotar la norma del operador de forma independiente de la dimensión. Primero nos ocupamos del operador maximal sobre medias en bolas euclídeas. Cuando este operador actúa sobre funciones radiales tiene cotas de tipo débil L^1 independientes de la dimensión (Teorema 1.9), como se probó en [54]. Cuando la acción se restringe a las funciones radiales y decrecientes la constante en este tipo débil es 1 en todas las dimensiones (Teorema 1.10). Esto lo probaron J.M. Aldaz y J. Pérez-Lázaro en [5]. Cuando se aplica a funciones integrables generales, E.M. Stein y J.O. Strömberg probaron en [74] que las constantes en la acotación de tipo débil L^1 crecen como mucho con la tasa $\mathcal{O}(n)$ as $n \to \infty$ cuando $n \to n$ (Teorema 1.8). Finalmente, también recogemos el argumento empleado por E.M. Stein ([70], [71] and [74]) para obtener cotas independientes de la dimensión para este operador maximal en L^p para $p > 1$. (Teorema 1.13).

La situación es más complicada cuando consideramos operadores maximales con medias sobre cuerpos convexos arbitrarios, que siempre podemos interpretar como bolas definidas por cierta norma. E.M. Stein y J.O. Strömberg [74] demostraron que las constantes en la desigualdad de tipo débil crecen como mucho como $\mathcal{O}(n \log n)$ cuando $n \to \infty$ (Teorema 1.16). J. Bourgain [10] probó que la acción sobre L^2 se puede acotar independientemente de la dimensión n y de la norma escogida (Teorema 1.17). Más tarde él (ver [11],[12]) e independientemente, A. Carbery [17] probaron que este resultado también se extiende a las cotas en L^p con $p > 3/2$ (Teorema 1.18). D. Müller [61] demostró que hay cotas independientes de la dimensión en L^p para $p > 1$ para los operadores maximales asociados a las bolas provenientes de las normas ℓ^q en $\mathbb{R}^n$ con $1 \leq q < \infty$ (Teorema 1.28). Aunque en este caso las constantes dependen de q. Veremos cómo los argumentos de [17] dan una prueba más sencilla del Teorema 1.17.

El capítulo 2 está dedicado a las cotas inferiores para el comportamiento asintótico de las constantes en las cotas de tipo débil L^1 del operador maximal. El primer paso en esta dirección fue el de M. de Guzmán [42], que demostró que la acotación débil en L^1 de un operador maximal de convolución es equivalente a la acotación sobre sumas finitas de deltas de Dirac. M.T. Menárguez y F. Soria [53] señalaron que en ambos casos las constantes deben ser las mismas (Teorema 2.1). Esta técnica de discretización da un método para buscar cotas inferiores de las constantes. De este modo en [53] se dan algunas cotas inferiores de constantes del operador unidimensional y del operador asociado a cubos. Este método ha sido seguido por los muchos investigadores que intentaron hallar el valor exacto de la constante del operador en una dimension. Volviendo a nuestro problema en altas dimensiones, el siguiente paso importante fue dado por J.M. Aldaz en [3] para el operador maximal asociado a cubos. Demostró que las constantes en la desigualdad de tipo débil L^1 crecen a infinito con la dimensión (Teorema 2.3). G. Aubrun probó en [7] que este crecimiento es al menos tan rápido como $C(\log n)^{1-\varepsilon}$ para todo $\varepsilon > 0$. Nosotros presentamos el Teorema 2.9, que es una extensión de este último resultado para acotaciones en espacios de Orlicz. Demostramos que las constantes en estas desigualdades para ciertos espacios de Orlicz también crecen hacia infinito con la dimensión y damos cotas inferiores de este crecimiento relacionadas con la función de Young que define el espacio.

En el resto de los capítulos consideraremos modificaciones de los problemas tratados anteriormente. Nos centraremos en operadores maximales asociados a medidas que no son la de Lebesue:

$$M_\mu f(x) = \sup_{\mu(B(x,r))>0} \frac{1}{\mu(B(x,R))} \int_{B(x,R)} |f(y)|\, d\mu(y).$$

Como consideramos operadores centrados, su acotación es una consecuencia sencilla del lema de recubrimiento de Besicovitch. Entonces podemos preguntarnos si las normas de operador de estas funciones maximales están acotadas independientemente de la dimensión. Tiene sentido que nos centremos en medidas con una densidad radial, ya que esas son medidas que se pueden definir en todas las dimensiones. En los Capítulos 3, 4 y 5 intentamos dar algunas respuestas a estas preguntas.

En el Capítulo 3, consideramos el caso en el que μ es una medida finita. Usando el método de discretización J.M. Aldaz probó in [2] que si μ es una medida finita con una densidad radial y acotada, entonces la norma como operador de tipo débil $L^1(\mu)$ de M_μ crece exponencialmente con la dimensión, con una cota inferior para ese crecimiento que solo depende de la dimensión n y no de μ (Teorema 3.3). En el Teorema 3.4 presentamos la siguiente extensión a L^p de este resultado. Si μ es una medida finita con una densidad acotada y radialmente decreciente, entonces la norma de M_μ como operador en $L^p(\mu)$ para algunos valores de p cerca de uno ($1 \leq p \leq 1.005$) crece exponencialmente hacia infinito con la dimensión, con una cota que no depende de μ incluso si se restringe la acción a funciones radiales decrecientes. J.M. Aldaz y J. Pérez-Lázaro [4] probaron independientemente el mismo resultado para un rango de p ligeramente mayor ($1 \leq p \leq 1.0378$) y clases más generales de medidas (el método funciona para algunas con densidades no acotadas). En ambos casos está demostrado que el método no proporciona resultados generales de no acotación para valores de p que no sean cercanos a 1, como veremos en unos ejemplos.

El Capítulo 4 se da una descripción completa de la situación para los operadores maximales asociados a dos medidas con densidades radialmente decrecientes y acotadas. Estas son la medida de Lebesgue restringida a la bola unidad y la medida gaussiana. Cuando μ es la medida de Lebesgue restringida a la bola unidad, demostramos que no hay acotaciones uniformes en $L^p(\mu)$ si $1 \leq p < 2$. Las constantes crecen exponencialmente aun cuando se considera la acción sobre funciones radiales decrecientes (Teorema 4.1). Sin embargo, si $p > 2$ demostramos que hay cotas uniformes en $L^p(\mu)$ para M_μ cuando actúa sobre funciones radialmente decrecientes (Teorema 4.3). La otra familia de medidas que tomamos como ejemplo es la que viene dada por la densidad gaussiana. En el Teorema 4.11 probamos que el operador maximal asociado no está (débilmente) acotado uniformemente en L^p para ningún $p \geq 1$ (excepto $p = \infty$). Este resultado también se puede extender a familias de medidas con densidades con decaimiento de tipo exponencial o doble exponencial (Teoremas 4.12 y 4.13).

Por último, en el Capítulo 5 estudiamos el caso en el que μ es una medida con cierta propiedad doblante. Demostramos que algunos de los resultados clásicos tratados en el Capítulo 2 se pueden extender a un contexto más general. Nuestro ejemplo modelo serán los pesos potencia, que son las medidas sobre $\mathbb{R}^n$ dadas por las densidades de tipo $|\cdot|^\alpha\, dx$ con $\alpha > -n$. En algún caso trabajaremos con medidas μ definidas en espacios métricos de medida generales. Brevemente recordamos que μ es n-microdoblante si la dilatación de una bola por el factor $(1 + 1/n)$ da otra bola de medida comparable. La medida μ es débilmente regular si cualesquiera dos bolas secantes con el mismo radio tienen medidas comparables. Decimos que una familia de medidas es uniformemente n-microdoblante o débilmente regular si las constantes en dichas comparaciones se pueden acotar uniformemente en toda la familia. Como veremos, los pesos potencia satisfacen estas condiciones de uniformidad. Si μ pertenece a una familia de medidas uniformemente n-microdoblantes y débilmente regulares, A. Naor and T. Tao probaron en [63] que M_μ satisface una desigualdad de tipo débil L^1 con una constante que como mucho es $\mathcal{O}(n \log n)$. Esto es reminiscente del Teorema 1.16 de Stein y Strömberg. El resultado se obtiene mediante un principio de localización (Teorema 5.3) para el que daremos una nueva prueba basada en argumentos geométricos en vez de en métodos probabilísticos (ver Teorema 5.4).

El otro resultado principal de este capítulo es el Teorema 5.9. Demostramos que existen familias de medidas (incluyendo los pesos potencia) cuyos operadores maximales asociados satisfacen cotas independientes de la dimensión en L^p para todo $p > 1$.

Notation and common facts.

As usual the symbol $|\cdot|$ will denote different things depending on the context. For a complex number a, $|a|$ will denote its absolute value; for $x \in \mathbb{R}^n$, $|x|$ will stand for the norm of x and if $E \subset \mathbb{R}^n$ is measurable, $|E|$ will denote the Lebesgue measure of the set. We will write the k-dimensional Hausdorff measure of $A \subset \mathbb{R}^n$ as $|A|_k$ for $0 < k \leq n$.

By $B(x,R)$ we will denote the ball in $\mathbb{R}^n$ centred at $x \in \mathbb{R}^n$ with radius $R > 0$ with the special notation $B_R := B(0,R)$ for balls centred at the origin. The unit sphere in $\mathbb{R}^n$ will be denoted by $\mathbb{S}^{n-1}$. We will write σ_{n-1} when referring to the measure that Lebesgue measure induces over $\mathbb{S}^{n-1}$ (this measure coincides with $|\cdot|_{n-1}$). We will also use the notation $\omega_{n-1} := \sigma_{n-1}(\mathbb{S}^{n-1})$. Recall that

$$\omega_{n-1} = \sigma_{n-1}(\mathbb{S}^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)},$$

where Γ is the Euler Gamma function defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty s^{x-1} e^{-s}\, ds.$$

We will use the well known identities

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 u^{a-1}(1-u)^{b-1}\, du, \tag{5}$$

for $a, b > 0$, and

$$\int_0^\infty \frac{\mathrm{e}^{(-a+bi)s}}{s^\alpha}\, ds = \frac{\Gamma(1-\alpha)}{(a-bi)^{1-\alpha}}, \tag{6}$$

for $a \geq 0$ and $b \in \mathbb{R}$ (with the exception $b = a = 0$). We recall here that by the log-convexity of Γ one has

$$\sqrt{\frac{n-2}{2\pi}} \leq \frac{\omega_{n-2}}{\omega_{n-1}} \leq \sqrt{\frac{n-1}{2\pi}}. \tag{7}$$

Last, we use the usual agreement that c and C denote positive constants that might have different values even in the same line. Subscripts will make explicit the dependence on a parameter, for example $C_{x,y,z}$ expresses that C might depend on x, y and z.

Chapter 1

Dimension-free bounds for classical operators arising in Harmonic Analysis.

As a motivation for the main results presented in this work, we analyse the behaviour of some classical operators that arise in Harmonic Analysis. Perhaps the simplest one to start with is the Fourier transform, which is defined for every integrable function f as

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} f(x)\mathrm{e}^{-2\pi i\langle x,\xi\rangle}\, dx.$$

It is easy to see that $\|\mathcal{F}f\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}$ and, for functions in $L^1 \cap L^2$, one has Plancherel's identity

$$\|\mathcal{F}f\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}.$$

This allows to extend $\mathcal{F}$ to the entire space L^2 and, by interpolation, to other L^p spaces as well as to the class of distributions. Moreover, the Hausdorff-Young's inequality says that

$$\|\mathcal{F}f\|_{L^{p'}(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} \quad \text{for} \quad 1 \leq p \leq 2. \tag{1.1}$$

Observe that these estimates work in any Euclidean space $\mathbb{R}^n$ independently of the dimension n. Now, while the first two are sharp, the last one for $1 < p < 2$ is not, in the sense that the exact norm of $\mathcal{F}$ as an operator mapping $L^p(\mathbb{R}^n)$ into $L^{p'}(\mathbb{R}^n)$ is smaller than 1. The sharp constant for the inequality (1.1) to hold was found by W. Beckner in [8]. The precise estimate says that

$$\|\mathcal{F}f\|_{L^{p'}(\mathbb{R}^n)} \leq \left(\frac{p^{1/p}}{(p')^{1/p'}}\right)^{n/2} \|f\|_{L^p(\mathbb{R}^n)}.$$

In particular, the operator norm of $\mathcal{F}$ depends on the dimension. What is important here, nevertheless, is to realise that these constants remain uniformly bounded as n goes to infinity. This is the main goal of our present work, to determine not the exact norm of our operators but rather whether these norms can be bounded above independently of the underlying space.

The examples that we consider initially, ergodic and semigroup maximal operators, the Mehler transform, some singular integrals, the Kakeya and the Hardy-Littlewood maximal functions, among others, represent only a small portion of what has been studied in this area in recent years. We hope that they will provide a flavour of the kind of question and problems that we are trying to address in this work.

1.1 General Semigroups.

One of the most powerful tools in the field of abstract harmonic analysis is given by the general theory of semi-groups. As we will see, some important semigroups satisfy maximal inequalities independent of the dimension. For others the question remains open, at least in the extremal cases.

The theory of semigroups can be developed in more abstract settings (see [68]), but for the sake of clarity, we will stick to the case when the integration space is $\mathbb{R}^n$ equipped with a Radon measure μ. A semi-group is a family of linear operators $\{T_t\}_{t\geq 0}$ mapping measurable functions on $\mathbb{R}^n$ onto themselves, satisfying the following properties. Given $s, t \geq 0$ one has $T_s \circ T_t = T_{s+t}$ and $T_0 = I$.

Suppose that L is a differential operator in the variable x and that the initial value problem

$$(A)\begin{cases} \frac{\partial}{\partial t}u(x,t) = Lu(x,t) \quad \text{for} \quad (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,0) = f(x), \end{cases}$$

has a unique solution u for each $f \in L^2$, such that $u(\cdot, t) \in L^2$ for each $t > 0$ and $u(x,t) \to f$ as $t \to 0$ in L^2. Then the operators $T_t : f \mapsto u(\cdot, t)$ is a semigroup. To see that $T_t \circ T_s = T_{t+s}$, given $f \in L^2$ call $g(x) = u(x,s) = T_s f(x)$ and call v to the solution of (A) with initial datum g. By uniqueness we have $v(x,t) = u(x, t+s)$ and so we get

$$T_{t+s}f(x) = T_t u(x,s) = T_t g(x) = v(x,t) = u(x,t+s) = T_{t+s}f(x).$$

The action of the semigroup can be written formally as $T_t f = \mathrm{e}^{tL} f$, which is a formal solution to (A) and also formally satisfies the semigroup conditions.

We are interested in L^p bounds for the maximal operator $f \mapsto \sup_{t>0} |T_t f|$ since it would imply pointwise convergence of the solutions to initial data in L^p.

For instance if we look at the heat equation

$$(B) \begin{cases} \frac{\partial}{\partial t} u(x,t) = \Delta u(x,t) \quad \text{for} \quad (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,0) = f(x), \end{cases}$$

we know that for a L^2 initial datum f, the solution $u(x,t)$ is given by the convolution $H_t * f(x)$, where

$$H(x) = \frac{1}{(4\pi)^{n/2}} \mathrm{e}^{-|x|^2/4},$$

is called the heat kernel and $H_t(x) = t^{-n/2} H(x/\sqrt{t})$. The semigroup $T_t f = H_t * f$ is known as the heat semigroup.

Another important semigroup given by a convolution formula is the Poisson semigroup. In this case we take the Poisson kernel

$$P(x) = \frac{c_n}{(1+|x|^2)^{(n+1)/2}},$$

where $c_n = \Gamma((n+1)/2)/\pi^{(n+1)/2}$ is chosen so that $\|P\|_{L^1} = 1$, and its dilations are $P_t(x) = t^{-n} P(x/t)$. Define the action of the semigroup by $T_t f = P_t * f$. The best way to see that the semigroup conditions are fulfilled in L^2 is to check them on the Fourier side, since $\hat{P}_t(\xi) = e^{-2\pi t|\xi|}$. By a standard density argument we have the same in L^p with $1 \le p \le \infty$. Finally let us recall that $P_t * f$ is the formal solution of the initial value problem

$$(C) \begin{cases} \frac{\partial^2}{\partial t^2} u(x,t) + \Delta u(x,t) = 0 \quad \text{for} \quad (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,0) = f(x). \end{cases}$$

In order to get the desired maximal inequalities for semigroups we have to require some additional properties. For all $f \in L^2$ we will always require $T_t f \in L^2$ for each $t \ge 0$ and $T_t \to f$ as $t \to 0$ in L^2. We say that a semigroup $\{T_t\}_{t>0}$ is contractive if

(I) $\|T_t f\|_{L^p} \le \|f\|_{L^p}$ for all $f \in L^p$ and $1 \le p \le \infty$.

We say that $\{T_t\}_{t>0}$ is symmetric if

(II) T_t is self-adjoint on L^2.

In our examples these properties are satisfied trivially. The convergence $T_t f \stackrel{t\to 0}{\longrightarrow} f$ in L^2 is easy on the Fourier side, (I) derives from the Young inequality for convolutions and (II) from Parseval identity.

Given a semigroup $\{T_t\}_{t\geq 0}$, the associated maximal operator over ergodic means is given by

$$M_T f(x) = \sup_{t>0} \left(\frac{1}{t} \int_0^t T_s f(x)\, ds \right).$$

The Hopf–Dunford–Schwartz Theorem (see [26]) ensures that this operator is bounded on L^p for $p > 1$ and weakly bounded in L^1 with absolute constants (in particular they do not depend on the dimension n).

Theorem 1.1. *[Hopf-Dunford-Schwartz ergodic theorem] Let $\{T_t\}_{t\geq 0}$ be a measurable and contractive semigroup. Then one has $\|M_T f\|_{L^p} \leq C_p \|f\|_{L^p}$ for $1 < p \leq \infty$ and $\mu(\{x \in \Omega : M_T f(x) > \lambda\}) \leq \frac{c}{\lambda}\|f\|_{L^1}$, where C_p and C are absolute constants (only depending on p).*

By adding the symmetry hypothesis, not only the $\sup$ over mean values can be bounded, but also the pointwise $\sup$. This is stated in the following

Theorem 1.2. *[Maximal theorem] Let $\{T_t\}_{t\geq 0}$ be a measurable, contractive and symmetric semigroup. Then*

$$\left\| \sup_{t>0} |T_t f| \right\|_{L^p} \leq C_p \|f\|_{L^p},$$

for $1 < p \leq \infty$. As a consequence if $f \in L^p$ and $1 < p < \infty$ one has

$$T_t f(x) \stackrel{t\to 0}{\longrightarrow} f(x) \quad a.e.$$

Observe that the case $p = 1$ is excluded in this second theorem.

Another important operator related to general semigroups is the Littlewood-Paley function defined by

$$g_1(f)(x) = \left(\int_0^\infty t \left| \frac{\partial}{\partial t} T_t f(x) \right|^2 dt \right)^{1/2}. \tag{1.2}$$

The following theorem is due to E.M. Stein [68] and gives the uniform bound on L^2 for this operator.

Theorem 1.3. *If $\{T_t\}_{t\geq 0}$ is a measurable semigroup, contractive in L^p and symmetric, one has*

$$\|g_1(f)\|_{L^2} \leq c\|f\|_{L^2},$$

where c is an absolute constant.

E.M. Stein [73] also proved that in the special case that T_t is the Poisson semigroup one gets the bound

$$\|g_1(f)\|_{L^p} \leq C_p\|f\|_{L^p}, \tag{1.3}$$

for all the range $1 < p \leq 2$, with C_p independent of n.

1.2 The Mehler kernel.

When we replace the Lebesgue measure by the Gaussian measure $d\gamma_n$ in $\mathbb{R}^n$, the usual Laplacian Δ is not a self-adjoint operator on $L^2(\gamma_n)$ any more. Taking the Gaussian density $\gamma_n(x) = \mathrm{e}^{-\pi|x|^2}$ it is easy to see that the adjoint operator for ∂_{x_i} on $L^2(\gamma_n)$ is $(\partial_{x_i})^* = -\partial_{x_i} + 2\pi x_i$. Hence to obtain a symmetric substitute for the Laplacian in this setting we can consider

$$L = \frac{1}{2}\sum_{i=1}^{n} \delta_{x_i}^* \delta_{x_i} = -\frac{1}{2}\Delta + \pi\langle x, \nabla\rangle,$$

a priori defined for test functions. Indeed L has a self-adjoint closure in L^2 and is a positive operator (in the sense that $\langle Lf, f\rangle \geq 0$).

If we consider the initial value problem

$$(\star) \quad \begin{cases} Lu(x,t) &= -\partial_t u(x,t) \quad \text{for} \quad (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,0) &= f(x) \qquad\quad\ \text{for} \quad x \in \mathbb{R}^n, \end{cases}$$

note that $u(x,t) = \mathrm{e}^{-Lt}f$ is a formal solution, at least for test functions. We can consider the Hermite semigroup $\{\mathrm{e}^{-Lt}\}_{t\geq 0}$ defined by $e^{-Lt} : f \mapsto u(\cdot,t)$.

For $f \in L^2(\gamma_n)$ there exist an explicit expression for $\mathrm{e}^{-Lt}f$ (see [67], [77] or [64]) it is known that the solution of $(\star)$ is given by

$$u(x,t) = \int_{\mathbb{R}^n} M_t(x,y) f(y)\, d\gamma_n(y),$$

where

$$M_t(x,y) = \frac{e^{-\frac{\pi|y-e^{-\pi t}x|^2}{1-e^{-2\pi t}}}}{(1-e^{-2\pi t})^{n/2}}.$$

By the change of integration variable $z = \frac{\pi(y-e^{-\pi t}x)^2}{1-e^{-2\pi t}}$ we obtain the dimension free formula:

$$\begin{aligned} \mathrm{e}^{-Lt}f(x) &= \frac{1}{(1-e^{-2\pi t})^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{\pi|y-e^{-\pi t}x|^2}{1-e^{-2\pi t}}} f(y)\,dy \\ &= \int_{\mathbb{R}^n} f(e^{-\pi t}x + \sqrt{1-e^{-2\pi t}}z)\mathrm{e}^{-\pi|z|^2}\,dz. \end{aligned}$$

As before, it is interesting to know whether the maximal operator

$$M^*f(x) = \sup_{t\geq 0} |e^{-tL}f(x)|,$$

is bounded on L^p, since this ensures that $e^{-tL}f(x) \overset{t\to 0}{\longrightarrow} f(x)$ for a.e.x. Given that the Hermite semigroup is a contractive and symmetric diffusion semigroup, Theorem 1.2 implies that for each $p > 1$

$$\|M^*f\|_{L^p(\gamma_n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)},$$

where C_p is a constant only depending on p.

A weak-type bound on L^1 was obtained by B. Muckenhoupt [58] in dimension 1. P. Sjögren [66] proved the weak-type bound in arbitrary dimension, but with constants growing to infinity with the dimension. A different proof for this result was produced by M.T. Menárguez, S. Pérez and F. Soria in [52], see also [37] and [67]. The question of whether dimension free weak-type bounds can be obtained remains an open problem.

One can also define Riesz transforms of arbitrary order in this setting. A related problem is the one of finding bounds for them. These Riesz transforms have been proved to satisfy dimension-free L^p estimates for $p > 1$ with different proofs, see [59], [39], [78], [65], [40], [41]. First and second order Riesz transforms are weakly bounded on L^1, see [59], [33], [34], but it is unknown if uniform estimates in dimension can be achieved. Third and higher order Riesz transforms are not even weakly bounded on L^1, see [34], [37], [51], [64].

1.3 The method of rotations.

The method of rotations allows to transfer one-dimensional boundedness results to larger dimensions producing dimension-free estimates. The most simple examples are the unidirectional operators in $\mathbb{R}^n$. Given a vector $u \in \mathbb{R}^n$ every $x \in \mathbb{R}^n$ can be written as $x = x_1 + x_2$ where $x_1 = su$ for some $s \in \mathbb{R}$ and $\langle x_2, u \rangle = 0$. If T is a bounded operator T on $L^p(\mathbb{R})$, for some $p \in [1, \infty]$, we can define the directional operator T_u on $L^p(\mathbb{R}^n)$ by $T_u f(x) = T(f((\cdot)u + x_2))(s)$.

Examples of this kind of operators are the unidirectional Hilbert transforms

$$H_u f(x) = \frac{1}{p}\, p.v. \int_{\mathbb{R}} \frac{f(x - su)}{s}\, ds,$$

or the unidirectional Hardy-Littlewood maximal operators

$$M_u f(x) = \sup_{r>0} \frac{1}{2r} \int_{-r}^{r} |f(x + su)|\, ds. \tag{1.4}$$

It is easy to show that if C_p is the operator norm of T on $L^p(\mathbb{R})$, then T_v is bounded on $L^p(\mathbb{R}^n)$ with at most the same norm

$$\begin{aligned} \|T_u\|^p_{L^p(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |T_u(x)|^p\, dx = \int_{u^\perp} \int_{\mathbb{R}u} |T(f((\cdot)u + x_2))(s)|^p\, ds\, dx_2 \\ &\leq C_p \int_{u^\perp} |f(su + x_2)|^p\, ds\, dx_2 = C_p \|f\|^p_{L^p(\mathbb{R}^n)}. \end{aligned}$$

We will bound Riesz transforms in L^p by writing them as a convex combination of unidirectional operators. Recall that for $j = 1, \ldots, n$ the Riesz transforms are defined by

$$R_j f(x) = a_n\, p.v. \int_{\mathbb{R}^n} \frac{y_j}{|y|^{n+1}} f(x - y)\, dy = a_n \lim_{\varepsilon \to 0^+} \int_{|y| > \varepsilon} \frac{y_j}{|y|^{n+1}} f(x - y)\, dy,$$

where $a_n = \Gamma((n+1)/2)\pi^{-(n+1)/2}$ is chosen so that

$$(R_j f)^\wedge(\xi) = -\frac{i\xi_j}{|\xi|} \hat{f}(\xi).$$

From the last expression it is obvious that R_j is bounded on $L^2(\mathbb{R}^n)$ with operator norm 1 for every n. For other values of $p > 1$ we apply the aforementioned method of rotations. To integrate in polar coordinates call $y' = y/|y| \in \mathbb{S}^{n-1}$ and σ_{n-1} to the measure on $\mathbb{S}^{n-1}$ induced by Lebesgue measure on $\mathbb{R}^n$, then

$$\begin{aligned} R_j f(x) &= a_n \lim_{\varepsilon\to 0^+} \int_{|y|>\varepsilon} \frac{y'_j}{|y|^n} f(x-y)\, dy \\ &= a_n \lim_{\varepsilon\to 0^+} \int_{\mathbb{S}^{n-1}} y'_j \int_{\varepsilon}^{\infty} \frac{f(x-sy')}{s}\, ds\, d\sigma_{n-1}(y') \\ &= a_n \pi \int_{\mathbb{S}^{n-1}} y'_j H_{y'} f(x)\, d\sigma_{n-1}(y'), \end{aligned} \tag{1.5}$$

where the last equality is a consequence of the fact that the truncated operators

$$H_\varepsilon f(x) = \frac{1}{\pi} \int_{|y|>\varepsilon} \frac{f(x-y)}{y}\, dy.$$

converge to the Hilbert transform H in L^p (see [30] for example).

From (1.5) we can now obtain a dimension-free estimate for Riesz transforms:

$$\begin{aligned} \|R_j f\|_{L^p(\mathbb{R}^n)} &= a_n \pi \Big\| \int_{\mathbb{S}^{n-1}} y'_j H_{y'} f(\cdot)\, d\sigma_{n-1}(y') \Big\|_{L^p(\mathbb{R}^n)} \\ &\le a_n \pi \int_{\mathbb{S}^{n-1}} |y'_j| \|H_{y'} f\|_{L^p(\mathbb{R}^n)}\, d\sigma_{n-1}(y') \\ &\le a_n \pi \int_{\mathbb{S}^{n-1}} |y'_j|\, d\sigma_{n-1}(y') \|f\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

Now we only have to see that the constant can be bounded uniformly in n. First we calculate

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} |y'_j|\, d\sigma_{n-1}(y') &= 2 \int_0^{\pi/2} \sin\beta \sigma_{n-2}(\cos\beta \mathbb{S}^{n-2})\, d\beta \\ &= 2\sigma_{n-2}(\mathbb{S}^{n-2}) \int_0^{\pi/2} \sin\beta\, \cos^{n-2}\beta\, d\beta = \frac{2\sigma_{n-2}(\mathbb{S}^{n-2})}{n-1}. \end{aligned}$$

And finally the operator norm of R_j on L^p with $p > 1$ is bounded by

$$a_n \pi \frac{2\sigma_{n-2}(\mathbb{S}^{n-2})}{n-1} = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{4\pi^{(n-1)/2}}{\Gamma((n-1)/2)(n-1)} = 2.$$

Still stronger results can be reached. If we define the vector valued operator

$$\mathcal{R} f(x) = (R_1(x), \ldots, R_n(x)),$$

one also has for $p > 1$

$$\||\mathcal{R}f|\|_{L^p(\mathbb{R}^n)} \le C_p \|f\|_{L^p(\mathbb{R}^n)},$$

where C_p only depends on p. Applying Fourier transform it is trivial that one has the isometry $\||\mathcal{R}f|\|_{L^2} = \|f\|_{L^2}$. For other values of p we again use the method of rotations. Fix $x \in \mathbb{R}^n$, there exist $v \in \mathbb{S}^{n-1}$ so that $|\mathcal{R}f(x)| = \langle v, \mathcal{R}f(x)\rangle$, and then one has

$$\begin{aligned} |\mathcal{R}f(x)| &= a_n\pi \sum_j v_j \int_{\mathbb{S}^{n-1}} y'_j H_{y'} f(\cdot)\, d\sigma_{n-1}(y') \\ &= a_n\pi \int_{\mathbb{S}^{n-1}} \langle v, y'\rangle H_{y'} f(\cdot)\, d\sigma_{n-1}(y') \\ &\le c_n\pi \left(\int_{\mathbb{S}^{n-1}} |\langle v, y'\rangle|^{p'}\, d\sigma_{n-1}(y') \right)^{1/p'} \left(\int_{\mathbb{S}^{n-1}} |H_{y'} f(x)|^p\, d\sigma_{n-1}(y') \right)^{1/p}. \end{aligned}$$

where the last step comes from Hölder's inequality. Now we have that by rotation invariance

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} |\langle v, y'\rangle|^{p'}\, d\sigma_{n-1}(y') &= \int_{\mathbb{S}^{n-1}} |y'_1|^{p'}\, d\sigma_{n-1}(y') \\ &= 2\int_0^{\pi/2} \sin^{p'} \beta \sigma_{n-2}(\cos\beta \mathbb{S}^{n-2})\, d\beta \\ &= 2\sigma_{n-2}(\mathbb{S}^{n-2}) \int_0^{\pi/2} \sin^{p'} \beta\, \cos^{n-2} \beta\, d\beta. \end{aligned}$$

By the change of variables $\sin^2 \beta = t$ one has

$$\begin{aligned} \int_0^{\pi/2} \sin^{p'} \beta\, \cos^{n-2} \beta\, d\beta &= \frac{1}{2} \int_0^1 t^{(p'-1)/2} (1-t)^{(n-3)/2}\, dt \\ &= \frac{\Gamma((p'+1)/2)\Gamma((n-1)/2)}{2\Gamma((p'+n)/2)}. \end{aligned}$$

Since

$$\begin{aligned} \left\| \left(\int_{\mathbb{S}^{n-1}} |H_{y'} f|^p\, d\sigma_{n-1}(y') \right)^{1/p} \right\|_{L^p} &= \left(\int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}^n} |H_{y'} f(x)|^p\, dx\, d\sigma_{n-1}(y') \right)^{1/p} \\ &\le b_p \left(\sigma_{n-1}(\mathbb{S}^{n-1}) \right)^{1/p} \|f\|_{L^p}, \end{aligned}$$

where b_p is the L^p operator norm of the Hilbert transform, one has

$$\||\mathcal{R}f|\|_{L^p} \le C_{n,p} \|f\|_{L^p}.$$

Now we have to prove that $C_{n,p}$ can be bounded from above independently of n

$$\begin{aligned} C_{n,p} &= \pi b_p c_n \left(2\sigma_{n-2}(\mathbb{S}^{n-2}) \frac{\Gamma(\frac{p'+1}{2})\Gamma(\frac{n-1}{2})}{2\Gamma(\frac{p'+n}{2})} \right)^{1/p'} \left(\sigma_{n-1}(\mathbb{S}^{n-1})\right)^{1/p} \\ &= \pi b_p \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}} \left(\frac{2\pi^{(n-1)/2}\Gamma(\frac{p'+1}{2})}{\Gamma(\frac{p'+n}{2})} \right)^{1/p'} \left(2\frac{\pi^{n/2}}{\Gamma(\frac{n}{2})} \right)^{1/p}. \end{aligned}$$

By applying Hölder's inequality to its definition Γ is log-convex and then $\Gamma((n+1)/2) \leq \Gamma((p'+n)/2)^{1/p'}\Gamma(n/2)^{1/p}$. This implies

$$C_{n,p} \leq 2\pi^{1/2} p' b_p.$$

Here we followed the scheme of the proof given in [28]. This result had been previously proved by E.M. Stein [73] using the Littlewood-Paley function. For a probabilistic approach see also [55].

1.4 Kakeya maximal function.

Fixed $N > 0$, we denote by $\mathcal{R}_N$ the family of all parallelepipeds in $\mathbb{R}^n$ with edge lengths $h \times h \times \cdots \times h \times hN$, where $h > 0$ is arbitrary. The Kakeya maximal operator is defined by

$$\mathcal{K}_N f(x) = \sup_{x \in R \in \mathcal{R}_N} \frac{1}{R} \int_R |f(y)|\, dy.$$

It is easy to prove that $\mathcal{K}_N f(x) \leq N^{(n-1)} M f(x)$ where Mf is here the usual maximal function over all rotated cubes. One just has to replace $R \in \mathcal{R}_N$ by the smallest cube that contains it. Then $\mathcal{K}_N$ is weakly bounded on $L^1(\mathbb{R}^n)$ with a constant growing with N at most at the rate N^{n-1}. By interpolation with the L^∞ case the operator norm on $L^p(\mathbb{R}^n)$ grows at most like $N^{(n-1)/p}$ for $1 < p < \infty$. However, it is conjectured that for $p = n$ it grows no faster than $C_\varepsilon N^\varepsilon$ for each $\varepsilon > 0$.

In the extremal case, where the eccentricity N is infinity, Kakeya's maximal operator is given by

$$\mathcal{K} f(x) = \sup_{u \in \mathbb{S}^{n-1}} M_u f(x),$$

where M_u is the directional maximal operator defined by (1.4). This universal Kakeya controls all the above $\mathcal{K}_N$ but turns out to be unbounded on every L^p, except for $p = \infty$ (see [42]).

A. Carbery, E. Hernández and F. Soria [19] proved that the conjecture on every $\mathcal{K}_N$ holds for radial functions and more than this, that $\mathcal{K} : L^{n,1}_{\text{rad}} \to L^{n,\infty}_{\text{rad}}$ (see footnote[1]). This means that the estimates for $\mathcal{K}_N$ are independent of N on $L^p(\mathbb{R}^n)$, if $p > n$. Later J. Duoandikoetxea, V. Naibo and O. Oruetxebarria [27] found a more geometrical proof of this result which extended to a wider class of maximal operators. The key of this proof was to compare the action of $\mathcal{K}$ with respect to the maximal operator over centred rings, $\mathcal{A}$, defined by

$$\mathcal{A}f(x) = \sup_{x \in A_{a,b}} \frac{1}{|A_{a,b}|_n} \int_{A_{a,b}} |f(y)|\, dy,$$

where $A_{a,b} = \{x \in \mathbb{R}^n : a \leq |x| \leq b\}$ is a ring. Although not related to N the obtained estimates were strongly dependant on n. We show here however that it is possible to use $\mathcal{A}$ to obtain dimension-free estimates for $\mathcal{K}$. This is presented in the next theorem.

Theorem 1.4. *Let f be a radial function over $\mathbb{R}^n$, then*

$$\|\mathcal{K}f\|_{L^{n,\infty}(\mathbb{R}^n)} \leq \frac{4n}{n-1} \|f\|_{L^{n,1}(\mathbb{R}^n)}.$$

We will first see that $\mathcal{K}$ is controlled pointwise by $\mathcal{A}$ over characteristic functions of radial sets.

Lemma 1.5. *Let E be a radial subset of $\mathbb{R}^n$. Then one has the pointwise inequality*

$$\mathcal{K}\chi_E(x) \leq 2(\mathcal{A}\chi_E(x))^{1/n}.$$

To conclude then, it is enough to find dimension-free estimates for $\mathcal{A}$.

Lemma 1.6. *For all $f \in L^1(\mathbb{R}^n)$*

$$|\{x \in \mathbb{R}^n : \mathcal{A}f(x) > \lambda\}| \leq \frac{2}{\lambda} \|f\|_{L^1(\mathbb{R}^n)}.$$

[1] The definition of the Lorentz spaces $L^{n,1}$, $L^{n,\infty}$, and its norms $\|\cdot\|_{L^{n,1}}$, $\|\cdot\|_{L^{n,\infty}}$ and quasi-norms $\|\cdot\|^*_{L^{n,1}}$, $\|\cdot\|^*_{L^{n,\infty}}$, respectively, can be found in the book by E.M. Stein and G. Weiss [75]. See also Chapter 4 for a brief introduction to the spaces $L^{p,q}$.

Assuming these two lemmas for the moment, we provide a proof of the above theorem.

Proof of Theorem 1.4. By density, we just need to prove the result for a simple function of the form

$$f(x) = \sum_{j=1}^{J} c_j \chi_{E_j}(x),$$

where $E_1 \supset \ldots \supset E_J$ are radial sets and $c_1, \ldots, c_J$ are positive reals. In view of the two lemmas, for a characteristic function of a radial set E one has

$$\begin{aligned} |\{x \in \mathbb{R}^n : \mathcal{K}\chi_E(x) > \lambda\}| &\leq |\{x \in \mathbb{R}^n : 2\mathcal{A}\chi_E(x)^{1/n} > \lambda\}| \\ &= |\{x \in \mathbb{R}^n : \mathcal{A}\chi_E(x) > (\lambda/2)^n\}| \leq \left(\frac{2}{\lambda}\right)^n |E|. \end{aligned}$$

Hence

$$\|\mathcal{K}\chi_E\|^*_{L^{n,\infty}} = \sup_{\lambda>0} \lambda |\{x \in \mathbb{R}^n : \mathcal{K}\chi_E(x) > \lambda\}|^{1/n} \leq 2|E|^{1/n},$$

and to prove the result for f we follow the standard argument

$$\begin{aligned} \|Kf\|^*_{L^{n,\infty}} &\leq \|Kf\|_{L^{n,\infty}} \leq \sum_{j=1}^{J} c_j \|K\chi_{E_j}\|_{L^{n,\infty}} \\ &\leq \frac{n}{n-1} \sum_{j=1}^{J} c_j \|K\chi_{E_j}\|^*_{L^{n,\infty}} \leq \frac{2n}{n-1} \sum_{j=1}^{J} c_j |E_j|^{1/n} \\ &= \frac{2n}{n-1} \|f\|^*_{L^{n,1}}. \end{aligned}$$

□

We continue with the proof of Lemma 1.6, which is similar to the way one proves the weak L^1 boundedness of the one dimensional uncentred maximal operator.

Proof of Lemma 1.6. Fix $f \in L^1(\mathbb{R}^n)$ (without loss of generality we can assume $f \geq 0$), $\lambda > 0$ and call $E_\lambda := \{x \in \mathbb{R}^n : \mathcal{A}f(x) > \lambda\}$. We have to prove that $|E_\lambda| \leq \frac{2}{\lambda}\|f\|_{L^1}$.

Since it makes no difference for $\mathcal{A}$, here we will assume that the rings $A_{a,b}$ are open. If $\mathcal{A}f(x) > \lambda$, there exist an open ring $A_x \ni x$ such that $\frac{1}{|A_x|}\int_{A_x} f(y)\,dy >$

λ. Then $A_x \subset E_\lambda$ and $|A_x| < \frac{1}{\lambda}\int_{A_x} f(y)\,dy$. Hence $E_\lambda = \bigcup_{x\in E_\lambda} A_x$ is an open and radial set. By the usual one dimensional argument we can select a numerable covering $\{A_{x_j}\}_{j\in J\subset\mathbb{N}}$ such that at most two coronas overlap at each point, that is $E_\lambda = \bigcup_{j\in J} A_{x_j}$ and $\sum_{j\in J}\chi_{A_{x_j}} \le 2$. Thus

$$\begin{aligned} |E_\lambda| &\le \sum_{j\in J}|A_{x_j}| \le \frac{1}{\lambda}\sum_{j\in J}\int_{A_{x_j}} f(y)\,dy = \frac{1}{\lambda}\int_{\mathbb{R}^n}\sum_{j\in J}\chi_{A_{x_j}}(y)f(y)\,dy \\ &\le \frac{2}{\lambda}\int_{\mathbb{R}^n} f(y)\,dy. \end{aligned}$$

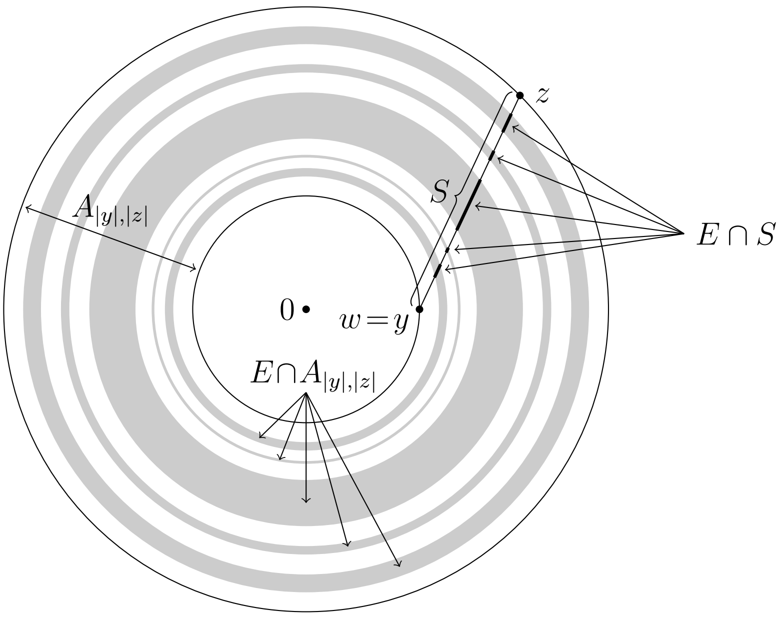
□

Proof of Lemma 1.5. Given a radial set $E \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$ we need to show that for any one dimensional segment S of length L containing x one has

$$\frac{|S\cap E|_1}{|S|_1} \le 2\mathcal{A}\chi_E(x)^{1/n}.$$

We need some notation, call w, z to the extremal points of S, we will always assume that $|w| \le |z|$, and call y to the point in S which is closest to the origin. We will consider as the closed ring $A = A_{|y|,|z|}$. Note that A is the minimal ring that contains the segment S. Finally we write $\ell = |S\cap E|_1$. It will be enough to prove that

$$\frac{\ell}{L} \le 2\left(\frac{|A\cap E|_n}{|A|_n}\right)^{1/n}.$$



We will work on three cases separately and the following three step scheme for each of them:

First step Defining E_0 as the the smallest radial set such that $|E_0 \cap S|_1 = \ell$, we will see that E_0 is a ring of the form $E_0 = A_{|y|,|u|}$ with $u \in S$ such that $|y| \leq |u| \leq |z|$. We will also prove that $|E_0|_n = |A \cap E_0|_n \leq |A \cap E|_n$. So proving the result for E_0 is enough because then we would have

$$\frac{\ell}{L} = \frac{|S \cap E_0|_1}{|S|_1} \leq 2\left(\frac{|A \cap E_0|_n}{|A|_n}\right)^{1/n} \leq 2\left(\frac{|A \cap E|_n}{|A|_n}\right)^{1/n}.$$

Second step We will show that the result is true when $n = 2$, that is we will prove the inequality

$$\frac{\ell}{L} \leq 2\left(\frac{|E_0|_2}{|A|_2}\right)^{1/2} = 2\left(\frac{|z|^2 - |y|^2}{|u|^2 - |y|^2}\right)^{1/2}.$$

Third step Finally we will show that it suffices to prove the result for $n = 2$ because then for $n \geq 2$ one has

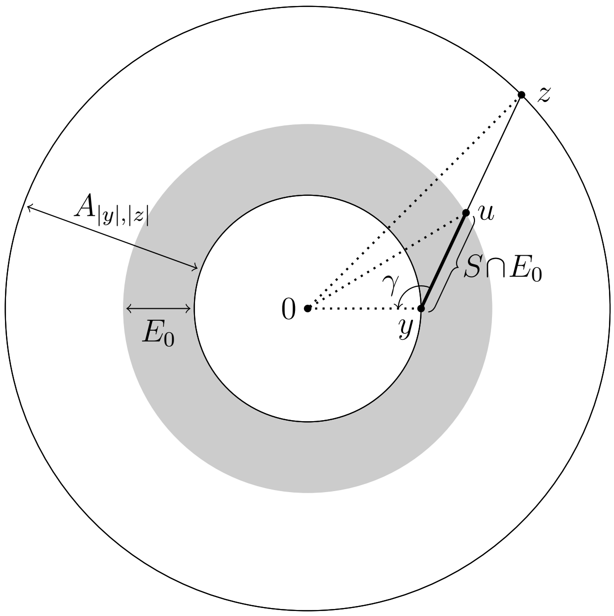
$$\left(\frac{|E_0|_n}{|A|_n}\right)^{1/n} = \left(\frac{|z|^n - |y|^n}{|u|^n - |y|^n}\right)^{1/n} \geq \left(\frac{|z|^2 - |y|^2}{|u|^2 - |y|^2}\right)^{1/2}. \tag{1.6}$$

CASE 1: $y = w$. We want E_0 to be the the smallest radial set such that $|E_0 \cap S|_1 = \ell$. We claim that $E_0 = A_{|y|,|u|}$, where u is the point in S so that $|y - u| = \ell$. It is obvious that $|A_{|y|,|u|} \cap S| = \ell$, there is left to see that $|A_{|y|,|u|}|_n \leq |E \cap A|_n$. Call γ to the angle determined by the origin, y and z. By the cosine law for any $v \in S$ we have

$$|v|^2 = |v - y|^2 + |y|^2 - 2|v - y||y| \cos \gamma.$$

In particular for u this means

$$|u|^2 = \ell^2 + |y|^2 - 2\ell|y| \cos \gamma.$$



Define the set of the possible radii in $E \cap A$ as $T = \{|v| : v \in E \cap A\}$ and the set $T^\star = \{|y - v| : v \in S \cap E\}$, then $|T^\star|_1 = \ell$. By the change of variables $t = (s^2 + |y|^2 - 2s|y| \cos \gamma)^{1/2}$ one has

$$\begin{aligned} |E \cap A|_n &= \omega_{n-1} \int_T t^{n-1}\, ds \\ &= \omega_{n-1} \int_{T^\star} (s^2 + |y|^2 - 2s|y| \cos \gamma)^{n/2-1} (s - 2|y| \cos \gamma)\, ds. \end{aligned}$$

The first integral represents the volume of $E \cap A$ as the result of integrating spherical caps along the way from y to $\frac{|z|}{|y|} y$. The second one corresponds to integrating spherical caps along S (from y to z). Note that $\pi/2 \leq \gamma \leq \pi$, so $\cos \gamma \leq 0$ and the function in the last integral is increasing in s. Therefore recalling that $|T^\star|_1 = \ell$ we have

$$\begin{aligned}\omega_{n-1}\int_{T^\star}&(s^2+|y|^2-2s|y|\cos\gamma)^{n/2-1}(s-2|y|\cos\gamma)\,ds\\ &\geq\ \omega_{n-1}\int_0^\ell(s^2+|y|^2-2s|y|\cos\gamma)^{n/2-1}(s-2|y|\cos\gamma)\,ds\\ &=\ \omega_{n-1}\int_{|y|}^{|u|}t^{n-1}\,dt=|A_{|y|,|u|}|.\end{aligned}$$

This finishes the first step. To fulfill the second one we have to prove

$$\frac{\ell^2}{L^2}\leq\frac{|z|^2-|y|^2}{|u|^2-|y|^2}=\frac{\ell^2-2\ell|y|\cos\gamma}{L^2-2L|y|\cos\gamma},$$

where the equality comes from the cosine law. This is equivalent to

$$\frac{\ell}{L}\leq\frac{\ell-2|y|\cos\gamma}{L-2|y|\cos\gamma},$$

which is obviously truth since $\ell\leq L$ and $\cos\gamma\leq 0$.

The third step is the same for all the cases. Dividing by $|y|$ in (1.6) and renaming $\alpha=(|z|/|y|)^2$ and $\beta=(|u|/|y|)^2$, the inequality that we have to prove is

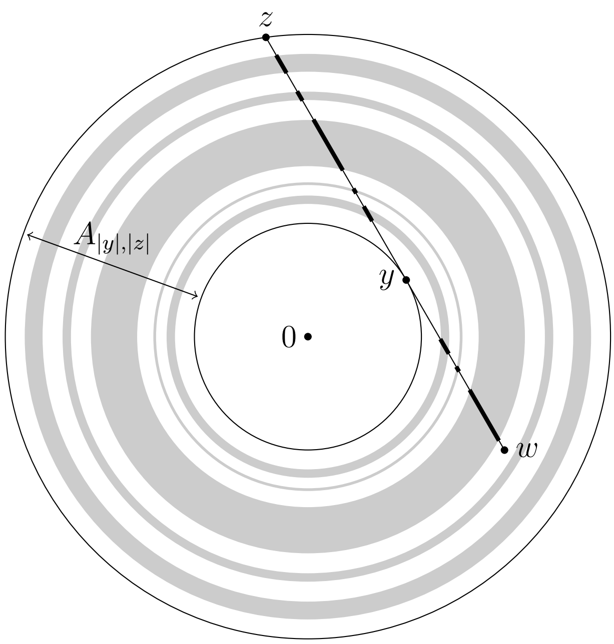
$$\left(\frac{\alpha^{n/2}-1}{\beta^{n/2}-1}\right)^{1/n}\geq\left(\frac{\alpha-1}{\beta-1}\right)^{1/2},$$

or equivalently

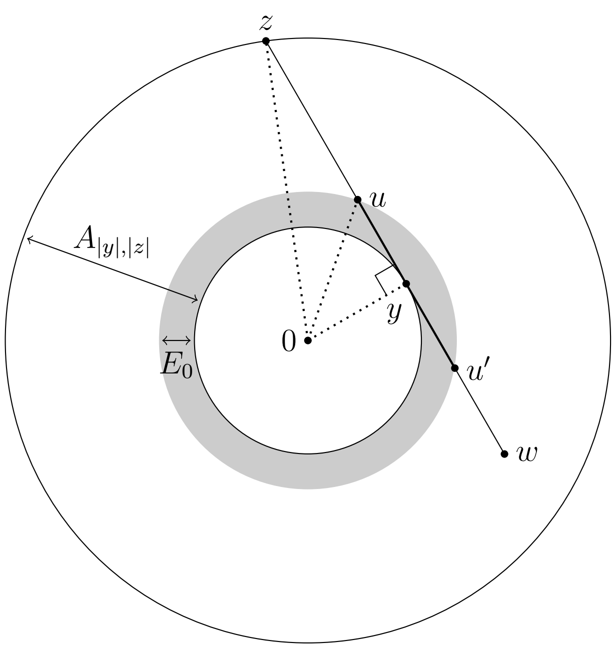
$$(\alpha-1)^{n/2}(\alpha^{n/2}-1)\geq(\beta-1)^{n/2}(\beta^{n/2}-1),$$

which is true for $\alpha>\beta\geq 1$ since $s\mapsto(s-1)^{n/2}(s^{n/2}-1)$ is clearly an increasing function for $s\geq 1$.

CASE 2: $y\neq w$ and $|y-w|\geq\ell/2$.



In this case we choose u in the part of S between y and z and so that $|y - u| = \ell/2$. In the part of S between y and w there is another point u' such that $|y - u'| = \ell/2$. Call S_0 to the segment joining u and u'. Obviously $|S_0|_1 = \ell$ and $A_{|y|,|u|} \cap S = S_0$. To be done with the first step we need to prove that the volume of $A_{|y|,|u|}$ is smaller than that of $A \cap E$.



By orthogonality for every point $v \in S$ we have $|v|^2 = |v-y|^2 + |y|^2$. By the change of variables $t = (s^2 + |y|^2)^{1/2}$

$$|E \cap A|_n = \omega_{n-1} \int_T t^{n-1}\, ds = \omega_{n-1} \int_{T^\star} (s^2 + |y|^2)^{n/2-1} s\, ds,$$

where T and $T^\star$ are defined as before. The second integral again corresponds to calculating the volume of $E \cap A$ by integrating spheres along a path from y to z. We are integrating an increasing function and claiming that $|T^\star|_1 \geq |u-y|$ we have

$$\begin{aligned} \omega_{n-1} \int_{T^\star} (s^2 + |y|^2)^{n/2-1} s\, ds &\geq \omega_{n-1} \int_0^{|u-y|} (s^2 + |y|^2)^{n/2-1} s\, ds \\ &= \omega_{n-1} \int_{|y|}^{|u|} t^{n-1}\, dt = |A_{|y|,|u|}|_n. \end{aligned}$$

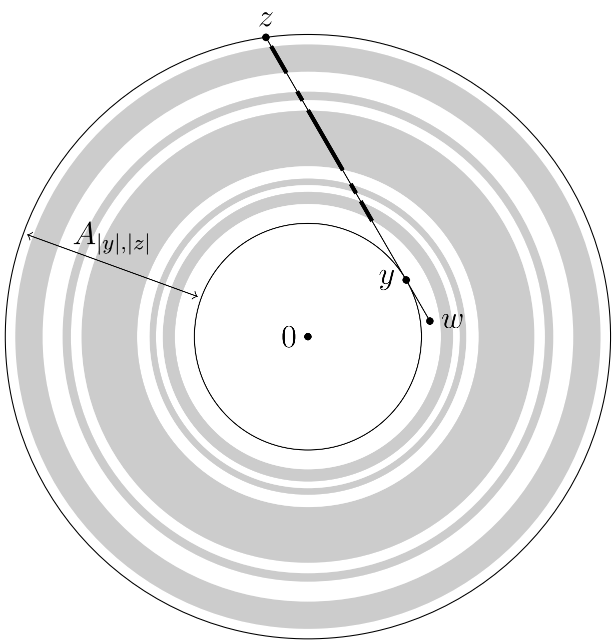
Now we have to justify our claim that $|T^\star|_1 \geq |u-y|$. We prove this by contradiction. Call S_1 to the segment joining w and y and S_2 to the one joining y and z. Obviously $S = S_1 \cup S_2$ is a disjoint union and $|S_1 \cap E|_1 = |T^\star|_1$. Supposing $|T^\star|_1 < |u-y|$ leads us to the contradiction.

$$\begin{aligned} \ell &= |S \cap E|_1 = |S_1 \cap E|_1 + |S_2 \cap E|_1 \leq |T^\star|_1 + |S_2|_1 \\ &< |u-y| + |y-u'| = |u-u'| = \ell. \end{aligned}$$

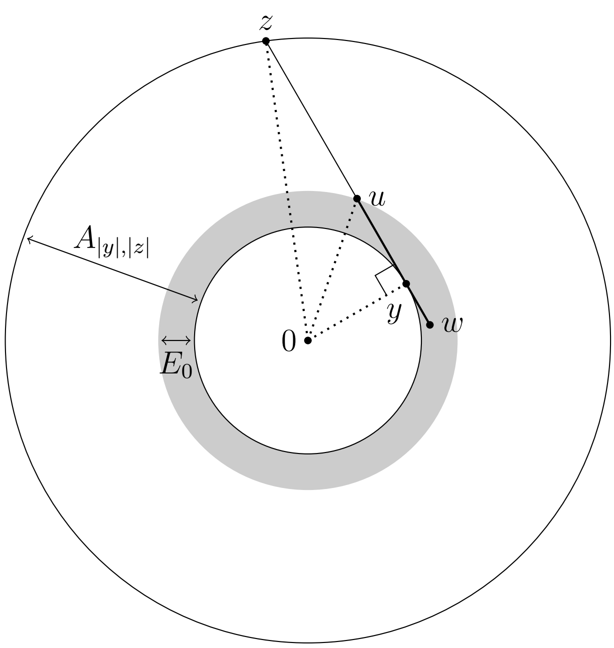
The second step in this case is easy since

$$2\left(\frac{|u|^2 - |y|^2}{|z|^2 - |y|^2}\right)^{1/2} = 2\left(\frac{(\ell/2)^2}{|z-y|^2}\right)^{1/2} = \frac{\ell}{|z-y|} \geq \frac{\ell}{L}.$$

CASE 3: $y \neq w$ and $|y-w| < \ell/2$.



In this case u is the point in S (more precisely between y and z) such that $|w-u| = \ell$. Note that $|u-y| = |u-w| - |y-w| \geq \ell/2$. Calling S_0 to the segment whose extremal points w and u, we have that $A_{|y|,|u|} \cap S = S_0$ and that $|S_0|_1 = \ell$. To complete the first step we have to prove that $|A_{|y|,|u|}|_n \leq |E \cap A|_n$ and this is can be done exactly as in the preceding case.

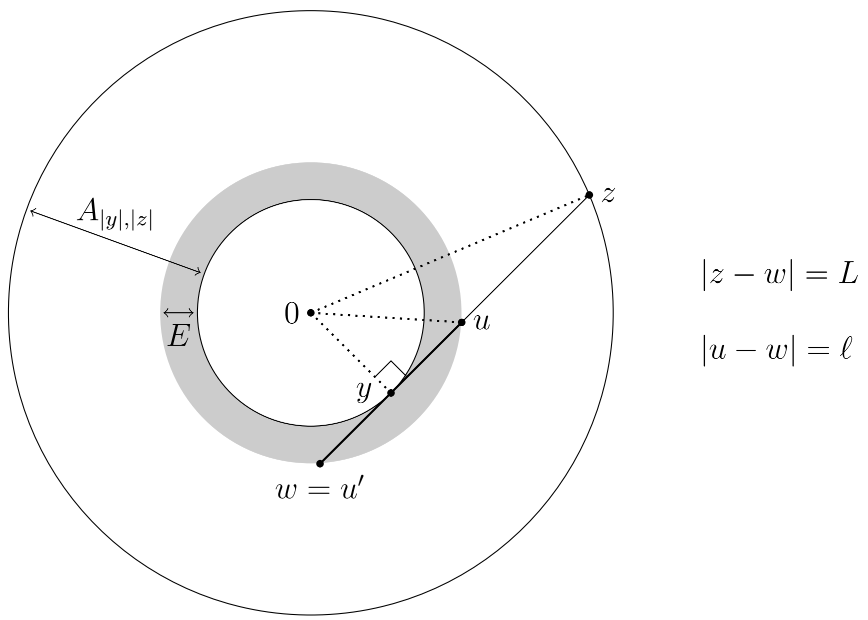


For the second step we use again that $u-y$ and $z-y$ are orthogonal to y and hence

$$2\left(\frac{|u|^2-|y|^2}{|z|^2-|y|^2}\right)^{1/2}=2\frac{|u-y|}{|z-y|}\geq 2\frac{|u-w|-|w-y|}{|z-w|}\geq\frac{\ell}{L}.$$

$\square$

Remark 1.7. *The constant 2 in this Lemma is optimal. To see this, take a segment S in $\mathbb{R}^2$ to be in the second case of the proof of the Lemma and let $E=E_0=A_{|y|,|u|}$ be in such a way that $u'=w$.*



Note that by orthogonality

$$\begin{aligned}\mathcal{A}\chi_E(z) &= \sup_{|y|\leq t\leq |u|}\frac{|A_{t,|z|}\cap E|}{|A_{t,|z|}|}=\sup_{|y|\leq t\leq |u|}\frac{|A_{t,|u|}|}{|A_{t,|z|}|}=\sup_{|y|\leq t\leq |u|}\frac{|u|^2-t^2}{|z|^2-t^2}\\ &= \frac{|u|^2-|y|^2}{|z|^2-|y|^2}=\frac{|u-y|^2}{|z-y|^2}=\frac{(\ell/2)^2}{(L-\ell/2)^2}.\end{aligned}$$

Let $C>0$ be a constant such that

$$\mathcal{K}\chi_E(z)\leq C\mathcal{A}\chi_E(z)^{1/2}=C\frac{\ell/2}{L-\ell/2}. \tag{1.7}$$

We also have

$$\mathcal{K}\chi_E(z) \geq \frac{|S \cap E|_1}{|S|_1} = \frac{\ell}{L},$$

and then inequality (1.7) implies

$$C \geq 2\frac{L - \ell/2}{L}.$$

Since ℓ can be taken as small as wanted, necessarily $C \geq 2$.

1.5 Maximal functions in high dimensions.

The remaining part of this chapter is devoted to the dimension dependence of operator norms of centred maximal functions associated to general convex bodies.

By B we will denote a bounded, convex body in $\mathbb{R}^n$ which is symmetric with respect to the origin, in the sense that if $x \in B$, then also $-x \in B$. Note that given any norm $|\cdot|$ in $\mathbb{R}^n$ and a radius $r > 0$, the ball $B(0, r)$ with respect to the norm is a bounded and convex set in $\mathbb{R}^n$. It is also symmetric with respect to the origin. Conversely, given an open, bounded, convex and symmetric set B, the function

$$x \mapsto |x| := \inf\{r > 0 : x \in rB\},$$

defines a norm in $\mathbb{R}^n$, called Poincaré's norm. Observe that B is $B(0, 1)$ with respect to this norm.

Fixed B, or equivalently a norm in $\mathbb{R}^n$, the maximal operator associated is defined by

$$Mf(x) = M_B f(x) = \sup_{r>0} \frac{1}{|rB|} \int_{rB} |f(x + y)|\, dy,$$

for each locally integrable function f. Sometimes if necessary we will use the notation M_B instead of M in order to make the dependence on B explicit.

It is trivial to prove that $\|Mf\|_{L^\infty(\mathbb{R}^n)} \leq \|f\|_{L^\infty(\mathbb{R}^n)}$, another example of a dimension-free bound.

It is also a well-known fact that although Mf is only integrable if $f \equiv 0$, one has the weak-type bound

$$|\{x \in \mathbb{R}^n : Mf(x) > \lambda\}| \leq \frac{c_{1,n}}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}, \tag{1.8}$$

for certain constant $c_{1,n}$ independent of f and $\lambda > 0$. After that it is standard to obtain via real interpolation the $L^p(\mathbb{R}^n)$ bounds

$$\|Mf\|_{L^p(\mathbb{R}^n)} \leq C_{p,n}\|f\|_{L^p(\mathbb{R}^n)}.$$

for $1 < p < \infty$.

As for the size of the constants involved, the usual proofs of (1.8) using either Vitali's or Besicovitch's covering Lemmas or the Calderón-Zygmund decomposition give constants $C_{1,n}$ growing exponentially with n (see Chapter 3). This causes that the values for $C_{p,n}$ obtained by interpolation may also have an exponential growth.

We consider here the problem of determining if these are the smallest possible constants, and if not, the question of deciding if the smallest possible constants will grow to infinity with the dimension. In this chapter we present some of the most important results giving upper bounds for these constants.

Note that if we have with respect to the dimension a uniform (weak) L^{p_0} bound for certain p_0, by interpolation with the L^∞ case we have indeed uniform L^p bounds for all $p > p_0$.

1.6 The case of Euclidean balls.

This section is devoted to some important results concerning the asymptotic behaviour as $n \to \infty$ of maximal functions associated to Euclidean balls. Here we have the advantage that Euclidean balls are rotation invariant, and that we can compare their characteristic functions with radial kernels for which good bounds are known.

1.6.1 A weak type L^1 inequality: the "n result".

By comparing the maximal operator with the heat semigroup, E.M. Stein and J.O. Strömberg obtained in [74] a much better estimate than the one obtained via Vitali's or Besicovitch's covering lemmas.

Theorem 1.8. *There exists an absolute constant $C > 0$ such that for all $n \in \mathbb{N}$ and all $f \in L^(\mathbb{R}^n)$ one has*

$$|\{x \in \mathbb{R}^n : Mf(x) > \lambda\} \leq \frac{Cn}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}. \tag{1.9}$$

We must say at this point that the problem of determining whether the weak type L^1 norm of M grows or not to infinity with the dimension is still open.

Proof. The idea of the proof is that the mean value of f over a ball is not very different from the convolution with the heat kernel and so, the maximal ergodic theorem can be applied. We will control Mf by the maximal operator

$$H^*f(x) = \sup_{t>0} \frac{1}{t}\int_0^t |G_s * f(x)|\, ds,$$

where G denotes the heat kernel $G(x) = (4\pi)^{-n/2}\mathrm{e}^{|x|^2/4}$, and $G_s(x) = s^{-n}G(x/s)$ $= (4\pi s)^{-n/2}\mathrm{e}^{|x|^2/(4s)}$. By Theorem 1.1 we know that

$$|\{x \in \mathbb{R}^n : H^*f(x) > \lambda\}| \leq \frac{1}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}.$$

In view of this, all we need to show is that there exist a constant $A > 0$ independent of n and f such that $Mf \leq An\, H^*f$. Of course it is enough to do so for a non-negative f, since otherwise we can separate f as a linear combination the positive and negative parts of its real and imaginary parts. Note that

$$\frac{1}{t}\int_0^t G_s * f(x)\, ds = H^t * f(x),$$

where $H^t(x) = 1/t\int_0^t G_s(x)\, ds$. Let x be an arbitrary point in $\mathbb{R}^n$, we have to see that for each $R > 0$ there exist a $t > 0$ so that

$$\frac{1}{|B_R|}\chi_R * f(x) \leq An\, H^t * f(x).$$

Since we are assuming that f is non-negative it is enough to check that

$$\frac{1}{|B_R|}\chi_R(y) \leq An\, H^t(y),$$

for all $y \in \mathbb{R}^n$. Actually, we just need to prove this for $R = 1$, i.e.

$$\frac{1}{|B_1|}\chi_1(y) \leq An\, H^t(y), \tag{1.10}$$

since dilating both sides of this inequality by an arbitrary $R > 0$ we get

$$\frac{1}{|B_R|}\chi_R(y) \leq An\, H^t_R(y) = \frac{An}{R^n} H^t(y/R) = An\, H^{tR^2}(y).$$

Note that (1.10) is trivial when $|y| > 1$. If (1.10) is true when $|y| = 1$, for $|y| < 1$, calling $y' = y/|y|$ one also has

$$\frac{1}{|B_1|}\chi_1(y) = \frac{1}{|B_1|}\chi_1(y') \leq An\, H^t(y') \leq An\, H^t(y),$$

since H^t is radially decreasing. Thus we only need to prove (1.10) when $|y| = 1$, i.e.

$$\frac{1}{|B_1|} \leq \frac{An}{t}\int_0^t \frac{1}{(4\pi s)^{n/2}}\, \mathrm{e}^{-1/4s}\, ds. \tag{1.11}$$

To estimate the integral on the right hand side we make the change of variables $1/4s = u$ to obtain

$$\begin{aligned}
\int_0^t \frac{1}{(4\pi s)^{n/2}}\, \mathrm{e}^{-1/4s}\, ds &= \frac{1}{4\pi^{n/2}}\int_{1/4t}^{\infty} u^{n/2-2}\, \mathrm{e}^{-u}\, du \\
&= \frac{1}{4\pi^{n/2}}\left(\int_0^{\infty} u^{n/2-2}\, \mathrm{e}^{-u}\, du - \int_0^{1/4t} u^{n/2}\, e^{-u}\, du\right) \\
&= \frac{1}{4\pi^{n/2}}\left(\Gamma(n/2-1) - \int_0^{1/4t} h(u)\, du\right),
\end{aligned}$$

where $h(u) = u^{n/2-2}\,\mathrm{e}^{-u}$. It is simple to see that $h'(u) > 0$ if $u \in (0, n/2-2)$. Then h increases in that interval and assuming $1/4t < n/2 - 2$ one can estimate

$$\int_0^{1/4t} h(u)\, du \leq \int_0^{1/4t} h(1/4t)\, du = \left(\frac{1}{4t}\right)^{n/2-1} \mathrm{e}^{-1/4t}.$$

Thus taking $t = 1/n$ (which is allowed since $1/4t = n/4 < n/2 - 2$) we obtain

$$\begin{aligned}
\frac{1}{t}\int_0^t \frac{1}{(4\pi s)^{n/2}}\, \mathrm{e}^{-1/4s}\, ds &\geq \frac{1}{4t\pi^{n/2}}\left(\Gamma(n/2-1) - \left(\frac{1}{4t}\right)^{n/2-1} \mathrm{e}^{-1/4t}\right) \\
&= \frac{n}{4\pi^{n/2}}\left(\Gamma(n/2-1) - \left(\frac{n}{4}\right)^{n/2-1} \mathrm{e}^{-n/4}\right) \\
&\geq Cn\frac{\Gamma(n/2-1)}{\pi^{n/2}}.
\end{aligned} \tag{1.12}$$

The last inequality is justified because

$$\left(\frac{n}{4}\right)^{n/2-1} \mathrm{e}^{-n/4} = o\left(\Gamma(n/2-1)\right),$$

which is easy to see in view of the Stirling's formula:

$$\Gamma(t) = \sqrt{\frac{2\pi}{t}} \left(\frac{t}{e}\right)^t (1 + O(1/t)),$$

On the other hand we have

$$\frac{1}{|B_1|} = \frac{n}{2}\frac{\Gamma(n/2)}{\pi^{n/2}} = \frac{n}{2}\left(\frac{n}{2} - 1\right)\frac{\Gamma(n/2-1)}{\pi^{n/2}},$$

and this together with (1.12) imply (1.11). □

1.6.2 Radial functions.

The problem about uniform bounds is completely solved if we restrict our attention to radial functions. In [54], M.T. Menárguez and F. Soria obtained the following dimension-free estimate.

Theorem 1.9. *If f is a radial function over $\mathbb{R}^n$ one has*

$$|\{x \in \mathbb{R}^n \ : \ Mf(x) > \lambda\}| \leq \frac{4}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}.$$

We will not give the proof of this theorem here, since in Chapter 5 we will prove a more general result containing this as a particular case. Instead, we will focus on a related result.

As pointed out by J.M. Aldaz and J. Pérez Lázaro [5], the constant in the previous inequality is 1 if we restrict further to positive and radially decreasing functions.

Theorem 1.10 (J.M Aldaz, J. Pérez Lázaro)**.** *Let f be a non-negative and radially decreasing function, then*

$$|\{x \in \mathbb{R}^n \ : \ Mf(x) > \lambda\}| \leq \frac{1}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}.$$

Note that this result is sharp in the sense that the constant 1 cannot be replaced by anything smaller.

We will prove Theorem 1.10 using essentially the same arguments as in [5]. We start comparing the action of M on radially decreasing functions with the one of the Hardy operator. We define the Hardy operator as

$$\mathcal{H}f(x) := \frac{1}{|B_{|x|}|}\int_{B_{|x|}} |f(x)|\, dx,$$

for a locally integrable function f. Theorem 1.10 is an immediate corollary of the following Lemmas.

Lemma 1.11. *Given an integrable function f over $\mathbb{R}^n$ one has*

$$|\{x \in \mathbb{R}^n \ : \ \mathcal{H}f(x) > \lambda\}| \leq \frac{1}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}.$$

Lemma 1.12. *For any integrable and radially decreasing function f one has $Mf \leq \mathcal{H}f$.*

Proof of Lemma 1.11. Consider the set $E_\lambda = \{x \in \mathbb{R}^n \ : \ \mathcal{H}f(x) > \lambda\}$. For each $x \in E_\lambda$ there exist a ball B_x centred at the origin such that

$$\frac{1}{|B_x|}\int_{B_x} |f(x)|\, dx > \lambda.$$

Note that $B_x \subset E_\lambda$ and that

$$|B_x| \leq \frac{1}{\lambda}\int_{B_x} |f(x)|\, dx \leq \frac{1}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}.$$

Since $E_\lambda = U_{x\in E_\lambda} B_x$, we have that E_λ is a ball centred at the origin, i.e. $E_\lambda = B_R$ for certain radius $R \geq 0$. By monotonicity

$$|E_\lambda| = |B_R| \leq \frac{1}{\lambda}\|f\|_{L^1(\mathbb{R}^n)}.$$

□

Proof of Lemma 1.12. Given a radially decreasing and integrable function f it is enough to prove that for each $x \in \mathbb{R}^n$ and $r > 0$ one has

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} f(x)\,dx \leq \frac{1}{|B_{|x|}|}\int_{B_{|x|}} f(x)\,dx.$$

By a density argument, it is enough to prove this for a simple function of the form $f = \sum_{i=1}^{N} c_i \chi_{B_i}$, where $c_i > 0$ and B_i is a ball centred at the origin for $i = 1, \ldots, N$. In order to do so, by linearity, it suffices to check it for the characteristic function of a ball centred at the origin. Taking $f = \chi_{B_R}$ for some $R \geq 0$, $x \in \mathbb{R}^n$ and $r > 0$, we want to prove

$$\frac{|B(x,r) \cap B_R|}{|B(x,r)|} \leq \frac{|B_{|x|} \cap B_R|}{|B_{|x|}|}.$$

If $|x| < R$ this inequality is trivial because the right hand side equals 1. If $|x| > r + R$ the inequality is also trivial because the left hand side is 0. Assuming that $R < |x| < r + R$, the case that $r > |x|$ is also easy because then also $r > R$ and we have

$$\frac{|B(x,r) \cap B_R|}{|B(x,r)|} \leq \frac{|B_R|}{|B_r|} \leq \frac{|B_R|}{|B_{|x|}|} = \frac{|B_{|x|} \cap B_R|}{|B_{|x|}|}.$$

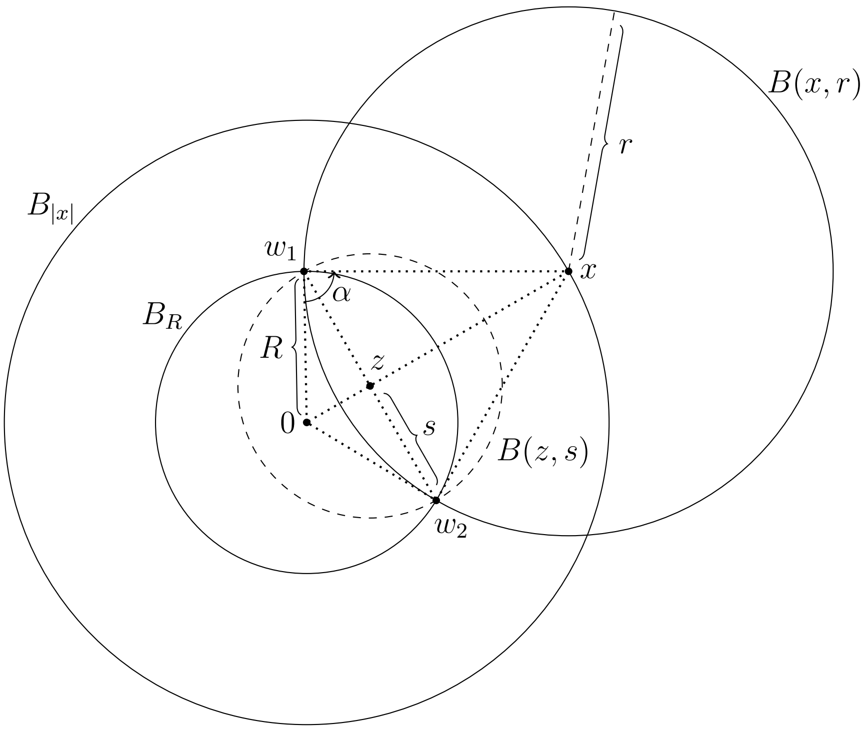
For the remaining case consider the minimal ball $B(z,s)$ containing $B(x,r) \cap B_R$. It is enough to prove that

$$\frac{|B(z,s)|}{|B(x,r)|} \leq \frac{|B_R|}{|B_{|x|}|},$$

or equivalently that

$$s|x| \leq rR.$$

Note that $S := \partial B(x,r) \cap \partial B_R \subset \partial B(z,s)$ is an $n-2$-dimensional sphere. We take w_1 and w_2 to be two points diametrically opposed in S.



Then, $s|x| = |w_1 - z||x|$ is the 2-dimensional area of the quadrilateral A whose vertices are 0, w_1, z and w_2. We call α to the angle determined at w_1 by the segments joining this point with x and with the origin. The area of A can also be expressed as $|w_1||w_1 - x| \sin\alpha = Rr\sin\alpha$. Hence, we have $s|x| = Rr\sin\alpha \leq Rr$.

□

1.6.3 L^p for $p > 1$.

E.M. Stein was the first one to realise that the L^p constants of the maximal operator associated to Euclidean balls do admit a bound independent of the dimension. This was announced in [70] and the details were given in [74]. He used a method of rotations combined with previous results on the spherical maximal operator to prove the following.

Theorem 1.13. *Let $p > 1$. Then there exists a universal constant C_p only depending on p such that*

$$\|Mf\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}, \tag{1.13}$$

for every $f \in L^p(\mathbb{R}^n)$.

As we have just said, the spherical maximal operator will arise in the proof of this Theorem. Let us briefly introduce it. Let $n \geq 2$, for a suitable smooth function we can define a maximal function where the means are taken over $n-1$-dimensional spheres instead of solid balls:

$$\mathcal{M}_n f(x) = \sup_{r>0} \frac{1}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} |f(x+ry)|\, dy.$$

This is called the spherical maximal function. We will employ the following result, proved by E.M. Stein [69] in the case $n \geq 3$ and by J. Bourgain [13] for $n = 2$.

Theorem 1.14. *Let $n \geq 2$ and $p > n/(n-1)$, then one has*

$$\|\mathcal{M}_n f\|_{L^p(\mathbb{R}^n)} \leq A_{n,p} \|f\|_{L^p(\mathbb{R}^n)},$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$, with $A_{n,p} > 0$ a constant that only depends on n and p.

This allows to define the maximal spherical operator over functions in $L^p(\mathbb{R}^n)$ with $p > n/(n-1)$.

We will also use the following technical lemma. Roughly speaking, it asserts that an integral mean over a ball in $\mathbb{R}^n$ can be transformed into an integral mean over a ball in $\mathbb{R}^k$ combined with all possible rotations in $\mathbb{R}^n$.

Lemma 1.15. *Let $k < n$ be natural numbers. For each $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$ we call $x_1 = (x^1, \ldots, x^k) \in \mathbb{R}^k$ and $x_2 = (x^{k+1}, \ldots, x^n) \in \mathbb{R}^{n-k}$. By abuse of the language we will write $x = (x_1, x_2)$. For any positive and measurable function f on $\mathbb{R}^n$ one has*

$$\frac{\int_{|x|<R} f(x)\, dx}{\int_{|x|<R} dx} = \frac{\int_{SO(\mathbb{R}^n)} \int_{|x_1|<R} f(\tau(x_1, 0)) |x_1|^{n-k}\, dx_1}{\int_{|x_1|<R} |x_1|^{n-k}\, dx_1},$$

where $SO(\mathbb{R}^n) = \{\tau \in \mathcal{M}_{n\times n}(\mathbb{R}) : \exists \tau^{-1} = \tau^t\}$, i.e. the special orthonormal group in $\mathbb{R}^n$.

First we show how to prove Theorem 1.13 using Lemma 1.15, then we will prove the Lemma.

Proof of Theorem 1.13. Take $p > 1$ and $k > \frac{p}{p-1}$, which is equivalent to $p > k/(k-1)$. We shall see first that the mean value of $|f|$ can be bounded in terms of a spherical maximal function. By Lemma 1.15

$$\begin{aligned} Mf(x) &= \sup_{R>0} \frac{1}{B_R} \int_{B_R} |f(x-y)|\, dy \\ &= \sup_{R>0} \frac{\int_{SO(\mathbb{R}^n)} \int_{|y_1|<R} |f(x-\tau(y_1,0))|\, |y_1|^{n-k}\, dy_1\, d\tau}{\int_{|y_1|} |y_1|^{n-k}\, dy_1}. \end{aligned}$$

For a function $g : \mathbb{R}^n \to \mathbb{R}$, we define $g_{x_2} : \mathbb{R}^k \to \mathbb{R}$ for each $x_2 \in \mathbb{R}^{n-k}$ as $g_{x_2}(x_1) = g(x_1, x_2)$. With this notation

$$Mf(x) = \sup_{R>0} \frac{\int_{SO(\mathbb{R}^n)} \int_{|y_1|<R} |(f\circ\tau)_{\tau^{-1}(x)_2}(\tau^{-1}(x)_1 - y_1)|\, |y_1|^{n-k}\, dy_1\, d\tau}{\int_{|y_1|<R} |y_1|^{n-k}\, dy_1},$$

and changing to the polar coordinates of $\mathbb{R}^k$ $Mf(x)$ can be written as

$$\begin{aligned} &\sup_{R>0} \int_{SO(\mathbb{R}^n)} \frac{\int_0^R \int_{\mathbb{S}^{k-1}} |(f\circ\tau)_{\tau^{-1}(x)_2}(\tau^{-1}(x)_1 - ry_1')\, d\sigma_{k-1}(y_1') r^{n-1}\, dr}{\omega_{k-1}\int_0^R r^{n-1}\, dr}\, d\tau \\ &\leq \sup_{R>0} \int_{SO(\mathbb{R}^n)} \frac{\int_0^R \mathcal{M}_k(f\circ\tau)_{\tau^{-1}(x)_2}(\tau^{-1}(x)_1) r^{n-1}\, dr}{\int_0^R r^{n-1}\, dr}\, d\tau \\ &= \int_{SO(\mathbb{R}^n)} \mathcal{M}_k(f\circ\tau)_{\tau^{-1}(x)_2}(\tau^{-1}(x)_1)\, d\tau. \end{aligned} \tag{1.14}$$

In order to estimate the L^p norm of Mf we will use Minkowski's inequality

$$\begin{aligned} \|Mf\|_{L^p} &\leq \int_{SO(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |\mathcal{M}_k(f\circ\tau)_{\tau^{-1}(x)_2}(\tau^{-1}(x)_1)|^p\, dx \right)^{1/p} d\tau \\ &= \int_{SO(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |\mathcal{M}_k(f\circ\tau)_{y_2}(y_1)|^p\, dy_1\, dy_2 \right)^{1/p} d\tau, \end{aligned}$$

and since $k > \frac{p}{p-1}$ we can also apply Theorem 1.14 in order to get

$$\begin{aligned} \|Mf\|_{L^p} &\leq \int_{SO(\mathbb{R}^n)} \left(\int_{\mathbb{R}^{n-k}} A_{k,p}^p \int_{\mathbb{R}^k} |(f\circ\tau)_{y_2}(y_1)|^p\, dy_1\, dy_2 \right)^{1/p} d\tau \\ &= A_{k,p}\|f\|_{L^p}. \end{aligned}$$

□

Proof of Lemma 1.15. By a density argument it is enough to prove the result for functions of the type $f(x) = F(|x|)g(x')$. For such a function one has

$$\frac{\int_{|x|<R} f(x)\, dx}{\int_{|x|<R} dx} = \frac{n}{R^n \omega_{n-1}} \int_0^R F(r) r^{n-1}\, dr \int_{\mathbb{S}^{n-1}} g(x')\, d\sigma_{n-1}(x'),$$

and on the other hand

$$\frac{\int_{SO(\mathbb{R}^n)} \int_{|x_1|<R} f(\tau(x_1,0)) |x_1|^{n-k}\, dx_1}{\int_{|x_1|<R} |x_1|^{n-k}\, dx_1},$$

equals

$$\frac{n}{R^n \omega_{k-1}} \int_0^R F(r) r^n\, dr \int_{SO(\mathbb{R}^n)} \int_{\mathbb{S}^{k-1}} g(\tau(x_1',0))\, d\sigma_{k-1}(x_1')\, d\tau.$$

So we just have to show that

$$\frac{1}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} g(x')\, d\sigma_{n-1}(x') = \frac{1}{\omega_{k-1}} \int_{SO(\mathbb{R}^n)} \int_{\mathbb{S}^{k-1}} g(\tau(x_1',0))\, d\sigma_{k-1}(x_1'). \tag{1.15}$$

This equality holds for characteristic functions of measurable sets on $\mathbb{S}^{n-1}$, since given a measurable $A \subset \mathbb{S}^{n-1}$, both

$$⨍_{\mathbb{S}^{n-1}} \chi_A(x')\, d\sigma_{n-1}(x') \quad \text{and} \quad \int_{SO(\mathbb{R}^n)} ⨍_{\mathbb{S}^{k-1}} \chi_A(\tau(x_1',0))\, d\sigma_{k-1}(x_1'),$$

define a unitary and rotation invariant measure on S^{n-1}, which is known to be unique. So, by linearity, (1.15) holds for simple functions and by density for general functions. □

1.7 Maximal operators associated to general convex bodies.

Now we turn to the case of maximal operators associated to general convex bodies. We assume B to be a bounded, convex body in $\mathbb{R}^n$ that is symmetric with respect to the origin and we consider the associated maximal operator M_B. The goal is to bound the L^p operator norm of M_B independently of n and if possible of B.

1.7.1 A weak type L^1 inequality: the "$n \log n$ result".

The following result, due to E.M. Stein and J.O. Strömberg [74], gives an estimate for the weak type L^1 that substantially improves the one obtained through the Besicovitch covering lemma.

Theorem 1.16. *For every $f \in L^1(\mathbb{R}^n)$,*

$$|\{x \in \mathbb{R}^n : Mf(x) > \lambda\}| \leq \frac{Cn \log n}{\lambda} \|f\|_{L^1(\mathbb{R}^n)}, \tag{1.16}$$

where $C > 0$ is an absolute constant not depending on the dimension n, neither on the underlying norm.

Later on, M.T. Menárguez and F. Soria [54] gave a different proof, based on the following principle: M is bounded by the supremum of at most $\mathcal{O}(n \log n)$ operators each of them satisfying a dimension-free weak type L^1 estimate. In Chapter 5 we will follow this method to prove a generalization of Theorem 1.16 as well as an extension of a recent result by A. Naor and T. Tao [63].

1.7.2 L^p bounds for $p > 1$

The first complete result for general convex bodies was presented by J. Bourgain in [10]. His result states that M_B is uniformly bounded on $L^2(\mathbb{R}^n)$. More precisely, we have the following

Theorem 1.17. *Let B be an open, convex and symmetric body. There exists an absolute constant C independent of the dimension and B, such that*

$$\|M_B f\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)},$$

for every $f \in L^2(\mathbb{R}^n)$.

This result was extended by J. Bourgain in [11], [12] and, independently, by A. Carbery in [17] to L^p whenever $p > 3/2$.

Theorem 1.18. *Let B be an open, convex and symmetric body. Let $f \in L^p(\mathbb{R}^n)$, with $p > 3/2$. Then*

$$\|M_B f\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)},$$

where the constant C_p might depend on p but not on n or B.

The main tools used to prove these results are Fourier analysis, geometry of convex bodies and fractional calculus. In this thesis we will present a proof of Theorem 1.17 using arguments inspired by the approach in [17]. To be more precise we will use a general principle appeared in [17] to prove the L^2 boundedness of a maximal convolution operator and some estimates from [10]. The proof of Theorem 1.18 requires some extensions of these results, together with a semi-orthogonality principle for maximal operators.

To begin with, we will need some preliminaries on geometrical properties of the body B, on the function $N = \chi_B$ and its Fourier transform as well as some concepts on fractional calculus. This will be presented in the following two technical sections.

1.7.3 Fractional Calculus.

We want to develop a differential calculus that includes arbitrary orders of differentiation for functions. Although there are many ways to do this, let us briefly introduce the approach based on the Fourier transform. In order to define derivatives of an arbitrary order recall the nice relation between derivatives and Fourier transforms. It is well-known that if f is a good function, for instance in the Schwartz class, one has

$$\left(\frac{d}{dx} f\right)^{\wedge}(\xi) = 2\pi i \xi \hat{f}(\xi),$$

and in general for any positive integer n

$$\left(\left(\frac{d}{dx}\right)^n f\right)^{\wedge}(\xi) = (2\pi i \xi)^n \hat{f}(\xi).$$

It seems natural for $\alpha \in \mathbb{R}$ with $\alpha > 0$ to define the derivative of order α by

$$\left(\left(\frac{d}{dx}\right)^\alpha f\right)^{\wedge}(\xi) = (2\pi i \xi)^\alpha \hat{f}(\xi). \tag{1.17}$$

In this way, we will say that $f \in L^2$ has α derivatives if $|\cdot|^\alpha \hat{f} \in L^2$, and if needed define Sobolev spaces of fractional order. Of course the order α does not have to be a fractional number, but the name fractional has been kept historically.

We are also interested in defining the inverse operator of the fractional derivative, that will be called by obvious reasons the fractional integral. Consider a function $f \in \mathcal{S}(\mathbb{R})$, integrating by parts repeatedly we obtain

$$\begin{aligned} f(x) &= -\int_x^\infty \frac{d}{dt} f(t)\,dt = \int_x^\infty (t-x)\left(\frac{d}{dt}\right)^2 f(t)\,dt \\ &= -\int_x^\infty \frac{(t-x)^2}{2}\left(\frac{d}{dt}\right)^3 f(t)\,dt = \cdots \\ &= \frac{(-1)^n}{(n-1)!}\int_x^\infty (t-x)^{n-1}\left(\frac{d}{dt}\right)^n f(t)\,dt. \end{aligned}$$

Thus, over regular functions the operator I^n defined by

$$I^n h(x) = \frac{(-1)^n}{(n-1)!}\int_x^\infty (t-x)^{n-1} h(t)\,dt,$$

and called the integral of order n inverts the derivative of order n. Then it seems natural to define I^α, the fractional integral of order α, as

$$I^\alpha h(x) = \frac{(-1)^\alpha}{\Gamma(\alpha)}\int_x^\infty (y-x)^{\alpha-1} h(y)\,dy,$$

that is, as the convolution with the function

$$\varphi(x) = \frac{(-1)^\alpha(-x)_+^{\alpha-1}}{\Gamma(\alpha)} = \begin{cases} (-1)^\alpha |x|^{\alpha-1}/\Gamma(\alpha) & \text{if } x < 0, \\ 0 & \text{if } x \geq 0. \end{cases}$$

Since the definition of fractional derivative based on the Fourier transform is not easy to handle when considering concrete cases, we will give an alternative definition. We will focus on the case that the order α is between 0 and 1. We then define the fractional derivative of order α as the result of integrating with order $1-\alpha$ and then apply the usual derivative, that is

$$\left(\frac{d}{dt}\right)^\alpha f(t) = \frac{d}{dt} I^{1-\alpha} f(t) = \frac{(-1)^{1-\alpha}}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^\infty \frac{f(s)}{(s-t)^\alpha}\,ds. \tag{1.18}$$

This definition has sense for bounded f with the decay $|f(t)| \leq C|t|^{\alpha-1-\varepsilon}$ for an $\varepsilon > 0$. By using identity (6) it is easy to see that for regular functions this definition and the one given by (1.17) coincide.

Next we check that for regular functions the fractional integral is the inverse of the fractional derivative defined this way.

Lemma 1.19. *Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and such that f and f' are bounded and $|f(x)| \leq C/|x|^{1-\alpha+\varepsilon}$ for some $\varepsilon > 0$ and $|f'(x)| \leq C/|x|$. Then we have*

$$I^\alpha \left(\frac{d}{dx}\right)^\alpha f(x) = f(x).$$

The proof is easy using formulas (5) and (6).

We shall introduce now another fractional differential operator as an extension of the operator $\langle \nabla, \cdot \rangle f(x) \langle \nabla f(x), x \rangle$. Note that

$$\langle \nabla f(x), x \rangle = \frac{d}{dt} f(tx)\Big|_{t=1} = \int_{\mathbb{R}^n} 2\pi i \langle x, \xi \rangle \, \check{f}(\xi) \, \mathrm{e}^{-2\pi i \langle x, \xi \rangle} \, d\xi.$$

At first for $\alpha \in [0, 1]$ and $f \in \mathcal{S}(\mathbb{R}^n)$ we define $\langle \nabla, \cdot \rangle^\alpha$ by

$$\langle \nabla, \cdot \rangle^\alpha f(x) = \left(\frac{d}{dt}\right)^\alpha f(tx)\Big|_{t=1} = \int_{\mathbb{R}^n} (-2\pi i \langle x, \xi \rangle)^\alpha \check{f}(\xi) \, \mathrm{e}^{-2\pi i \langle x, \xi \rangle} \, d\xi. \quad (1.19)$$

It is easy to see that the last equality holds employing (6).

We will still use another different expression for this operator when acting on differentiable functions:

$$\begin{aligned} \left(\frac{d}{dt}\right)^\alpha f(tx)\Big|_{t=1} &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^\infty \frac{f(sx)}{(s-t)^\alpha} \, ds\Big|_{t=1} \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^\infty \frac{f((r+t)x)}{r^\alpha} \, dr\Big|_{t=1} \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^\infty \frac{\langle \nabla f((r+t)x), x \rangle}{r^\alpha} \, dr\Big|_{t=1} \\ &= \frac{1}{\Gamma(1-\alpha)} \int_1^\infty \frac{\langle \nabla f(sx), sx \rangle}{(s-1)s^\alpha} \, ds. \end{aligned} \quad (1.20)$$

The third inequality is justified by dominated convergence. To see this note that for $f \in \mathcal{S}$, one has that $|\langle \nabla f(x), x \rangle| \leq |\nabla f(x)|^{1/2} \, |x|^{1/2}$ is uniformly bounded and so

$$\Big| \frac{\langle \nabla f((r+t)x), x \rangle}{r^\alpha} \Big| \leq \frac{C}{(r+t) r^\alpha},$$

that is integrable uniformly in t for t near 1.

1.7.4 Convex and symmetric bodies.

Recall that we are taking B as a convex body, symmetric with respect to the origin and such that $|B| = 1$. In this section we will give bounds for the function $\hat{\chi}_B$ and its derivatives.

We have the following invariance by linear transformations. Note that if $\tau \in SL(\mathbb{R}^n)$ and $B = \tau(B')$, then M_B and $M_{B'}$ have the same $L^p(\mathbb{R}^n)$ operator norm. To see this just make the change $z = \tau^{-1}(y)$ in the definition of $M_B f$ to obtain

$$M_B f(x) = \sup_{R>0} \int_B f(x+Ry)\, dy = \sup_{R>0} \int_{B'} f(x+R\tau(z))\, dy = M_{B'} f(\tau(z)),$$

since $\det \tau = 1$. Thus

$$\|M_B f\|_{L^p(\mathbb{R}^n)} = \|M_{B'} f \circ \tau\|_{L^p(\mathbb{R}^n)} = \|M_{B'} f\|_{L^p(\mathbb{R}^n)}.$$

We say that a convex body B is isotropic if there exists a constant $L > 0$ such that

$$L^2 = \int_B |x \cdot \xi|^2\, dx \tag{1.21}$$

for every $\xi \in \mathcal{S}^{n-1}$. Usually L is referred to as the isotropy constant. From now on we will assume that B is isotropic with constant L, this may seem a strong assumption but every convex body can be transformed into an isotropic one via a linear transformation with determinant 1.

Lemma 1.20. *Let B' be a symmetric and convex body in $\mathbb{R}^n$. There exists a linear transformation $\tau \in SL(\mathbb{R}^n) = \{\tau \in \mathcal{M}_{n\times n} : \mathbb{R}^n \to \mathbb{R}^n \text{ with } \det(\tau) = 1\}$ so that $B = \tau(B')$ is isotropic.*

Proof. We will show that for some $\tau \in SL(\mathbb{R}^n)$ and $L > 0$ we have

$$\int_{\tau(B')} |\langle x, \xi\rangle|^2\, dx = L^2 |\xi|^2,$$

for each $\xi \in \mathbb{R}^n$. This is equivalent to or equivalently

$$\int_{B'} |\langle \tau(y), \xi\rangle|^2\, dy = \int_{B'} |\langle y, \tau^t(\xi)\rangle|^2\, dy = L^2 |\xi|^2,$$

for every ξ. Writing $\tau^*(\xi) = z$, we have to show that

$$\int_{B'} |\langle y, z\rangle|^2\, dy = L^2|(\tau^t)^{-1}(z)|^2.$$

for every z. This equality is possible since both sides define non-degenerate quadratic forms for z, and we can choose $\tau \in SL(\mathbb{R}^n)$ so that they agree up to a constant. $\square$

It will be useful to know some properties of the intersections of B and $n-1$-dimensional hyperplanes. Fixed a unit vector ξ, we define these intersections with the hyperplanes orthogonal to ξ as the sets $D^t = D^t_\xi = \{x \in B : x \cdot \xi = t\}$ for $t \in \mathbb{R}$.

Consider the function $\varphi(t) = \varphi_\xi(t) := |D^t|_{n-1}$, where $|\cdot|_{n-1}$ denotes $n-1$-dimensional Hausdorff measure. It is obvious that φ is a pairwise function over $\mathbb{R}$, moreover $\varphi^{1/(n-1)}$ is concave in its support. Let $a, b > 0$ such that $\varphi(a), \varphi(b) > 0$ and $0 \le \alpha \le 1$, since B is convex one has that $\alpha D^a + (1-\alpha)D^b \subset D^{\alpha a + (1-\alpha)b}$, then by Brunn's Theorem

$$\begin{aligned}\varphi(\alpha a + (1-\alpha)b)^{\frac{1}{n-1}} &= \left|D^{\alpha a+(1-\alpha)b}\right|_{n-1}^{\frac{1}{n-1}} \ge |\alpha D^a + (1-\alpha)D^b|_{n-1}^{\frac{1}{n-1}} \\ &\ge |\alpha D^a|_{n-1}^{\frac{1}{n-1}} + |(1-\alpha)D^b|_{n-1}^{\frac{1}{n-1}} \\ &= \alpha\, \varphi(a)^{\frac{1}{n-1}} + (1-\alpha)\varphi(b)^{\frac{1}{n-1}}.\end{aligned}$$

It is also easy to see that φ is radially decreasing. If $0 < a < b$, take $\alpha \in (0,1)$ so that $\alpha(-b) + (1-\alpha)b = a$. As B is convex, $\alpha D^{-b} + (1-\alpha)D^b \subset D^a$. Then by Brunn's Theorem again

$$\begin{aligned}\varphi(a)^{1/(n-1)} &= |D^a|_{n-1}^{1/(n-1)} \ge |\alpha D^{-b} + (1-\alpha)D^b|_{n-1}^{1/(n-1)} \\ &\ge \alpha|D^{-b}|_{n-1}^{1/(n-1)} + (1-\alpha)|D^b|_{n-1}^{1/(n-1)} = \varphi(b)^{1/(n-1)}.\end{aligned}$$

Thus, φ is differentiable almost everywhere.

We will use the following lemmas appeared in [10]. The first one gives an upper bound for the decay of φ_ξ independent of ξ, n and B.

Lemma 1.21. *Let $n \ge 2$ and $B \subset \mathbb{R}^n$ be a convex body as above, then for every $\xi \in \mathbb{S}^{n-1}$ and $t \in \mathbb{R}$ one has $\varphi_\xi(t) \le e\varphi_\xi(0)\, \mathrm{e}^{\phi(0)|t|/e}$.*

The second one asserts that the size of the $n-1$-dimensional sections of the body B containing the origin is controlled by the inverse of isotropy constant:

Lemma 1.22. *There exist constants $a, A > 0$ independent of ξ, n and B so that*

$$0 < a \leq L\,\varphi_\xi(0) \leq A < \infty, \tag{1.22}$$

for every $\xi \in \mathcal{S}^{n-1}$.

The following set of estimations is derived from the previous lemmas in [10].

Lemma 1.23. *If $N = \chi_B$, for every non-zero $\xi \in \mathbb{R}^n$ one has*

i) $|\hat{N}(\xi)| \leq \dfrac{C}{|\xi| L}$,

ii) $|\hat{N}(\xi) - 1| \leq C|\xi|L$,

iii) $|\langle \nabla \hat{N}(\xi), \xi \rangle| \leq C$,

where C always denotes an absolute constant.

We introduce the following extension of part $iii)$ for the fractional operator $\langle \nabla, \cdot \rangle^\alpha$.

Lemma 1.24. *If $N = \chi_B$, and $0 \leq \alpha \leq 1$ for every non-zero $\xi \in \mathbb{R}^n$ one has*

$$|\langle \nabla, \cdot \rangle^\alpha \hat{N}(\xi)| \leq C \min\{(L|\xi|)^\alpha, (L|\xi|)^{\alpha-1}\},$$

where $C > 0$ is an absolute constant.

Proof. First we write

$$\langle \nabla, \cdot \rangle^\alpha \hat{N}(\xi) = \int_B (2\pi i \langle x, \xi \rangle)^\alpha \mathrm{e}^{-2\pi i \langle x, \xi \rangle}\, dx = C|\xi|^\alpha \int_{-\infty}^{\infty} t^\alpha \varphi(t) \mathrm{e}^{-2\pi i |\xi| t}\, dt,$$

where we called $t = \langle x, \xi' \rangle$ and as a consequence $\langle x, \xi \rangle = |\xi| t$. On the one hand

$$\begin{aligned} |\langle \nabla, \cdot \rangle^\alpha \hat{N}(\xi)| &\leq C|\xi|^\alpha \int_0^\infty t^\alpha \varphi(t)\, dt = C|\xi|^\alpha \int_0^\infty t^\alpha \varphi(0) \mathrm{e}^{-\varphi(0)t/e}\, dt \\ &= C \frac{|\xi|^\alpha}{\varphi(0)^\alpha} \int_0^\infty u^\alpha \mathrm{e}^{-u}\, du \leq C(L|\xi|)^\alpha. \end{aligned}$$

On the other one, integrating by parts

$$\int_{-\infty}^{\infty} t^{\alpha}\varphi(t)\mathrm{e}^{-2\pi i t|\xi|}\,dt = \int_{-\infty}^{\infty} (\alpha t^{\alpha-1}\varphi(t) + t^{\alpha}\varphi'(t))\frac{\mathrm{e}^{-2\pi i t|\xi|}}{2\pi i|\xi|}\,dt.$$

Then, recalling that the sign of φ' remains constant over $(0,\infty)$, since φ is a decreasing function there, and that φ is pairwise, we take absolute values to find

$$\begin{aligned} |\langle\nabla,\cdot\rangle^{\alpha}\hat{N}(\xi)| &\leq C|\xi|^{\alpha-1}\int_0^{\infty} (\alpha t^{\alpha-1}\varphi(t) - t^{\alpha}\varphi'(t))\,dt \\ &= C|\xi|^{\alpha-1}\int_0^{\infty} t^{\alpha-1}\varphi(t)\,dt, \end{aligned}$$

the last equality justified by integrating by parts again. Then, by Lemmas 1.21 and 1.22,

$$\begin{aligned} |\langle\nabla,\cdot\rangle^{\alpha}\hat{N}(\xi)| &\leq C|\xi|^{\alpha-1}\int_0^{\infty} t^{\alpha-1}\varphi(0)\mathrm{e}^{\varphi(0)t/e}\,dt \\ &= C\frac{|\xi|^{\alpha-1}}{\varphi(0)^{\alpha-1}}\int_0^{\infty} u^{\alpha-1}\mathrm{e}^{-u}\,du \leq \frac{C}{(L|\xi|)^{1-\alpha}}. \end{aligned}$$

□

1.7.5 A proof for Theorem 1.17.

This section is devoted to giving a proof of Theorem 1.17. We will combine the arguments in [10] and in [17]. The main tool that we will use is the following principle appeared in [17].

Proposition 1.25. *Let $K \in L^1$ be a convolution kernel such that $\hat{K}(s\xi)/s$ verifies the hypothesis of Lemma 1.19 as a function of s. Then we have*

$$\|\sup_{r>1}|K * f|\,\|_{L^2(\mathbb{R}^n)} \leq \sup_{\xi\in\mathbb{S}^{n-1}}\left(\int_0^{\infty}|m_u^{\alpha}(\xi)|^2\frac{du}{u}\right)^{1/2}\|f\|_{L^2(\mathbb{R}^n)},$$

where $m_s^{\alpha}(\xi) = s^{1+\alpha}(d/ds)^{\alpha}(\hat{K}(s\xi)/s)$.

We will apply this result to the kernel $K = N - P_L$, where P is the Poisson kernel, to obtain the bound

$$\|\sup_{t>0}|K_t * f|\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)},$$

with C an absolute constant. Then we can apply the triangle inequality and get

$$\|\sup_{t>0}|N_t*f|\|_{L^2(\mathbb{R}^n)} \leq \|\sup_{t>0}|K_t*f|\|_{L^2(\mathbb{R}^n)}+\|\sup_{t>0}|P_{Lt}*f|\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)},$$

because the Poisson maximal operator is known to be bounded in L^2 independently of n.

We will also need the following estimates, whose proof we leave for the end of the section.

Lemma 1.26. *Let $K = N - P_L$, where P is the Poisson kernel. Then we have the following:*

i) $|\hat{K}(\xi)| \leq C\min(L|\xi|, (L|\xi|)^{-1})$,

ii) $|\langle \nabla \hat{K}(\xi), \xi\rangle| \leq C\min(L|\xi|, 1)$,,

iii) $|\langle \nabla, \cdot\rangle^\alpha \hat{K}(\xi)| \leq C\min((L|\xi|)^\alpha, (L|\xi|)^{\alpha-1})$, *for* $0 < \alpha < 1$.

It is clear that we would be done with Theorem 1.17 if we are able to bound

$$\sup_{\eta\in\mathbb{S}^{n-1}} \int_0^\infty |m_s^\alpha(\eta)|^2 \frac{ds}{s} = \sup_{\eta\in\mathbb{S}^{n-1}} \int_0^\infty \left| s^{\alpha+1}\left(\frac{d}{ds}\right)^\alpha \left(\frac{\hat{K}(s\eta)}{s}\right)\right|^2 \frac{ds}{s},$$

independently of the dimension. For this let us first find an expression for m_s^α without fractional derivatives

$$\begin{aligned} m_s^\alpha(\xi) &= u^{\alpha+1}\left(\frac{d}{du}\right)^\alpha\left(\frac{\hat{K}(u\xi)}{u}\right) \\ &= \frac{u^{\alpha+1}}{\Gamma(\alpha)}\frac{d}{du}\int_u^\infty \frac{\hat{K}(t\xi)}{t(t-u)^\alpha}\,dt \\ &= \frac{u^{\alpha+1}}{\Gamma(\alpha)}\frac{d}{du}\int_1^\infty \frac{\hat{K}(us\xi)}{s(s-1)^\alpha u^\alpha}\,ds \\ &= \frac{1}{\Gamma(\alpha)}\int_1^\infty \frac{\langle\nabla\hat{K}(us\xi), us\xi\rangle - \alpha\hat{K}(us\xi)}{s(s-1)^\alpha}\,ds. \end{aligned}$$

The last equality is justified by dominated convergence since in view of Lemma 1.26

$$\left|\frac{\langle\nabla\hat{K}(us\xi), us\xi\rangle - \alpha\hat{K}(us\xi)}{s(s-1)^\alpha}\right| \leq \frac{C}{s(s-1)^\alpha},$$

that is integrable over $(0,\infty)$. From (1.20) we have now that

$$m_s^\alpha(\xi) = \langle \nabla, \cdot \rangle^\alpha \hat{K}(s\xi) - \frac{\alpha}{\Gamma(\alpha)} \int_1^\infty \frac{\hat{K}(st\xi)}{t(t-1)^\alpha}\, dt. \tag{1.23}$$

Then by (1.23), and the triangle inequality

$$\left(\sup_{\eta \in \mathbb{S}^{n-1}} \int_0^\infty |m_s^\alpha(\xi)|^2 \frac{ds}{s} \right)^{1/2} \leq I_1 + I_2,$$

where

$$\begin{aligned} I_1 &= \left(\sup_{\eta \in \mathbb{S}^{n-1}} \int_0^\infty \left| \langle \nabla, \cdot \rangle^\alpha \hat{K}(s\xi) \right|^2 \frac{ds}{s} \right)^{1/2} \\ I_2 &= \frac{\alpha}{\Gamma(\alpha)} \left(\sup_{\eta \in \mathbb{S}^{n-1}} \int_0^\infty \left| \int_1^\infty \frac{\hat{K}(st\xi)}{t(t-1)^\alpha}\, dt \right|^2 \frac{ds}{s} \right)^{1/2}. \end{aligned}$$

Using part $iii)$ of Lemma 1.26 we can bound I_1 in the following way:

$$I_1 \leq \left(C \int_0^{1/L} (Ls)^{2\alpha} \frac{ds}{s} + C \int_{1/L}^\infty \frac{1}{(Ls)^{2-2\alpha}} \frac{ds}{s} \right)^{1/2} \leq C.$$

To bound I_2 we use Minkowski inequality and then part $i)$ of Lemma 1.26

$$\begin{aligned} I_2 &\leq C \int_1^\infty \left(\int_0^\infty |\hat{K}(st\xi)|^2 \frac{ds}{s} \right)^{1/2} \frac{dt}{t(t-1)^\alpha} \\ &\leq C \int_1^\infty \left(\int_0^{1/Lt} (Lst)^2 \frac{ds}{s} + \int_{1/Lt}^\infty \frac{1}{(Lst)^2} \frac{ds}{s} \right)^{1/2} \frac{dt}{t(t-1)^\alpha} \leq C. \end{aligned}$$

This finishes the proof of Theorem 1.17. It remains yet to provide the proofs of Lemma 1.26 and Proposition 1.25. We finish this section with them.

Proof of Proposition 1.25. On the Fourier side, by Lemma 1.19

$$\begin{aligned} (K_r * f)^\wedge(\xi) &= \hat{K}(r\xi)\hat{f}(\xi) = r\frac{\hat{K}(r\xi)}{r}\hat{f}(\xi) \\ &= \frac{r(-1)^\alpha}{\Gamma(\alpha)} \int_r^\infty \frac{1}{(s-r)^{1-\alpha}} \left(\frac{d}{ds} \right)^\alpha \left(\frac{\hat{K}(s\xi)}{s} \right) ds\, \hat{f}(\xi). \end{aligned}$$

Then we can write

$$K_r * f(x) = \frac{r(-1)^\alpha}{\Gamma(\alpha)} \int_r^\infty \frac{1}{(s-r)^{1-\alpha} s^\alpha} P_s^\alpha f(x) \, \frac{ds}{s}, \tag{1.24}$$

where $(P_s f)^\wedge(\xi) = m_s^\alpha(\xi) \hat{f}(\xi)$. By Hölder inequality

$$\begin{aligned} |K_r * f(x)| &\leq \frac{r}{\Gamma(\alpha)} \left(\int_r^\infty \left| \frac{1}{(s-r)^{1-\alpha} s^\alpha} \right|^2 \frac{ds}{s} \right)^{1/2} \left(\int_r^\infty |P_s^\alpha f(x)|^2 \, \frac{ds}{s} \right)^{1/2} \\ &\leq C_\alpha \left(\int_0^\infty |P_s^\alpha f(x)|^2 \, \frac{ds}{s} \right)^{1/2}, \end{aligned}$$

with the last inequality coming from the following. By the change of integration variable $s = ru$ we have

$$\begin{aligned} \frac{r}{\Gamma(\alpha)} \left(\int_r^\infty \frac{1}{(s-r)^{1-\alpha} s^\alpha} \, \frac{ds}{s} \right)^{1/2} &= \frac{1}{\Gamma(\alpha)} \left(\int_1^\infty \frac{1}{(u-1)^{2-2\alpha} u^{2\alpha+1}} \, du \right)^{1/2} \\ &\leq C_\alpha, \end{aligned}$$

where C_α is a constant that only depends on α. For this integral to be finite it has been necessary the hypothesis $\alpha > 1/2$.

Thus, taking L^2 norms and using Fubini Theorem

$$\begin{aligned} \| \sup_{r>0} |K_r * f| \, \|_{L^2(\mathbb{R}^n)} &\leq C_\alpha \left\| \left(\int_0^\infty |P_s^\alpha f(x)|^2 \, \frac{ds}{s} \right)^{1/2} \right\|_{L^2(\mathbb{R}^n)} \\ &= C_\alpha \left(\int_0^r \int_{\mathbb{R}^n} |P_s^\alpha f(x)|^2 \, dx \, \frac{ds}{s} \right)^{1/2}. \end{aligned}$$

Now we apply Plancherel's identity and Fubini Theorem again to get

$$\begin{aligned} \int_0^r \int_{\mathbb{R}^n} |P_s^\alpha f(x)|^2 \, dx \, \frac{ds}{s} &= \int_0^r \int_{\mathbb{R}^n} |(P_s^\alpha f)^\wedge(\xi)|^2 \, d\xi \, \frac{ds}{s} \\ &= \int_{\mathbb{R}^n} \int_0^\infty |m_s^\alpha(\xi)|^2 \, \frac{ds}{s} \, |\hat{f}(\xi)|^2 \, d\xi. \end{aligned} \tag{1.25}$$

Recalling (1.23) we have

$$m_s^\alpha(\xi) = \langle \nabla, \cdot \rangle^\alpha \hat{K}(s|\xi|\xi') - \frac{\alpha}{\Gamma(\alpha)} \int_1^\infty \frac{\hat{K}(st|\xi|\xi')}{t(t-1)^\alpha} \, dt,$$

and the change $|\xi|s = u$ gives

$$\begin{aligned}\int_0^\infty |m_s^\alpha(\xi)|^2 \frac{ds}{s} &= \int_0^\infty |m_u^\alpha(\xi')|^2 \frac{du}{u} \\ &\le \sup_{\eta\in\mathbb{S}^{n-1}} \left(\int_0^\infty |m_u^\alpha(\eta)|^2 \frac{du}{u}\right).\end{aligned}$$

By plugging this into (1.25), we obtain:

$$\begin{aligned}\int_0^r \int_{\mathbb{R}^n} |P_s^\alpha f(x)|^2 \, dx \frac{ds}{s} &\le \int_{\mathbb{R}^n} \sup_{\eta\in\mathbb{S}^{n-1}} \left(\int_0^\infty |m_u^\alpha(\eta)|^2 \frac{du}{u}\right) |\hat{f}(\xi)|^2 \, d\xi \\ &= \sup_{\eta\in\mathbb{S}^{n-1}} \left(\int_0^\infty |m_u^\alpha(\eta)|^2 \frac{du}{u}\right) \|f\|_{L^2(\mathbb{R}^n)}^2.\end{aligned}$$

□

Proof of Lemma 1.26. In view of Lemma 1.23, part i), and recalling that $\hat{P}_L(\xi) = \mathrm{e}^{-2\pi L|\xi|}$ we have

$$|\hat{K}(\xi)| \le |\hat{N}(\xi)| + |\mathrm{e}^{-2\pi L|\xi|}| \le \frac{C}{L|\xi|},$$

and

$$|\hat{K}(\xi)| \le |\hat{N}(\xi) - 1| + |1 - \mathrm{e}^{-2\pi L|\xi|}| \le CL|\xi|.$$

This proves part i) of the Lemma. For ii) note that $\langle \nabla \hat{P}_L(\xi), \xi\rangle = -2\pi L|\xi|\mathrm{e}^{-2\pi L|\xi|}$. From this we get easily

$$\begin{aligned}|\langle \nabla \hat{K}(\xi), \xi\rangle| &\le |\langle \nabla \hat{N}(\xi), \xi\rangle| + |\langle \nabla \hat{P}_L(\xi), \xi\rangle| \\ &= |\langle \nabla \hat{N}(\xi), \xi\rangle| + |2\pi L|\xi|\mathrm{e}^{-2\pi L|\xi|}| \le C\min(L|\xi|, 1).\end{aligned}$$

Finally,

$$\begin{aligned}\langle \nabla, \cdot\rangle^\alpha \hat{P}_L(\xi) &= \frac{1}{\Gamma(1-\alpha)} \int_1^\infty \frac{\langle \nabla \hat{P}_L(s\xi), s\xi\rangle}{(s-1)^\alpha s} \, ds \\ &= \frac{-2\pi L|\xi|}{\Gamma(1-\alpha)} \int_1^\infty \frac{\mathrm{e}^{-2\pi L(s-1)|\xi|}}{(s-1)^\alpha} \, ds \, \mathrm{e}^{-2\pi L|\xi|} \\ &= -(2\pi L|\xi|)^\alpha \, \mathrm{e}^{-2\pi L|\xi|},\end{aligned}$$

which together with Lemma 1.24 yields easily part iii). □

1.7.6 Back to L^p boundedness for $p > 1$.

Although we will not enter into the details of the proof of Theorem 1.18, let us remark here that if we apply the reasonings in Proposition 1.25 and Theorem 1.17 for $p < 2$, then we would only obtain the following

Lemma 1.27. *Let $p > 3/2$, then*

$$\sup_{j\in\mathbb{Z}}\Big\| \sup_{2^j\le r\le 2^{j+1}} |K_r * f| \Big\|_{L^p(\mathbb{R}^n)} \le C_p\|f\|_{L^p(\mathbb{R}^n)}, \tag{1.26}$$

where the constant C_p might depend on p but not on n and B.

It is obvious that (1.26) is weaker than the bound we need:

$$\| \sup_{r>0} |K_r * f| \, \|_{L^p(\mathbb{R}^n)} \le C_p\|f\|_{L^p(\mathbb{R}^n)},$$

with C_p only depending on p. The proof of this last estimate follows from (1.26) and a stronger L^2 bound via a semi-orthogonality principle.

We want to finish this section with yet another positive result on L^p. D. Müller proved in [61] that for certain families of convex bodies B the associated maximal operators M_B admit uniform boundedness on $L^p(\mathbb{R}^n)$ for every $p > 1$. The constant in this case may depend on the family of convex bodies. Let us briefly describe this approach.

Two new quantities related to B are to be considered. Given $\xi \in \mathbb{S}^{n-1}$ we denote by π_ξ the orthogonal projection onto the hyperplane $\xi^\perp$. Define

$$Q := \max\{|\pi_\xi(B)|_{n-1} \,:\, \xi \in \mathbb{S}^{n-1}\}.$$

This quantity is linearly invariant, that is, if $\nu \in SL(\mathbb{R}^n)$, then $Q(B) = Q(\nu(B))$ because ν preserves volumes. Then applying the adequate transformation ν we may always assume that B is isotropic with constant L.

With a much subtler study of the boundedness of $\langle \nabla, \cdot\rangle^\alpha \hat{K}$ as an L^p multiplier D. Müller proved the following.

Theorem 1.28. *Let $p > 1$. There exist a constant $C_{p,L,Q}$ such that*

$$\|M_B f\|_{L^p(\mathbb{R}^n)} \le C_{p,L,Q}\|f\|_{L^p(\mathbb{R}^n)},$$

for every $f \in L^p(\mathbb{R}^n)$. The constant $C_{p,L,Q}$ only may depend on p, L and Q and not on the dimension n and might grow with L and Q.

If $1 \leq q < \infty$ and we consider the family of ℓ^q norms in $\mathbb{R}^n$ for each $n \in \mathbb{N}$ it is easy to check (see [61] for details) that L and Q are uniformly bounded in n for each q. So Theorem 1.28 ensures dimension-free bounds on $L^p(\mathbb{R}^n)$ for the family of maximal operators associated to the balls given by the ℓ^q for each $1 \leq q < \infty$. When considering cubes, that are the balls given by the ℓ^∞ norm, Q grows to infinity with n at the rate $\sqrt{n}$. Therefore Theorem 1.28 does not solve the question of uniformity with respect to the dimension in this case. It is still an open and important problem to determine if it is possible to bound the maximal function associated to cubes on $L^p(\mathbb{R}^n)$ independently of the dimension for $1 < p < 3/2$.

Chapter 2

High dimensional weak type bounds for maximal functions associated to cubes.

In this chapter, we will focus on the maximal operator associated to cubes. Recall that for a locally integrable function f on $\mathbb{R}^n$ we define it by

$$M_n f(x) = \sup_{R>0} \frac{1}{|Q(x,R)|} \int_{Q(x,R)} |f(y)|\, dy,$$

where $Q(x,R)$ is the cube of edge length R, centred at x. The operator M_n satisfies a weak L^1 estimate:

$$|\{x \in \mathbb{R}^n : M_n f(x) > \lambda\}| \leq \frac{C}{\lambda} \|f\|_{L^1(\mathbb{R}^n)}.$$

We denote by $C_{1,n}$ the best constant in this inequality. By the Stein and Strömberg [74] result $C_{1,n} = O(n \log n)$ when n grows to infinity. This chapter is devoted to the most significant results giving lower bounds for this constant. Let us first summarise them briefly.

The starting point in this study was the idea of discretisation set up by M. de Guzmán in [42]. He realised that convolution operators act very similarly on L^1 functions and on Dirac deltas. Indeed, assuming mild conditions, he proved that weak L^1 boundedness of maximal functions associated to integrable convolution kernels is equivalent to boundedness on finite sums of Dirac deltas. Later on, M.T. Menárguez and F. Soria [53] pointed out that the best constant in the weak type inequality for integrable functions and for Dirac deltas must be exactly the same. They used this fact to produce lower bounds for $C_{1,n}$. This approach is treated in the next two sections.

Departing from the ideas of these results, J.M. Aldaz [3] proved that $C_{1,n}$ grows to infinity with the dimension n, solving the long time open question stated in [74]. The proof is elementary but gives no lower bounds for the growth of the constant.

With far more sophisticated tools, G. Aubrun [7] gave a shorter proof of Aldaz's result. We will deal with this in Section 2.3.

Also in [7] G. Aubrun showed that $C_{1,n}$ has a growth faster than $(\log n)^{1-\varepsilon}$ for any $\varepsilon > 0$. In Section 2.4, we present an extension of this last result to Orlicz spaces. Roughly speaking, we prove that if $\Phi(t) = o(t(\log\log t)^{1-\varepsilon})$ for some $\varepsilon > 0$ then the best constant C_n^Φ in the L^Φ weak inequality grows to infinity with the dimension n. For this growth we give a lower bound related to the function Φ defining the space.

2.1 The discretisation idea.

Let $\{K_j\}_{j\in\mathbb{N}}$ be a discrete family of integrable functions on $\mathbb{R}^n$. For an integrable function f, we define the associated maximal operator by

$$K^*f(x) = \sup_{j\in\mathbb{N}} |K_j * f(x)|.$$

We can also define the action of this maximal operator on finite sums of Dirac deltas. Let $\{x_i\}_{i=1,\dots,I} \subset \mathbb{R}^n$, and consider the discrete measure

$$\mu = \sum_{i=1}^{I} \delta_{x_i}. \tag{2.1}$$

Since $K * \delta_{x_0}(x) = K(x - x_0)$ for an integrable K, we have

$$K^*\mu(x) = \sup_{j\in\mathbb{N}} |K_j * \mu(x)| = \sup_{j\in\mathbb{N}} \left|\sum_{i=1}^{I} K_j(x - x_i)\right|.$$

We will say that K^* is weakly bounded over finite sums of Dirac deltas if for each measure μ of the form (2.1) one has

$$|\{x \in \mathbb{R}^n : K^*\mu(x) > \lambda\}| \le \frac{C}{\lambda}\mu(\mathbb{R}^n) = \frac{C}{\lambda} I,$$

where $C > 0$ is an absolute constant.

M. de Guzmán [42] proved that if the K_j are integrable, then K^* is weakly bounded over integrable functions if and only if it is weakly bounded over finite sums of Dirac deltas. The observation that the constant must be the same in both cases if in addition every K_j is non-negative, is due to T. Menárguez and F. Soria [53].

Theorem 2.1. *Let $\{K_j\}_{j\in\mathbb{N}}$ be a sequence of positive integrable functions on $\mathbb{R}^n$. For an integrable function f, we define the associated maximal operator as $K^*f(x) = \sup_j |K_j * f(x)|$. Given a constant $A > 0$ the following statements are equivalent*

(a) for every integrable function f over $\mathbb{R}^n$ one has $|\{x \in \mathbb{R}^n : K^*f(x) > \lambda\}| \leq \frac{A}{\lambda}\|f\|_{L^1}$.

(b) for any measure μ of the form $\mu = \sum_{i=1}^{I} \delta_{x_i}$, with $\{x_i\}_{i=1,\ldots,I}$ different points in $\mathbb{R}^n$, one has $|\{x \in \mathbb{R}^n : K^*\mu(x) > \lambda\}| \leq \frac{A}{\lambda}I$.

Proof. Let us first prove "$(a) \Rightarrow (b)$". Take a measure μ of the aforementioned form. Calling $K_J^*f(x) = \sup_{j\leq J} |K_j * f(x)|$ and $E_J = \{x \in \mathbb{R}^n : K_J^*\mu(x) > \lambda\}$ it is enough to prove that for each J we have $|E_J| \leq \frac{A}{\lambda}I$. Then, by monotonicity, since E_J is an increasing sequence of sets

$$|\{x \in \mathbb{R}^n : K^*(x) > \lambda\}| = \left|\bigcup_{J=1}^{\infty} E_J\right| = \lim_{J\to\infty} |E_J| \leq \frac{A}{\lambda}I.$$

We approximate μ by functions of the form

$$f_R(x) = \frac{1}{R^n}\sum_{i=1}^{I} \chi_{Q(x_i,R)}(x),$$

with R small enough so that the cubes $Q(x_i, R)$ are pairwise disjoint. Fix $\lambda > 0$. By assumption the level sets $F_R = \{x \in \mathbb{R}^n : K^*f_R(x) > \lambda\}$, satisfy $|F_R| \leq \frac{A}{\lambda}\int f_R(x)\,dx = \frac{A}{\lambda}I$. For each j and for almost every x one has

$$\begin{aligned}\lim_{R\to 0} K_j * f_R(x) &= \lim_{R\to 0}\sum_{i=1}^{I}\frac{1}{R^n}\int_{Q(x_i,R)} K_j(x-y)\,dy = \sum_{i=1}^{I} K_j(x-x_i)\\ &= K_j * \mu(x),\end{aligned}$$

by Lebesgue differentiation theorem. Hence $\lim_{R\to 0} K_J^* f_R = K_J^*\mu$ and $\chi_{E_J} = \lim_{R\to 0}\chi_{F_R}$ almost everywhere. Finally, by monotonicity

$$|E_J| = \left|\bigcup_{T>0}\bigcap_{R\geq T} F_R\right| = \lim_{T\to\infty}\left|\bigcap_{R\geq T} F_R\right| \leq \lim_{T\to 0}|F_T| \leq \frac{A}{\lambda} I.$$

Now we show "$(b) \Rightarrow (a)$". Since $K^*f \leq K^*|f|$ it is enough to prove the result only for non-negative functions. This is the only place where we use that the K_j are non-negative. Moreover, by a density argument, it is enough to prove it for positive linear combinations of characteristic functions of dyadic cubes. We do it in three steps.

STEP 1. First we check that if $\mu = \sum_{i=1}^I n_i\delta_{x_i}$, where the n_i are positive integers, then

$$|\{x\in\mathbb{R}^n : K_J^*(x) > \lambda\}| \leq \frac{A}{\lambda}\mu(\mathbb{R}^n) = \frac{A}{\lambda}\sum_{i=1}^I n_i,$$

for any J. For each $i = 1,\ldots,I$ take n_i points x_i^ℓ, with $\ell = 1,\ldots,n_i$, all of them different but close to x_i in a way that we will precise later. Consider the measure

$$\nu = \sum_{i=1}^I\sum_{\ell=1}^{n_i}\delta_{x_i^\ell}.$$

We will use ν as an approximation to μ. Whenever $0 < \varepsilon < \lambda$ we have

$$|\{x\in\mathbb{R}^n : K_J^*\mu(x) > \lambda\}| \leq |E_1| + |E_2|,$$

where

$$\begin{aligned} E_1 &= \{x\in\mathbb{R}^n : K_J^*\nu(x) > \lambda-\varepsilon\},\\ E_2 &= \{x\in\mathbb{R}^n : K_J^*(\mu-\nu)(x) > \varepsilon\}. \end{aligned}$$

By assumption

$$|E_1| \leq \frac{A}{\lambda-\varepsilon}\nu(\mathbb{R}^n) = \frac{A}{\lambda-\varepsilon}\mu(\mathbb{R}^n).$$

On the other hand

$$\begin{aligned} |E_2| &\leq \frac{1}{\varepsilon}\int \sup_{j\leq J}\left|\sum_{i=1}^I\sum_{\ell=1}^{n_i}(K_j(x-x_i) - K_j(x-x_i^\ell))\right| dx\\ &\leq \frac{1}{\varepsilon}\sum_{j\leq J}\sum_{i=1}^I\sum_{\ell=1}^{n_i}\int |K_j(x-x_i) - K_j(x-x_i^\ell)|\,dx \leq \varepsilon, \end{aligned}$$

if we choose each x_i^ℓ so that

$$\int |K_j(x-x_i) - K_j(x-x_i^\ell)|\, dx \le \frac{\varepsilon^2}{J\mu(\mathbb{R}^n)}.$$

With this choice

$$|\{x \in \mathbb{R}^n : K_J^*\mu(x) > \lambda\}| \le \frac{A}{\lambda - \varepsilon}\mu(\mathbb{R}^n) + \varepsilon,$$

as ε is arbitrary, by letting it go to zero, we are done with step 1.

STEP 2. We are going to see now that K^* is weakly bounded on real positive linear combinations of Dirac deltas. From the previous step, by homogeneity it is easy to see that K^* is weakly bounded over rational positive linear combinations of Dirac deltas. Define the measure

$$\mu = \sum_{i=1}^{I} r_i \delta_{x_i},$$

with r_i positive real numbers. For each $i = 1, \dots, I$ take a rational q_i close to r_i in a way that will be determined later. We approximate μ by the measure

$$\nu = \sum_{i=1}^{I} q_i \delta_{x_i}.$$

For any $\varepsilon \in (0, \lambda)$

$$|\{x \in \mathbb{R}^n : K_J^*\mu(x) > \lambda\}| \le |E_1| + |E_2|,$$

with

$$\begin{aligned} E_1 &= \{x \in \mathbb{R}^n : K_J^*\nu(x) > \lambda - \varepsilon\}, \\ E_2 &= \{x \in \mathbb{R}^n : K_J^*(\mu - \nu)(x) > \varepsilon\}. \end{aligned}$$

The measure ν is a positive rational linear combination of Dirac deltas, and by the comment above

$$|E_1| \le \frac{A}{\lambda - \varepsilon}\nu(\mathbb{R}^n) = \frac{A}{\lambda - \varepsilon}\mu(\mathbb{R}^n).$$

We also have

$$\begin{aligned} |E_2| &\le \frac{1}{\varepsilon}\int \sup_{j \le J}\left|\sum_{i=1}^{I}(r_i - q_i)K_j(x - x_i)\right| dx \\ &\le \frac{1}{\varepsilon}\sum_{j \le J}\sum_{i=1}^{I}|r_i - q_i|\int |K_j(x - x_i)|\, dx \le \varepsilon, \end{aligned}$$

whenever the choice of the q_i is such that $|r_i - q_i| \leq \dfrac{\varepsilon^2}{JI \sup_{j\leq J} \|K_j\|_{L^1}}$.

STEP 3. We conclude by proving that K_J^* is weakly bounded over non-negative functions that are constant over dyadic cubes. Consider

$$f(x) = \sum_{i=1}^{I} r_i \chi_{Q_i},$$

where the Q_i are pairwise disjoint dyadic cubes and the r_i, positive real numbers, $i = 1, \dots, I$. Since the result is independent of the number of cubes I, by subdividing the bigger cubes we may assume that the edge length of every Q_i is 2^k for a certain integer k. Take a measure

$$\mu = 2^{-kn} \sum_{i=1}^{I} r_i \delta_{x_i},$$

where x_i is the centre of the cube Q_i. Let ε be such that $0 < \varepsilon < \lambda$. One has

$$|\{x \in \mathbb{R}^n : K_J^* \mu(x) > \lambda\}| \leq |E_1| + |E_2|,$$

where

$$\begin{aligned} E_1 &= \{x \in \mathbb{R}^n : K_J^* \nu(x) > \lambda - \varepsilon\}, \\ E_2 &= \{x \in \mathbb{R}^n : K_J^*(\mu - \nu)(x) > \varepsilon\}. \end{aligned}$$

Since μ is a positive linear combination of Dirac deltas

$$|E_1| \leq \frac{A}{\lambda - \varepsilon} \mu(\mathbb{R}^n) = \frac{A}{\lambda - \varepsilon} \|f\|_{L^1}.$$

Now by Chebychev and Minkowski inequalities

$$\begin{aligned} |E_2| &\leq \frac{1}{\varepsilon} \int \sup_{j \leq J} \left| \sum_{i=1}^{I} \left(\int_{Q_i} r_i K_j(x-y)\, dy - 2^{-kn} r_i K_j(x - x_i) \right) \right| dx \\ &\leq \frac{1}{\varepsilon} \int \sum_{j\leq J} \left| \sum_{i=1}^{I} r_i \int_{Q_i} \Big(K_j(x-y) - K_j(x-x_i) \Big)\, dy \right| dx \\ &\leq \frac{1}{\varepsilon} \sum_{j\leq J} \sum_{i=1}^{I} r_i \int_{Q_i} \|K_j(\cdot - y) - K_j(\cdot - x_i)\|_{L^1}\, dy \leq \varepsilon. \end{aligned}$$

To have the last inequality choose k such that $\|z-w\| \leq 2^k$ guarantees that

$$\|K_j(\cdot - z) - K_j(\cdot - w)\| \leq \frac{\varepsilon^2}{2^{kn} J \sum_{i=1}^{I} r_i},$$

for all $j = 1, \ldots, J$. Summarising, we have obtained

$$|\{x \in \mathbb{R}^n : K_J^* f(x) > \lambda\}| \leq \frac{A}{\lambda - \varepsilon} \|f\|_{L^1} + \varepsilon,$$

for an arbitrary ε. Letting ε go to zero we get precisely what we need. This finishes step 3. □

We are interested in maximal operators of the following kind. Let $K \in L^1(\mathbb{R}^n)$. For $t > 0$ we define $K_t(x) = t^{-1}K(x/t)$ and for $f \in L^1(\mathbb{R}^n)$ the associated maximal operator as $K^* f(x) = \sup_{t>0} |K_t * f(x)|$. The previous theorem asserts that

$$f \mapsto K_{\mathbb{Q}}^* f = \sup_{t>0, t \in \mathbb{Q}} |K_t * f|,$$

is weakly bounded on L^1 if and only if it is weakly bounded over finite sums of Dirac deltas. Is the same true for K^*, where the supremum is taken over all real t? The following example shows that the answer to this question is negative in general.

Take K as the Dirichlet function $K(x) = \chi_{\mathbb{Q}}(x)$, i.e. $K(x) = 1$ if $x \in \mathbb{Q}$ and $K(x) \equiv 0$ for $x \in \mathbb{R} \setminus \mathbb{Q}$. For each $f \in L^1$ given, $K^* f = 0$ because for each $t > 0$ one has $K_t * f \equiv 0$. Then K^* is weakly bounded on L^1. Nevertheless when considering its action over Dirac deltas we find the following. Given an $N > 0$, for each $x \in \mathbb{R}$ take $a \in \mathbb{Q}$ such that $a/x > N$. Then $K_{x/a} * \delta_0(x) = K_{x/a}(x) = a/xK(a) > N$. Thus $K * \delta_0 \equiv \infty$ and K^* cannot be weakly bounded over Dirac deltas.

However, the equivalence works for the Hardy-Littlewood maximal operator M_B, with B a convex body symmetric with respect to the origin. In this case $K = \chi_B$ and by the regularity of Lebesgue measure, $M_B = K^* = K_{\mathbb{Q}}^*$, with this equalities understood both over functions and over finite sums of Dirac deltas.

Further information and extensions of these results can be found in [42] and [20].

2.2 First lower bounds. The best one-dimensional constant.

By Theorem 2.1 and the last comment of the previous section, we may study the value of the constants $C_{1,n}$ in the weak type L^1 inequality of the operator M_n by looking at the action of this operator over finite sums of Dirac deltas. This was first used by M.T. Menárguez and F. Soria [53] for giving lower bounds of $C_{1,n}$. They proved the following

Proposition 2.2. $C_{1,n} \geq \left(\frac{1+2^{1/n}}{2}\right)^n$ *for every* $n \in \mathbb{N}$.

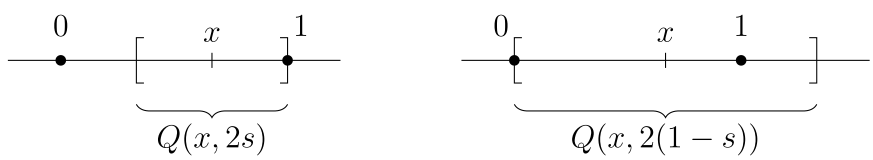
Proof. For each $N \in \mathbb{N}$ consider the discrete measure μ_N consisting in placing a Dirac delta at each point with integer coordinates in the cube $[0, N]^n \subset \mathbb{R}^n$, i.e.

$$\mu_N = \sum_{z \in [0,N]^n \cap \mathbb{Z}^n} \delta_z.$$

Write $\theta_n = \inf_{x \in [0,1]^n} M_n \mu_1(x)$. It is obvious that whenever $x \in [0, N]^n$, one has $M_n \mu_N(x) \geq \theta_n$. Hence

$$\begin{aligned} C_{1,n} &\geq \sup_{N \geq 1} \frac{\theta_n |\{x \in \mathbb{R}^n : M_n \mu_N(x) > \theta_n\}|}{\mu_N(\mathbb{R}^n)} \geq \sup_{N \geq 1} \frac{\theta_n \,|[0,N]^n|}{(N+1)^n} \\ &= \theta_n \sup_{N \geq 1} \left(\frac{N}{N+1}\right)^n = \theta_n. \end{aligned}$$

Now we have to show just that $\theta_n = \left(\frac{1+2^{1/n}}{2}\right)^n$. Fix an $x \in [0,1]^n$ and take $s = \inf_{z \in \mathbb{Z}^n} \|x - z\|_\infty$. It is clear that $s \in [0, 1/2]$ and that $\mu_1(Q(x, 2s)) \geq 1$. It is also easy to see that the cube $Q(x, 2(1-s))$ grabs at least two Dirac deltas and so $\mu_1(Q(x, 2(1-s))) \geq 2$.



Then we have that

$$M_n\mu_1(x) \geq \max\left(\frac{1}{(2s)^n}, \frac{2}{(2(1-s))^n}\right) = F(s).$$

Since $1/(2s)^n$ is decreasing in s and $2/(2(1-s))^n$ is increasing in s one, $F(s)$ attains its minimal value for $s \in [0,1/2]$ when $1/(2s)^n = 2/(2(1-s))^n$, that is when $s = (1+2^{1/n})^{-1} =: s_0$. Then for all $x \in [0,1]^n$

$$M_n\mu_1(x) \geq \min_{0\leq s\leq 1/2} F(s) = F(s_0) = \left(\frac{1+2^{1/n}}{2}\right)^n.$$

We have proved that $\theta_n \geq \left(\frac{1+2^{1/n}}{2}\right)^n$ and in fact this is all that we need for the proof the theorem. However, it is easy to obtain the reverse inequality. Just taking $x = (s_0, 0, \ldots, 0)$ one has $M_n\mu_1(x) = \left(\dfrac{1+2^{1/n}}{2}\right)^n$, because $\mu_1(Q(x, 2s_0)) = 1$, $\mu_1(Q(x, 2(1-s_0))) = 2$ and $\mu_1(Q(x,2)) = 2^n$ but $|Q(x,2)| = 2^n$. □

Observe that the lower bound obtained $((1+2^{1/n})/2)^n$ is decreasing and tends to $\sqrt{2}$ when n tends to infinity.

The best one dimensional constant For the one-dimensional operator the bound obtained in Theorem 2.2 is $C_{1,1} \geq 3/2$. It is easy to see that $C_{1,1} \leq 2$ (see [30], [38] or [53]). To know the exact value of $C_{1,1}$ became an important problem, to which much effort was devoted. Let us summarise the main advances. The value $3/2$ for $C_{1,1}$ was proposed by A. Carbery in [14]. J.M. Aldaz [1] disproved this conjecture by showing that $1.541 \sim 37/24 \leq C_{1,1} \leq (9+\sqrt{41})/8 \sim 1.925$, improving both the known upper and lower bounds. This also answered the question asked in [53] of whether the constant in the centred and uncentred case could be the same. J. Manfredi and F. Soria [46] improved Aldaz's lower bound showing that $C_{1,1} \geq 5/3 - 2\sqrt{7}/3\sin(\arctan(3\sqrt{3})^{-1}/3) \sim 1.555$ with an iterative argument conforming a dynamical system. A. Melas further sharpened previous bounds in [47] obtaining $1.567 \sim (11+\sqrt{61})/12 \leq C_{1,1} \leq 5/3 = 1.\overline{6}$. With a new covering argument in [48], he showed that $C_{1,1} \leq 1 + 1/\sqrt{3} \sim 1.577$, and finally proved in [49] that $C_{1,1} = (11+\sqrt{61})/12$, thus settling completely the problem of finding the best constant. As far as we know, no best bounds have been found in greater dimensions.

2.3 Maximal function associated to cubes in high dimensions. Weak type for L^1.

The next step in the study of dimension dependence of the constants $C_{1,n}$ was the work of J.M. Aldaz and J.L Varona [6] where it was established that $C_{1,n}$ is an increasing sequence in n. However, it was still unknown whether the sequence was bounded or not. This question was finally solved by J.M. Aldaz [3] who proved

Theorem 2.3. *$C_{1,n}$ grows to infinity with the dimension n.*

The proof takes as starting point a equally distributed configuration of Dirac deltas

$$\mu = \sum_{z \in \mathbb{Z}^n} \delta_z$$

like the ones considered in Proposition 2.2. It is based on a better estimation of the size of the maximal function and its levels sets, and the introduction of a probabilistic point of view that allows to use the Central Limit Theorem. Here we will follow the shorter, despite not so elementary, proof given in [7].

First we observe that the mean values $\mu(Q(x,t))/|Q(x,t)| \leq (t+1)^n/t^n$ tend to 1 when t tends to infinity. Thus for estimating $M_n\mu$ we can disregard means taken over big cubes. Indeed

$$\sup_{t \geq s} \frac{\mu(Q(x,t))}{|Q(x,t)|} \leq \sup_{t \geq s} \frac{(t+1)^n}{t^n} = \sup_{t \geq s} \left(1 + \frac{1}{t}\right)^n \leq \left(1 + \frac{1}{s}\right)^n \leq \mathrm{e}^{n/s}. \quad (2.2)$$

This allows us to reduce to the unit cube $[0,1]^n$ in the following sense.

Lemma 2.4. $C_{1,n} \geq \sup_{\lambda > 0} \lambda |\{x \in [0,1]^n : M_n\mu(x) > \lambda\}|$.

Proof. The idea of the proof is similar to the one used for Theorem 2.2. For $R > 0$ write μ_R to denote μ restricted to the cube $[0,R]^n$.

$$\begin{aligned} C_{1,n} &\geq \sup_{\lambda>0} \sup_{R>0} \frac{\lambda |\{x \in \mathbb{R}^n : M_n\mu_R > \lambda\}|}{|\mu_R|} \\ &\geq \sup_{s>0} \sup_{N \in \mathbb{N}} \frac{\mathrm{e}^{n/s} |\{x \in \mathbb{R}^n : M_n\mu_{N+2s}(x) > \mathrm{e}^{n/s}\}|}{|\mu_{N+2s}|}. \end{aligned}$$

In view of (2.2)

$$\begin{aligned} E_s &:= \{x \in \mathbb{R}^n : M_n\mu_{N+2s}(x) > \mathrm{e}^{n/s}\} \\ &\supset \{x \in [0,N]^n : \sup_{0<t<s} \frac{\mu_{N+2s}(Q(x,t))}{t^n} > \mathrm{e}^{n/s}\} \\ &= \{x \in [0,N]^n : \sup_{0<t<s} \frac{\mu(Q(x,t))}{t^n} > \mathrm{e}^{n/s}\} \\ &= \{x \in [0,N]^n : M_n\mu(x) > \mathrm{e}^{n/s}\}. \end{aligned}$$

Since the behaviour of $M_n\mu$ is the same in every cube of the grid $\mathbb{Z}^n$, we have

$$|E_s| = N^n|\{x \in [0,1]^n : M_n\mu(x) > \mathrm{e}^{n/s}\}|.$$

Hence

$$\begin{aligned} C_{1,n} &\geq \sup_{s>0}\sup_{N\in\mathbb{N}} \frac{\mathrm{e}^{n/s}|E_s|}{|\mu_{N+2s}|} \\ &\geq \sup_{s>0} \mathrm{e}^{n/s}|\{x \in [0,1]^n : M_n\mu(x) > \mathrm{e}^{n/s}\}| \sup_{N\in\mathbb{N}} \frac{N^n}{(N+2s+1)^n} \\ &= \sup_{s>0} \mathrm{e}^{n/s}|\{x \in [0,1]^n : M_n\mu(x) > \mathrm{e}^{n/s}\}| \\ &= \sup_{\lambda>0} \lambda|\{x \in [0,1]^n : M_n\mu(x) > \lambda\}|. \end{aligned}$$

□

Now we see that for almost every point in the unit cube, $M_n\mu$ is greater than any prescribed bound, provided that the dimension n is big enough.

Proposition 2.5. *For each $\lambda > 0$ one has*

$$|\{x \in [0,1]^n : M_n\mu(x) > \lambda\}| \overset{n\to\infty}{\longrightarrow} 1.$$

Once we have proved this, Theorem 2.3 is a simple consequence of Lemma 2.4.

The rest of the section is devoted to the proof of Proposition 2.5. With this in mind, we will obtain a lower bound for $M_n\mu(x)$ related to the relative position of x with respect to the grid $\mathbb{Z}^n$. For each $x \in [0,1]^n$, consider the Euclidean coordinates $x = (x_1, \dots, x_n)$. We will say that a coordinate x_i is centred, with respect to a parameter $\tau \in (0,1)$, if $x_i \in (\tau/2, 1-\tau/2)$. We shall see in the following lemma that for high dimensions $M_n\mu$ takes 'large' values at the points with a 'large' enough amount of centred coordinates. This principle appeared originally in [3], but the way it is presented here is closer to [7].

Before stating this lemma we need to introduce a probabilistic point of view. Considering the interval $[0,1]$ as a probability space, the random variable

$$U_\tau(y) := \begin{cases} 0 & \text{if } y \text{ is centred,} \\ 1 & \text{if } y \text{ is not centred,} \end{cases}$$

follows a Bernoulli distribution of parameter τ. Thinking of $[0,1]^n$ as the probability space product of n copies of $[0,1]$, the number of non-centred coordinates for a point $x = (x_1, \ldots, x_n) \in [0,1]^n$, is the sum of n independent Bernoulli variables of parameter τ, so it is a random binomial variable of parameters n and τ, $V_{n,\tau}$. To be more precise

$$V_{n,\tau}(x) := U_\tau(x_1) + \ldots + U_\tau(x_n).$$

The expectation of $V_{n,\tau}$ is τn and its typical deviation, $\sigma_{n,\tau} = \sqrt{\tau(1-\tau)n}$. We consider the normalised random variable $\alpha_{n,\tau}(x) = (V_{n,\tau}(x) - \tau n)/\sigma_{n,\tau}$ and for each $T > 0$ the set $E^T_{n,\tau} = \{x \in [0,1]^n : \alpha_{n,\tau}(x) \leq -T\}$. For large dimensions $M_n\mu$ will be bounded by below in this set.

Lemma 2.6. *Let $\delta > 0$, there exists an $A > 0$ such that for each $x \in E^T_{n,\tau}$ the condition $\sigma_{n,\tau}/T > A$ implies*

$$\frac{\mu(Q(x,t))}{|Q(x,t)|} \geq \mathrm{e}^{(1-\delta)T^2/2},$$

for a certain $t < \sqrt{n}/2T$. As a consequence

$$E^T_{n,\tau} \subset \{x \in [0,1]^n : M_n\mu(x) > \mathrm{e}^{(1-\delta)T^2/2}\},$$

whenever $\sigma_{n,\tau}/T > A$.

Proof. Let us first make a one-dimensional reasoning. For each $s \in \mathbb{N}$ and $y \in [0,1]$ consider the interval $I = [y - (s - \tau/2), y + (s - \tau/2)]$. If y is centred with respect to τ, the interval I contains $2s$ integers, if y is not centred, I contains at least $2s - 1$ integers. Let $x \in [0,1]^n$ have k centred coordinates. By the product structure of the cubes, the cube $Q(x, 2s - \tau)$ contains at least $(2s)^k(2s-1)^{n-k}$ points of $\mathbb{Z}^n$. If $x \in E^T_{n,\tau}$ then $k > \ell := n - \tau n + T\sigma_{n,\tau}$, and

$$\begin{aligned} M_d\mu(x) &\geq \frac{\mu(Q(x,2s-\tau))}{|Q(x,2s-\tau)|} \geq \frac{(2s)^k(2s-1)^{n-k}}{(2s-\tau)^n} \\ &\geq \frac{(2s)^\ell(2s-1)^{n-\ell}}{(2s-\tau)^n} =: H(s), \end{aligned}$$

for each integer $s \geq 1$.

Considering now $H(s)$ defined by the above formula over all positive real s, we find that

$$\frac{d}{ds}\log H(s) = \frac{\ell}{s} + \frac{2(n-\ell)}{2s-1} - \frac{2n}{2s-\tau},$$

is positive for $s < s_n$, negative for $s > s_n$ and equals 0 at $s = s_n$, where

$$s_n = \frac{\tau\ell}{2(\ell-(1-\tau)n)} = \frac{\tau((1-\tau)n + T\sigma_{n,\tau})}{2T\sigma_{n,\tau}} = \frac{\sigma_{n,\tau}}{2T} + \frac{\tau}{2}.$$

Therefore H attains its maximum at $s = s_n$. This maximal value is

$$\begin{aligned} \max_{s>0} H(s) &= H(s_n) = \frac{(2s_n)^\ell}{(2s_n-1)^\ell}\frac{(2s_n-1)^n}{(2s_n-\tau)^n} \\ &= \left(1+\frac{\tau T}{\sigma_{n,\tau}}\right)^\ell \left(1-\frac{(1-\tau)T}{\sigma_{n,\tau}}\right)^{n-\ell} = (1+h_1)^\ell(1-h_2)^{n-\ell}. \end{aligned}$$

To estimate this quantity we will use the numerical estimates for $0 < h < 1$

$$\begin{aligned} \mathrm{e}^{1-h/2} \leq &\quad (1+h)^{1/h} &\leq \mathrm{e}^{1-h/2+h^2/(2+2h)}, \\ \mathrm{e}^{1+h/2} \leq &\quad (1-h)^{-1/h} &\leq \mathrm{e}^{1+h/2+h^2/(2-2h)}. \end{aligned}$$

From above it is clear that if $0 < h < 1/2$ one has $(1-h)^{1/h} \geq \mathrm{e}^{-1-h/2-h^2}$. Then

$$H(s_n) \geq \mathrm{e}^{(1-h_1/2)h_1\ell}\,\mathrm{e}^{-(1+h_2/2+h_2^2)h_2(n-\ell)}.$$

Writing $\ell = (1-\tau)n + T\sigma_{n,\tau}$ and after some calculations the last quantity becomes

$$\exp\left(\frac{T^2}{2} + \frac{(2\tau-1)T^3}{2\sigma_{n,\tau}} + \frac{(1-\tau)^3T^4}{\sigma^2_{n,\tau}}\right) \geq \exp\left(\frac{T^2}{2} - \frac{T^3}{2\sigma_{n,\tau}}\right).$$

Recall that, as far as we know, the comparison $Mg(x) > H(s)$ works only if s is an integer. However, $H(\lfloor s_n \rfloor)$ does not differ too much from $H(s_n)$. If $\lfloor s_n \rfloor \leq s \leq s_n$, then

$$2s \;\geq\; 2s - \tau \geq 2s - 1 \geq 2\lfloor s_n \rfloor - 1 \geq 2s_n - 3 \geq \frac{\sigma_{n,\tau}}{T} - 3.$$

and consequently

$$\begin{aligned}\left|\frac{d}{ds}\log H(s)\right| &= \left|\frac{2s((1-\tau)n-\ell)+\tau\ell}{2s(2s-1)(2s-\tau)}\right| = \left|\frac{\sigma_{n,\tau}^2-(2s-\tau)T\sigma_{n,\tau}}{2s(2s-1)(2s-\tau)}\right| \\ &\leq \frac{\sigma_{n,\tau}^2}{(\sigma_{n,\tau}/T-3)^3}+\frac{T\sigma_{n,\tau}}{(\sigma_{n,\tau}/T-3)^2}.\end{aligned}$$

Then by the mean value theorem

$$\begin{aligned}\log H(\lfloor s_n\rfloor) &\geq \log H(s_n) - \sup_{\lfloor s_n\rfloor\leq s\leq s_n}\left|\frac{d}{ds}\log H(s)\right|(\lfloor s_n\rfloor - s_n) \\ &\geq \frac{T^2}{2}-\frac{T^3}{2\sigma_{n,\tau}}-\frac{\sigma_{n,\tau}^2}{(\sigma_{n,\tau}/T-3)^3}-\frac{T\sigma_{n,\tau}}{(\sigma_{n,\tau}/T-3)^2} \\ &= \frac{T^2}{2}\left(1-\frac{1}{\sigma_{n,\tau}/T}-\frac{\sigma_{n,\tau}^2/T^2}{(\sigma_{n,\tau}/T-3)^3}-\frac{\sigma_{n,\tau}/T}{(\sigma_{n,\tau}/T-3)^2}\right).\end{aligned}$$

If $\sigma_{n,\tau}/T > A$ for an appropriate A the quantity in brackets is greater than $1-\delta$, and so

$$H(\lfloor s_n\rfloor) \geq \mathrm{e}^{(1-\delta)T^2/2}.$$

Finally, note that $t = 2s_n - \tau = \sigma_{n,\tau}/T \leq \sqrt{n}/2T$. □

In order to estimate the measure (or probability) of the sets $E_{n,\tau}^T$, we will apply the following multivariate Central Limit Theorem (see [31]) to the random variables $\alpha_{n,\tau}$.

Theorem 2.7. *Let $\{X_k\}_{k\in\mathbb{N}}$ a sequence of independent and identically distributed random vectors in $\mathbb{R}^J$, i.e. $Y_k = (Y_k^1,\ldots,Y_k^J)$ where the X_k^j are random variables. Call $m := E(X_k)$ and $\Gamma_{ji} := E(X_k^jX_k^i) < \infty$, noting that both the vector m and the matrix $\Gamma = (\Gamma_{ji})$ are independent of k. Then the random vector*

$$S_n = \frac{1}{\sqrt{n}}\sum_{k=1}^n X_k,$$

as n tends to infinity, converges in distribution to a random vector Z, following a multivariate normal distribution with expectation m and covariances given by Γ.

For $k = 1,\ldots,n$ take $X_{k,\tau}$ as a copy of the normalised $(U_\tau - \tau)/\sqrt{\tau(1-\tau)}$. Then $X_{k,\tau}$ has mean 0 and variance 1. Both $\alpha_{n,\tau}$ and

$$S_{n,\tau} = \frac{1}{\sqrt{n}}\sum_{k=1} X_{k,\tau},$$

have the same distribution. By the Multivariate Central Limit Theorem, as n tends to infinity the vector $(\alpha_{n,\tau_1},\dots,\alpha_{n,\tau_D})$, with $\tau_1,\dots,\tau_D \in (0,1)$, converges in distribution to a normal vector $Z=(Z_{\tau_0},\dots,Z_{\tau_D})$ with mean 0 and covariances

$$\begin{aligned} E(Z_sZ_t) &= E(U_sU_t) \\ &= \int_0^1 \frac{(\chi_{(0,s/2)\cup(1-s/2,1)}(x)-s)(\chi_{(0,t/2)\cup(1-t/2,1)}(x)-t)}{\sqrt{st(1-s)(1-t)}}\,dx \\ &= \frac{\min(s,t)-st}{\sqrt{st(1-s)(1-t)}}. \end{aligned} \tag{2.3}$$

The Brownian motion will appear as the limit of $\alpha_{n,\tau}$ as n goes to infinity (note that $\alpha_{n,\tau}$ can be interpreted as a random walk of n steps). We briefly recall the definition and most important properties of the Brownian motion. Given a rich enough probability space (Ω,P), we say that $\{W_t\}_{t\geq 0}$ defines a Brownian motion if for each $t\geq 0$ W_t is a real random variable enjoying the following properties:

1. $W_0(\omega)=0$ for all $\omega\in\Omega$.
2. $W_t(\omega)$ is continuous with respect to t almost surely.
3. For any choice $s,t,u,v\geq 0$ the increments W_t-W_s and W_u-W_v are independent random variables.
4. if $t\geq s$ the increment W_t-W_s follows a Gaussian distribution centred at the origin with typical deviation $t-s$.

A standard result ensures that conditions 3 and 4 are equivalent to

3'. W_t follows a Gaussian distribution and $E(W_t)=0$.

4'. $E(W_tW_s)=\min\{s,t\}$.

This and many other related results about the Brownian motion and other stochastic processes can be found in [57] or [31].

Proof of Proposition 2.5. Given $L>0$, take $T>0$ so that $\mathrm{e}^{T^2/4}>L$. Set $\tau_d=d/(d+1)$ for $d\in\mathbb{N}$. By Lemma 2.6, there exists $A>0$ such that if $\sqrt{\tau_D(1-\tau_D)n}>AT$ for a $D\in\mathbb{N}$, then

$$\{x\in[0,1]^n : M_n\mu(x)>L\}\supset\{x\in[0,1]^n : M_n\mu(x)>e^{T^2/4}\}\supset\bigcup_{d=1}^{D}E^T_{n,\tau_d}.$$

By definition,

$$\begin{aligned}\bigcup_{d=1}^{D} E_{n,\tau_d}^{T} &= \bigcup_{d=1}^{D} \{x \in [0,1]^n : \alpha_{n,\tau_d}(x) \le -T\} \\ &= \{x \in [0,1]^n : \min_{d=1,\dots,D} \alpha_{n,\tau_d}(x) \le -T\}.\end{aligned}$$

By the observation made before this proof, given $\varepsilon > 0$ and provided n is large enough

$$|\{x \in [0,1]^n : \min_{d=1,\dots,D} \alpha_{n,\tau_d}(x) \le -T\}| \ge P(\{\min_{d=1\dots,D} Z_{\tau_d} \le -T\}) - \frac{\varepsilon}{2},$$

where $(Z_{\tau_1}, \dots, Z_{\tau_D}) = Z$ is a multivariate Gaussian random vector with covariances for $i > j$

$$E(Z_{\tau_i} Z_{\tau_j}) = \frac{\min(\tau_i, \tau_j) - \tau_i \tau_j}{\sqrt{\tau_i \tau_j (1-\tau_i)(1-\tau_j)}} = \sqrt{\frac{j}{i}}.$$

Consider the Brownian motion W_t for $t > 0$. We claim that the vectors Z and $(W_1, W_2/\sqrt{2}, \dots, W_D/\sqrt{D})$ coincide in distribution. Indeed, as both are multivariate Gaussian vectors, all we have to check is that they share the same covariances. For $i > j$

$$E\left(\frac{W_i}{\sqrt{i}} \frac{W_j}{\sqrt{j}}\right) = \frac{E(W_i W_j)}{\sqrt{ij}} = \frac{\min\{i,j\}}{\sqrt{ij}} = \sqrt{\frac{j}{i}},$$

and our claim is justified. Since $\liminf_{d\to\infty} \frac{W_d}{\sqrt{d}} \le -T$ is a tail event, the Blumenthal 0 -1 law (see [57]) ensures that

$$P\left(\left\{\liminf_{d\to\infty} \frac{W_d}{\sqrt{d}} \le -T\right\}\right) = 0 \text{ or } 1.$$

By monotonicity, we have that

$$\begin{aligned}P\left(\left\{\liminf_{d\to\infty} \frac{W_d}{\sqrt{d}} \le -T\right\}\right) &= P\left(\bigcap_{d=1}^{\infty} \bigcup_{j=d}^{\infty} \left\{\frac{W_j}{\sqrt{j}} \le -T\right\}\right) \\ &= \lim_{d\to\infty} P\left(\bigcup_{j=d}^{\infty} \left\{\frac{W_j}{\sqrt{j}} \le -T\right\}\right) \\ &\ge \lim_{d\to\infty} P\left(\left\{\frac{W_d}{\sqrt{d}} \le -T\right\}\right) \\ &= P(\{W_1 \le -T\}) > 0.\end{aligned}$$

The last identity comes from the dilation invariance property:

$$P\left(\left\{\frac{W_d}{\sqrt{d}} \leq -T\right\}\right) = \int_{-\infty}^{-T\sqrt{d}} \frac{\mathrm{e}^{-\pi x^2/d}}{\sqrt{d}}\, dx = \int_{-\infty}^{-T} \mathrm{e}^{-\pi y^2}\, dy = P(\{W_1 \leq -T\}).$$

Hence,

$$P\left(\left\{\liminf_{d\to\infty} \frac{W_d}{\sqrt{d}} \leq -T\right\}\right) = 1.$$

Finally, given $\varepsilon > 0$, if D is large enough (only depending on ε and T)

$$\begin{aligned} |\{x \in [0,1]^n : M_n\mu(x) > L\}| &\geq P(\{\min_{d=1\ldots,D} Z_{\tau_d} \leq -T\}) - \frac{\varepsilon}{2} \\ &= P\left(\left\{\min_{d=1,\ldots,D} \frac{W_d}{\sqrt{d}} \leq -T\right\}\right) - \frac{\varepsilon}{2} \geq 1 - \varepsilon, \end{aligned}$$

for all large enough n (which we chose depending on D, ε and T). Since $\varepsilon > 0$ is arbitrary, the proposition is proved. □

2.4 Quantitative bounds for the constants. Weak Orlicz type inequalities.

In the previous section it was shown that $C_{1,n}$ grows to infinity with n, but no estimates of the speed of the growth were given. A first and quite naïv look on the original proof by J.M. Aldaz [3], replacing the Central Limit Theorem by the Berry-Esséen bounds, gives $C_{1,n} > \sqrt{\log n}/\log\log n$. G. Aubrun [7] gave the much sharper bound contained in the next theorem.

Theorem 2.8. *For every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that $C_{1,n} \geq C_\varepsilon(\log n)^{1-\varepsilon}$.*

In this section we present an extension in the form of failure of the maximal operator to be bounded on certain Orlicz spaces. Let us first define these spaces, which are a generalization of L^p spaces. We will focus on the fact that they build an intermediate scale between L^1 and L^p with $p > 1$.

The Orlicz space $L_\Phi(\mathbb{R}^n) = L_\Phi$ consists of the measurable functions f for which

$$\int_{\mathbb{R}^n} \Phi(|f(x)|)\, dx < \infty, \tag{2.4}$$

where $\Phi : [0, \infty) \longrightarrow [0, \infty)$ is a convex and increasing function such that $\Phi(0) = 0$ and for certain $C > 0$ one has $\Phi(2t) \leq C\Phi(t)$ for each $t > 0$. The integral in (2.4) does not define a norm. However Orlicz spaces are Banach spaces with the Luxembourg's norm

$$\|f\|_{L_\Phi} = \inf \left\{ a > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{a}\right) dx \leq 1 \right\}.$$

We will use the notation $\varphi(t) = \Phi(t)/t$. Note that φ is an increasing function. To see this, suppose the converse, i.e. that there exist positive real numbers x and y such that $x < y$ but $\varphi(x) > \varphi(y)$. By the convexity of Φ in $[0, y]$ we have $\Phi(t) \leq (1 - t/y)\Phi(0) + (t/y)\Phi(y) = t\varphi(y)$ for each $t \in [0, y]$. If we take $t = x$ we arrive to the contradiction

$$\Phi(x) \leq x\varphi(y) < x\varphi(x) = \Phi(x).$$

A typical example of an Orlicz space is the $L \log L$ space, which is an intermediate space between L^1 and L^p with $p > 1$. Its associated function is $\Phi(t) = t \log^+ t$, with $\log^+(x) = \log x$ for $x \geq e$ and $\log^+(x) = 1$ for $0 \leq x <$ e. The usual example of a function in this space is

$$f(x) = \frac{1}{x \log^3(1/x)} \chi_{(0,1/2)}(x).$$

This function is integrable but does not belong to any L^p with $p > 1$. However this function enjoys a better integrability than just being in L^1 since $f \in L \log L$, i.e.

$$\int_{\mathbb{R}} f(x) \log^+ f(x)\, dx < \infty.$$

Given Φ as described above the operator M_n is known to be of weak type L_Φ. That means that for each $f \in L_\Phi(\mathbb{R}^n)$,

$$|\{M_n f > \lambda\}| \leq A \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx, \tag{2.5}$$

where $A > 0$ is a constant that does not depend on λ. We can prove the weak L_Φ boundedness from the weak L^1 boundedness using Jensen Inequality as follows

$$\begin{aligned} |\{M_n f > \lambda\}| &= |\{\Phi(M_n f/\lambda) > \Phi(1)\}| \leq |\{M_n \Phi(f/\lambda) > \Phi(1)\}| \\ &\leq \frac{C_{1,n}}{\Phi(1)} \int \Phi\left(\frac{|f(x)|}{\lambda}\right) dx. \end{aligned}$$

The weak type L_Φ constant, C_n^Φ, is the least $A > 0$ for which (2.5) holds. Alternatively, C_n^Φ can be defined as:

$$C_n^\Phi := \sup \left\{ \frac{|\{M_n f > \lambda\}|}{\int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right) dx} : f \in L_\Phi(\mathbb{R}^n), \lambda > 0 \right\}. \tag{2.6}$$

Our result shows that for certain Orlicz spaces L_Φ, the constants C_n^Φ grows to infinity with the dimension, and gives a lower bound for this growth related to the function Φ defining the space.

Theorem 2.9. *Let Φ be a function as above and write $\varphi(t) = \Phi(t)/t$. If for some $\varepsilon > 0$*

$$\varphi(n^n) = o\left((\log n)^{1-\varepsilon}\right), \tag{2.7}$$

as $n \longrightarrow \infty$, then there exists a constant $C > 0$ only depending on ε such that

$$C_n^\Phi \geq C \frac{(\log n)^{1-\varepsilon}}{\varphi(n^n)}.$$

Remark 2.10. *It is easy to check that the condition (2.7) is equivalent to*

$$\varphi(n) = o\left((\log\log n)^{1-\varepsilon}\right).$$

Roughly speaking, the result asserts that there is no dimension-free bound for the weak type $L(\log\log L)^{1-\varepsilon}$ of the centred maximal operator associated with cubes, for any $\varepsilon > 0$.

The proof of Theorem 2.9 is a generalization of the one of Theorem 2.8 given in [7]. Instead of Dirac deltas we use approximations of the identity in the form of characteristic functions of small cubes divided by their measures. Consider the $L^\infty(\mathbb{R}^n)$ function

$$g = \sum_{z \in \mathbb{Z}^n} L_n^{-n} \chi_{Q(z, L_n)},$$

where $0 < L_n < \frac{1}{2}$ will be fixed later. The function g is an approximation to the measure μ used previously.

The following lemma gives a method to obtain lower bounds for the growth of C_n^Φ.

Lemma 2.11. *Suppose that* $\{a_n\}_{n\in\mathbb{N}}$ *is a positive sequence increasing to infinity and that there exists a constant* $K \in (0,1)$ *such that*

$$|\{x \in [0,1]^n : M_n g(x) > a_n\}| \geq K, \tag{2.8}$$

for all n*. Then if* $\varphi(n^n) = o(a_n)$ *as* $n \to \infty$*, the constant* C_n^Φ *grows to infinity with the lower bound*

$$C_n^\Phi \geq K \frac{a_n}{\varphi(n^n)}.$$

Theorem 2.9 will follow from the fact that $a_n = (\log n)^\alpha$ satisfies (2.8) for each $\alpha < 1$.

Proof of Lemma 2.11. For each $N > 0$ consider the function $g_N = g\chi_{[-n,N+n]^n}$. By (2.6) we have the bound

$$C_n^\Phi \geq \sup_{N\in\mathbb{N}} \frac{|\{x \in \mathbb{R}^n : M_n g_N(x) > a_n\}|}{\int_{\mathbb{R}^n} \Phi\left(\frac{|g_N(x)|}{a_n}\right)\, dx}. \tag{2.9}$$

For the denominator we have

$$\begin{aligned}\int_{\mathbb{R}^n} \Phi\left(\frac{|g_N(x)|}{a_n}\right) dx &= (N+2n)^n \int_{[-L_n/2,L_n/2]^n} \frac{1}{L_n^n a_n}\ \varphi\left(\frac{1}{L_n^n a_n}\right) dx \\ &= (N+2n)^n \frac{1}{a_n}\ \varphi\left(\frac{1}{L_n^n a_n}\right).\end{aligned}$$

We can take $L_n = Q/(n a_n^{1/n})$ with $0 < Q < 1$ to be determined later and then

$$\varphi\left(\frac{1}{L_n^n a_n}\right) = \varphi(Qn^n) \leq \varphi(n^n).$$

As usual we can reduce the problem to the unit cube, as we can see in the following

Lemma 2.12. *Let* $\lambda > e$*, then*

$$|\{x \in \mathbb{R}^n : M_n g_N(x) > \lambda\}| \geq N^n |\{x \in [0,1]^n : M_n g(x) > \lambda\}|.$$

In view of the last Lemma, if $a_n > e$ we obtain from (2.9)

$$\begin{aligned} C_n^{\Phi} &\geq \sup_{N\in\mathbb{N}} \frac{N^n}{(N+2n)^n} \frac{|\{x \in [0,1]^n : M_n g(x) > a_n\}|}{\varphi(n^n)} \\ &\geq K \sup_{N\in\mathbb{N}} \frac{N^n}{(N+2n)^n} \frac{a_n}{\varphi(n^n)}. \end{aligned}$$

Since $\sup_{n\in\mathbb{N}} N^n/(N+2n)^n = 1$ we have

$$C_n^{\Phi} \geq K \frac{a_n}{\varphi(n^n)}.$$

□

Proof of Lemma 2.12. Fix $N \in \mathbb{N}$. It it obvious that

$$\{x \in \mathbb{R}^n : M_n g_N(x) > \lambda\} \supset \{x \in [0,N]^n : \sup_{0<t<n} ⨍_{Q(x,t)} g_N > \lambda\}.$$

In order to replace g_N by g, note that mean values of g in very big cubes are close to 1. Indeed, similarly as in (2.2) we have

$$\sup_{t\geq s} ⨍_{Q(x,t)} g(x)\, dx \leq \sup_{t\geq s} \frac{(t+1)^n}{t^n} = \sup_{t\geq s} \left(1+\frac{1}{t}\right)^n \leq \left(1+\frac{1}{s}\right)^n \leq \mathrm{e}^{n/s}. \tag{2.10}$$

Taking $s = n$

$$\begin{aligned} \{x \in [0,N]^n : \sup_{0<t<n} ⨍_{Q(x,t)} g_N > \lambda\} &= \{x \in [0,N]^n : \sup_{0<t<n} ⨍_{Q(x,t)} g > \lambda\} \\ &= \{x \in [0,N]^n : M_n g(x) > \lambda\}. \end{aligned}$$

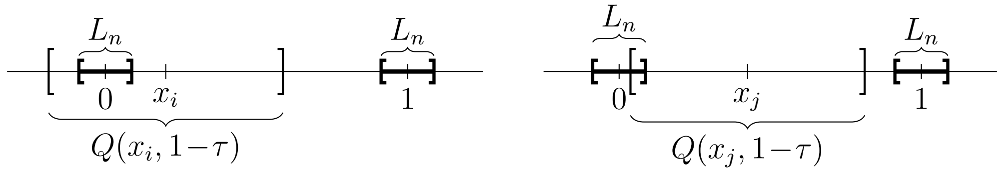
Since $M_n g$ is a $\mathbb{Z}^n$-periodic function

$$|\{x \in [0,N]^n : M_n g(x) > \lambda\}| = N^n |\{x \in [0,1]^n : M_n g(x) > \lambda\}|.$$

□

As we said before, to prove Theorem 2.9 we just need to check that $a_n = (\log n)^{\alpha}$ with $\alpha < 1$ satisfies estimate (2.8). The remaining of this chapter is devoted to that purpose. For this we will study the level sets of $M_n g$ in a similar way as we did with the ones of $M_n \mu$ using Lemma 2.6. We begin with an arbitrary sequence $\{a_n\}_{n\in\mathbb{N}}$ of positive numbers growing to infinity. Additional assumptions will be added when needed.

By the translation invariance of the underlying setting we can assume that $x \in [0,1]^n$. Fix $\tau \in (0,1)$, we shall say that a coordinate $x_i \in [0,1]$ is 'congruent' with respect to the parameter τ if for each $s \in \mathbb{N}$ and $k \in \mathbb{Z}$ the intersection of the intervals $Q(x_i, 1-\tau)$ and $Q(k, L_n)$ is either full or empty. Note that restricting to $s = 1$ and $k = 0, 1$ does not change this definition.



x_i is 'congruent' x_j is 'incongruent'

We will denote by D_n the subset of $[0,1]^n$ with at least one 'incongruent' coordinate. In order to estimate $|D_n|$, note that if x_j is 'incongruent' then $x_j \in [\tau/2 - \frac{L_n}{2}, \tau/2 + \frac{L_n}{2}] \cup [1 - \tau/2 - \frac{L_n}{2}, 1 - \tau/2 + \frac{L_n}{2}]$. Thus

$$|D_n| = 1 - |[0,1]^n \setminus D_n| \le 1 - (1 - 2L_n)^n \le 2nL_n, \tag{2.11}$$

where the last inequality was obtained via Bernoulli's inequality: $(1+x)^k \le 1 + kx$, valid for each $x \ge -1$ and $k \in \mathbb{N}$.

Now we are ready to state the following analogue of Lemma 2.6

Lemma 2.13. *Let $\delta > 0$, there exists an $A > 0$ such that if $x \in E^T_{n,\tau} \setminus D_n$ and $\sigma_{n,\tau}/T > A$ then*

$$\frac{1}{|Q(x,t)|} \int_{Q(x,t)} g(x)\, dx \ge \mathrm{e}^{(1-\delta)T^2/2},$$

for a certain $t < \sqrt{n}/4T$. This implies that

$$E^T_{n,\tau} \setminus D_n \subset \{x \in [0,1]^n : M_n g(x) > \mathrm{e}^{(1-\delta)T^2/2}\},$$

whenever $\sigma_{n,\tau}/T > A$.

Proof. Consider $x \in [0,1]^n \setminus D_n$, and s a positive integer. As x has only 'congruent' coordinates, the cube $Q(x, 2s-\tau)$ contains exactly $2s$ mass cubes of g for each given centred coordinate and at least $2s-1$ mass cubes for each non-centred coordinate of x. Assuming that x has k centred coordinates, we have

$$\int_{Q(x,2s-\tau)} g(x)\, dx \ge (2s)^k (2s-1)^{n-k}.$$

If $x \in E_{\tau,T}$, then $k > \ell = (1-\tau)n + T\sigma_{n,\tau}$ and

$$\begin{aligned} M_n g(x) &\geq \frac{1}{|Q(x,2s-\tau)|}\int_{Q(x,2s-\tau)} g(x)\,dx \geq \frac{(2s)^k(2s-1)^{n-k}}{(2s-\tau)^n} \\ &\geq \frac{(2s)^\ell(2s-1)^{n-\ell}}{(2s-\tau)^n} = H(s). \end{aligned}$$

And now everything goes on exactly as in the proof of Lemma 2.6. □

In view of Lemma 2.13, given a $\delta > 0$ there exist an $A > 0$ such that

$$\sigma_{n,\tau_0} > AT, \tag{2.12}$$

guarantees that

$$E^T_{n,\tau_0} \setminus D_n \subset \{x \in [0,1]^n : M_n g(x) > \mathrm{e}^{(1-\delta)T^2/2}\},$$

Fix $\tau_0 \in (0,1/2)$. If $\tau_0 \leq \tau \leq 1-\tau_0$, then $\sigma_{n,\tau} \geq \sigma_{n,\tau_0} > AT$, and then (2.12) implies

$$\bigcup_{\tau_0\leq\tau\leq1-\tau_0} E^T_{n,\tau} \setminus D_n \subset \{x \in [0,1]^n : M_n g(x) > \mathrm{e}^{(1-\delta)T^2/2}\}.$$

Taking $T = T(n)$ in such a way that

$$\mathrm{e}^{(1-\delta)T^2/2} \geq a_n, \tag{2.13}$$

we have

$$\bigcup_{\tau_0\leq\tau\leq1-\tau_0} E^T_{n,\tau} \setminus D_n \subset \{x \in [0,1]^n : M_n g(x) > a_n\}.$$

All we need to show now is that for certain choices of $\tau_0 = \tau_0(n)$ and $T = T(n)$ compatible with (2.12), (2.13) and $a_n = (\log n)^\alpha$ with $\alpha < 1$, one has

$$\left|\bigcup_{\tau_0\leq u\leq1-\tau_0} E^T_{n,\tau}\right| = P\left(\inf_{\tau_0\leq\tau\leq1-\tau_0} \alpha_{n,\tau} \leq -T\right) > K, \tag{2.14}$$

for some $K > 0$ not depending on n.

To see this, observe that by (2.11)

$$\begin{aligned} |\{x \in [0,1]^n : M_n g(x) > a_n\}| &\geq \left|\bigcup_{\tau_0<\tau<1-\tau_0} E^T_{n,\tau} \setminus D_n\right| \\ &\geq K - 2nL_n \geq K - 2Qa_n^{-1/n} \geq K/2, \end{aligned}$$

with an appropriate choice of Q, because $a_n > 1$ for large enough n.

To prove (2.14) we will use that $\alpha_{n,\tau}$ is an empirical process, and because of this, it converges to a Brownian bridge as n tends to infinity. Let us explain briefly the meaning of all these probabilistic concepts.

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables. The associated empirical distribution function is defined for $\tau \geq 0$ by

$$F_n(\tau) = \frac{\sharp\{k : X_k \leq \tau\}}{n} = \frac{1}{n}\sum_{k=1}^{n} I_{k,\tau},$$

where $I_{k,\tau} = \chi_{\{X_k \leq \tau\}}$. This variable follows a Bernoulli distribution of parameter $F(\tau)$ for certain $F(\tau)$ independent of k. An empirical process is a process of the form

$$G_{n,\tau} = \sqrt{n}(F_n(\tau) - F(\tau)).$$

By the Central Limit Theorem $G_{n,\tau}$ converges in distribution to a normal variable with expectation 0 and variance $F(\tau)(1 - F(\tau))$. Indeed, as we will later see, $G_{n,\tau}$ converges to a process related to the Brownian motion.

The relation to our problem is easy to see. Recall that

$$\sqrt{\tau(1-\tau)}\alpha_{n,\tau} = \sqrt{n}\left(\frac{1}{n}\sum_{k=1}^{n} U_{k,\tau} - \tau\right),$$

where each $U_{k,\tau}$ is an independent copy of U_τ, that follows a Bernoulli distribution of parameter τ. Taking $X_1, X_2, \ldots$ uniformly distributed over the interval $[0, 1]$, we would have that each $I_{k,\tau}$ coincides in distribution with $U_{k,\tau}$ and that $F(\tau) = \tau$. Hence, $\sqrt{\tau(1-\tau)}\alpha_{n,\tau}$ and $G_{n,\tau} = \sqrt{n}(F_n(\tau) - \tau)$ also share the same distribution.

Now we define the Brownian bridge. Let $\{W_t\}_{t\geq 0}$ be the standard Brownian motion. We call Brownian bridge the process defined by

$$B_t := W_t \mid W_1 = 0,$$

for $0 \leq t \leq 1$. This means that we consider only the realisations of the Brownian motion with value 0 at the time-point 1. The Brownian bridge can be represented in terms of the Brownian motion by

$$B_t = W_t - tW_1,$$

for $0 \leq t \leq 1$. For the same range of t another possible representation is

$$B_t = (1-t)W_{t/(1-t)}. \tag{2.15}$$

Then B_t is almost surely continuous on t. Moreover, for each $t \geq 0$ the variable B_t has a normal distribution, $E(B_t) = 0$ and for each $s \geq 0$ one has $E(B_s B_t) = \min\{s,t\} - st$. See [15] or [57] for more details.

Donsker Theorem [24] (see also [9] or [79]) roughly asserts that an empirical process of the form $G_{n,\tau}$ converges in distribution as n tends to infinity towards the process given by $B_{F(\tau)}$, where B denotes the Brownian bridge. Hence $\alpha_{n,\tau}$ converges in distribution to the process $B_\tau/\sqrt{\tau(1-\tau)}$. However Donsker Theorem needs some subtle considerations and does not contain any bound of the speed of convergence. We will employ instead the quantitative version of the theorem due to J. Komlós, P. Major and G. Tusnády [45], in the form presented by J. Bretagnolle and P. Massart in [16]:

Theorem 2.14. *Let $\{X_k\}_{k\in\mathbb{N}}$ be a sequence of independent random variables uniformly distributed over the interval $[0,1]$. For each $\tau \in [0,1]$ and $k \in \mathbb{N}$ consider $I_{k,\tau} = \chi_{\{X_k \leq \tau\}}$. For each $n \in \mathbb{N}$ call*

$$G_{n,\tau} := \frac{1}{\sqrt{n}} \sum_{k=1}^{n} (I_{k,\tau} - \tau).$$

For each n there exist a Brownian bridge $\{B_{n,\tau}\}_{0\leq\tau\leq 1}$ such that for any $x > 0$

$$P\left(\sup_{0\leq\tau\leq 1} |G_{n,\tau} - B_{n,\tau}| \geq \frac{x + 12\log n}{\sqrt{n}} \right) \leq 2\,\mathrm{e}^{-x/6}.$$

We will take $\beta_{n,\tau} = B_{n,\tau}/\sqrt{\tau(1-\tau)}$ where $B_{n,t}$ for $0 \leq t \leq 0$ is the Brownian bridge given by the previous theorem. Writing $T = a - b$ where a and b will be chosen later one has

$$\begin{aligned}
P_0 &:= P\left(\inf_{\tau_0\leq\tau\leq 1-\tau_0} \alpha_{n,\tau} \leq -T \right) \\
&= P\left(\inf_{\tau_0\leq\tau\leq 1-\tau_0} \alpha_{n,\tau} - \beta_{n,\tau} + \beta_{n,\tau} \leq b - a \right) \\
&\geq P\left(\left\{ \sup_{\tau_0\leq\tau\leq 1-\tau_0} \alpha_{n,\tau} - \beta_{n,\tau} \leq b \right\} \cap \left\{ \inf_{\tau_0\leq\tau\leq 1-\tau_0} \beta_{n,\tau} \leq -a \right\} \right) \\
&\geq P_1 - P_2,
\end{aligned}$$

where

$$\begin{aligned} P_1 &= P\left(\inf_{\tau_0\le\tau\le 1-\tau_0} \beta_{n,\tau} \le -a\right), \\ P_2 &= P\left(\sup_{\tau_0\le\tau\le 1-\tau_0} |\alpha_{n,\tau}-\beta_{n,\tau}| \ge b\right). \end{aligned}$$

By Theorem 2.14 taking

$$b = \frac{x+12\log n}{\sqrt{n\tau_0(1-\tau_0)}},$$

we have

$$\begin{aligned} P_1 &\le P\left(\sup_{\tau_0\le\tau\le 1-\tau_0} \frac{|G_{n,\tau}-\mathcal{B}_{n,\tau}|}{\sqrt{\tau(1-\tau)}} \ge \frac{x+12\log n}{\sqrt{n\tau_0(1-\tau_0)}}\right) \\ &\le P\left(\sup_{\tau_0\le\tau\le 1-\tau_0} |G_{n,\tau}-\mathcal{B}_{n,\tau}| \ge \frac{x+12\log n}{\sqrt{n}}\right) \\ &\le 2\,\mathrm{e}^{-x/6}, \end{aligned}$$

which we can make as small as needed just by taking x large enough. We estimate the other term with the following version of the iterated logarithm law appeared in [7].

Lemma 2.15. *Let B_t for $0 \le t \le 1$ be a Brownian bridge. If $0 < \eta < 2$, there exists a positive constant C_η so that for whenever $0 < t_0 < 1/e^e$,*

$$P\left(\sup_{t_0\le t\le 1-t_0} \frac{B_t}{\sqrt{t(1-t)}} \ge \sqrt{\eta\log\log(1/t_0)}\right) > C_\eta.$$

Then, taking $a = \sqrt{\eta\log\log 1/\tau_0}$ we get

$$\begin{aligned} P_2 &= P\left(\sup_{\tau_0\le\tau\le 1-\tau_0} \beta_{n,\tau} \ge a\right) \\ &= P\left(\sup_{\tau_0\le\tau\le 1-\tau_0} \frac{B_\tau}{\sqrt{\tau(1-\tau)}} \ge \sqrt{\eta\log\log(1/\tau_0)}\right) \ge C_\eta. \end{aligned}$$

To finish, we have to choose carefully the values of the several parameters involved.

Given $1 < \eta < 2$ (which we are interested in taking very close to 2), the value of C_η is determined. We first choose x large enough for $2e^{-x/6} \le C_\eta/2$ to hold. In this way we get

$$\left| \bigcup_{\tau_0 \le \tau \le 1-\tau_0} E_{n,\tau}^T \right| > \frac{C_\eta}{2}.$$

But we have to be careful to restrict ourselves to large enough values of n in order to make b small and have $a - b$ close to a in the sense that

$$a - b \ge \sqrt{\gamma \log\log 1/\tau_0},$$

where $\gamma = 2\eta - 2$ (in general we could take any γ such that $\gamma < \eta$ and that $\gamma \to 2$ as $\eta \to 2$). It is necessary that $a - b \ge T$ and that condition (2.13) holds. The latter is equivalent to

$$T \ge \sqrt{\frac{2}{1-\delta} \log a_n}.$$

It would be enough if we could choose $a - b$ and T so that

$$a - b \ge \sqrt{\gamma \log\log 1/\tau_0} \ge T \ge \sqrt{\frac{2}{1-\delta} \log a_n}.$$

For this it is necessary that

$$\sqrt{\gamma \log\log 1/\tau_0} \ge \sqrt{\frac{2}{1-\delta} \log a_n},$$

or equivalently that

$$\tau_0 \le \exp(-a_n^{2/\gamma(1-\delta)}).$$

On the other hand, since $\tau_0 \in (0, 1/2)$, condition (2.12) is equivalent to

$$\tau_0 \ge C \frac{\log a_n}{n},$$

for some absolute constant $C > 0$. Thus, the sequence $\{a_n\}$ has to verify

$$C \frac{\log a_n}{n} \le \tau_0 \le \exp(-a_n^{2/\gamma(1-\delta)}). \tag{2.16}$$

This will determine the maximal growth for $\{a_n\}$. If $a_n = (\log n)^\alpha$ for some $\alpha > 0$ we have

$$\exp(-a_n^{2/\gamma(1-\delta)}) = \exp(-(\log n)^{2\alpha/\gamma(1-\delta)-1} \log n) = n^{-\ell_n},$$

where $\ell_n = (\log n)^{2\alpha/\gamma(1-\delta)-1}$. Now (2.16) reads

$$C\alpha \frac{\log\log n}{n} \geq n^{-\ell_n},$$

and holds for large enough n if and only if $\ell_n = (\log n)^{2\alpha/\gamma(1-\delta)-1} < 1$. This implies that $2\alpha/\gamma(1-\delta) - 1 < 0$, i.e.

$$\alpha < \frac{\gamma(1-\delta)}{2}.$$

This means that $\alpha < 1$ necessarily, but we can make it as close to 1 as wanted by taking γ and δ close enough to 2 and to 0 respectively.

This finishes de proof of Theorem 2.9.

Chapter 3

On centred maximal functions associated to general measures.

Throughout the remaining part of this work we will mostly study maximal functions on Euclidean balls when the underlying measure is not that of Lebesgue. To that end, let us define for a Radon measure μ in $\mathbb{R}^n$ its associated maximal function as

$$M_\mu g(x) := \sup_{\substack{R>0 \\ \mu(B(x,R))>0}} \frac{1}{\mu(B(x,R))} \int_{B(x,R)} |g(y)| \, d\mu(y).$$

Since we are considering centred maximal operators, the boundedness on $L^p(\mu)$ for $p > 1$ and the weak boundedness on $L^1(\mu)$ are out of the question. Very little is known about the dependency with the dimension of these bounds. In fact, by using the standard Besicovitch covering Lemma to prove these estimates we can only ensure that the growth of this constants is at most exponential.

We will restrict our attention to maximal operators associated to rotation invariant measures. When μ is the Lebesgue measure, we know by work of M.T. Menárguez and F. Soria [54] (see also Chapter 1) that the action of the maximal function on radial functions is weakly bounded on L^1 independently of the dimension. If μ is a radially increasing measure (in the sense that $\mu(B(x,R)) \leq \mu(B(y,R))$ if $|x| \leq |y|$) the argument applies and the same can be proved. This was pointed out by A. Infante [44].

For finite measures μ with a bounded density J.M. Aldaz proved that the weak $L^1(\mu)$ norm of M_μ grows exponentially with n with a lower bound independent of μ. Here we present an extension of this last result.We show that, if in addition the density of μ is radially decreasing, for a small range of p above 1 ($1 < p < p_0$ with $p_0 \approx 1.005$), the $L^p(\mu)$ norm of M_μ grows exponentially to infinity with lower bounds independent of μ. This is true even when restricting the action to radially decreasing functions.

It is natural to ask ourselves how sharp the value of p_0 is. We will see that the same technique applied to some concrete families of measures gives the result for slightly higher values of p_0. The study of these measures shows however the limitation of the method since, as we will see, the values of p_0 that one can obtain in this way cannot exceed 1.049.

3.1 Besicovitch covering lemma and the boundedness of centred maximal functions.

This section deals with some general facts concerning maximal operators associated to some given measures. Let μ be a Radon measure in $\mathbb{R}^n$. With the above notation it is obvious to see that

$$\|M_\mu g\|_{L^\infty(\mu)} \leq \|g\|_{L^\infty(\mu)}. \tag{3.1}$$

For $1 \leq p < \infty$ we have the following

Theorem 3.1. *Let μ be a Radon measure in $\mathbb{R}^n$ and M_μ the associated maximal operator. Then we have the following bounds*

$$\mu\left(\{y \in \mathbb{R}^n : M_\mu g(y) > \lambda\}\right) \leq \frac{c}{\lambda}\|g\|_{L^1(\mu)}, \tag{3.2}$$

$$\|M_\mu g\|_{L^p(\mu)} \leq C\|g\|_{L^p(\mu)}, \text{ for } 1 < p < \infty, \tag{3.3}$$

This is a consequence of the following lemma due to Besicovitch. We say that a family $\mathcal{F}$ of balls is a Besicovitch covering of a set $E \subset \mathbb{R}^n$ if each point of E is the centre of a ball of $\mathcal{F}$.

Theorem 3.2 (Besicovitch covering Theorem). *Let $n \in \mathbb{N}$ and E be a bounded subset of $\mathbb{R}^n$. There exists an integer b_n only depending on n, such that if $\mathcal{F}$ is a Besicovitch covering of E, one can cover E with at most b_n subfamilies of $\mathcal{F}$, namely $A_1, \ldots, A_{b_n}$, each one consisting of pairwise disjoint balls.*

An elegant proof of this result can be found in [43]. Theorem 3.1 follows from this in a standard way.

Denote by $c_{\mu,1}$ and $C_{\mu,p}$, respectively, the best constants in (3.2) and in (3.3). A natural question in this context is that of determining whether there are bounds independent of the dimension for these constants. In the proof of Theorem 3.1 by the Besicovitch covering lemma one gets $c_{\mu,1} \leq b_n$. Z. Füredi and P.A. Loeb proved that for balls given by arbitrary norms $b_n \leq (2.691 + o(1))^n$; in the particular case of Euclidean balls it was obtained by J.M. Sullivan [76] that $b_n \leq (2.641 + o(1))^n$. This settles that $c_{1,\mu}$ grows at most exponentially with the dimension.

The same can be said for the constants $C_{\mu,p}$ for $1 < p < \infty$, since by the Marcinkiewicz interpolation Theorem

$$C_{\mu,p} \leq 2 \left(\frac{p}{p-1} \right)^{1/p} c_{\mu,1}^{1/p} \leq 2 \left(\frac{p}{p-1} \right)^{1/p} b_n^{1/p}.$$

In the sequel, it will be very convenient to consider instead of (3.3) the weak $L^p(\mu)$ bounds

$$\mu \left\{ (y : M_\mu g(y) > \lambda) \right\}^{1/p} \leq \frac{c}{\lambda} \|g\|_{L^p(\mu)}, \quad \lambda > 0.$$

The best constant in this inequality, $c_{\mu,p}$, satisfies

$$C_{\mu,p} \geq c_{\mu,p} \geq \frac{\lambda \, \mu(\{y \in \mathbb{R}^n : M_\mu g(y) \geq \lambda\})^{1/p}}{\|g\|_{L^p(\mu)}}, \tag{3.4}$$

for all $\lambda > 0$ and all $g \in L^p(\mu)$, with $g \neq 0$. We will bound $C_{\mu,p}$ from below by means of these two inequalities. Although $C_{\mu,p}$ might be significantly larger than $c_{\mu,p}$, they cannot have a very different behaviour with respect to the dimension. Indeed, if $c_{\mu,p}$ is bounded uniformly in the dimension, then by real interpolation $C_{\mu,q}$ is also bounded with respect to the dimension for all $q > p$.

The upper bound $c_{\mu,p} \leq b_n^{1/p}$ is obtained from Jensen inequality as follows

$$\begin{aligned} \mu \left\{ (y : M_\mu g(y) > \lambda) \right\} &= \mu \left\{ (y : M_\mu g(y)^p > \lambda^p) \right\} \\ &\leq \mu \left\{ (y : M_\mu (g^p)(y) > \lambda^p) \right\} \\ &\leq \frac{b_n}{\lambda^p} \|g^p\|_{L^1(\mu)} = \frac{b_n}{\lambda^p} \|g\|_{L^p(\mu)}^p. \end{aligned}$$

3.2 Maximal operators associated to finite and radially decreasing measures

In this section we will present negative results for the uniform bound of the weak L^p constants of maximal operators, as the dimension goes to infinity. We start with the following result by J.M. Aldaz [2]:

Theorem 3.3. *Let μ be a finite measure in $\mathbb{R}^n$ with a bounded density. Then one has*

$$c_{\mu,1} \geq C \left(\frac{2}{\sqrt{3}}\right)^{n/6},$$

where $C > 0$ is an absolute constant depending neither on μ nor on the dimension n.

This says that the best constants in the weak L^1 bounds grow exponentially to infinity with the dimension and so, roughly speaking, Besicovitch covering lemma gives the best result we can hope. In particular, we see that Theorem 1.16 by E.M. Stein and J.O. Strömgerg is not valid in this context. We present the following extension of Theorem 3.3 to the weak type L^p for values of p greater than 1, that originally appeared in [21].

Theorem 3.4. *There exists a $p_0 > 1$ such that for each p in the range $1 \leq p < p_0$, we can find a constant $a_p > 1$, depending only on p so that for each finite measure μ in $\mathbb{R}^n$ with a bounded and radially decreasing density one has*

$$c_{\mu,p} \geq a_p^n.$$

Obviously, since $C_{\mu,p} \geq c_{\mu,p}$, we have the same bound for the best constants in the strong $L^p(\mu)$ bounds. This shows that Theorem 1.13 by E.M. Stein does not hold for general measures.

Denote by B_r the ball with radius r centred at the origin, and let $\chi_r = \chi_{B_r}$ be its characteristic function. The starting point in the proof of Theorem 3.4 is the following proposition.

Lemma 3.5. *Let μ be a rotation-invariant Radon measure in $\mathbb{R}^n$. Then for each $x \in \mathbb{R}^n$ and $r, R > 0$ such that $\mu(B(x,R)) > 0$, we have that*

$$c_{\mu,p} \geq M_\mu \chi_r(x) \left(\frac{\mu(B_{|x|})}{\mu(B_r)}\right)^{1/p} \geq \frac{\mu(B(x,R) \cap B_r)}{\mu(B(x,R))} \left(\frac{\mu(B_{|x|})}{\mu(B_r)}\right)^{1/p} =: T_{\mu,p}(x,r). \tag{3.5}$$

In order to show this proposition we will use the following result, valid for a general measure μ.

Lemma 3.6. *Let μ be a Radon measure on $\mathbb{R}^n$. Then the maximal function $M_\mu\chi_r$ is decreasing on each ray from the origin. That is, for any $x \in \mathbb{R}^n$ and $y = \alpha x$ with $0 < \alpha < 1$ we have that*

$$M_\mu\chi_r(x) \leq M_\mu\chi_r(y). \tag{3.6}$$

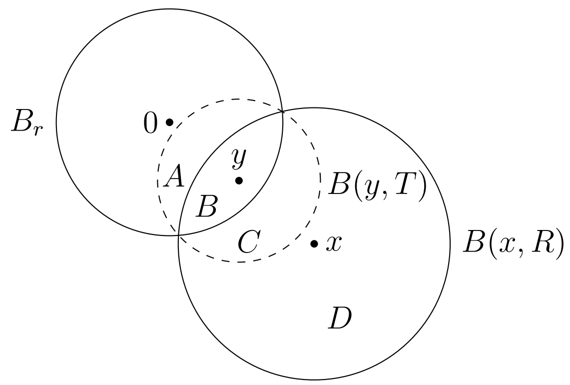
Proof that Lemma 3.6 implies Lemma 3.5. The first inequality in (3.5) follows if we let $g = \chi_r$ and $\lambda = M_\mu\chi_r(x)$ in (3.4), since Lemma 3.6 implies that $M_\mu\chi_r(y) \geq M_\mu\chi_r(x)$ for any $y \in B_{|x|}$. and as a consequence $B_{|x|} \subset \{y : M_\mu\chi_r(y) \geq M_\mu\chi_r(x)\}$. For the second inequality, just realise that $M_\mu\chi_r(x)$ is greater than or equal to any integral mean around x. □

Proof of Lemma 3.6. First we discard the trivial case when $y \in B_r$, since then $M_\mu\chi_r(y) = 1$ and we always have $M_\mu\chi_r(x) \leq 1$. Assume that y and (consequently) x are not in B_r. It would be enough to show that for each $R > 0$ we can find a $T > 0$ such that

$$\frac{\mu(B(x,R) \cap B_r)}{\mu(B(x,R))} \leq \frac{\mu(B(y,T) \cap B_r)}{\mu(B(y,T))}.$$

Take T so that $B(y,T)$ is the minimal ball centred at y containing $B(x,R) \cap B_r$. We call

$$\begin{aligned} A &= \mu(B(y,T) \setminus B(x,R)), & B &= \mu(B(x,R) \cap B_r), \\ C &= \mu(B(y,T) \setminus B_r), & D &= \mu(B(x,R) \setminus B(y,T)). \end{aligned}$$



Now it is clear that

$$\begin{aligned}\frac{\mu(B(x,R)\cap B_r)}{\mu(B(x,R))} &= \frac{B}{B+C+D} \leq \frac{B}{B+C} \leq \frac{A+B}{A+B+C} \\ &= \frac{\mu(B(y,T)\cap B_r)}{\mu(B(y,T))}.\end{aligned}$$

□

Remark 3.7. *In order to obtain lower bounds for $c_{\mu,p}$ using Lemma 3.5, there is no point in considering the case $|x| \leq r$, since it will never lead to a lower bound greater than 1. If $|x| \leq r$, the inclusions $B_r \subset B(x, |x|+r)$ and $B_{|x|} \subset B_r$ lead to*

$$T_{\mu,p}(x,r) = \frac{\mu(B_r)}{\mu(B(x,|x|+r))}\left(\frac{\mu(B_{|x|})}{\mu(B_r)}\right)^{1/p} \leq 1.$$

Proof of Theorem 3.4. Consider $r > 0$ and $x \in \mathbb{R}^n$ so that $|x| > r$. It is obvious that

$$M_\mu \chi_r(x) \geq \frac{\mu(B(x,|x|+r)\cap B_r)}{\mu(B(x,|x|+r))} = \frac{\mu(B_r)}{\mu(B(x,|x|+r))}.$$

Then, by Proposition 3.5 we have

$$\begin{aligned} c_{\mu,p} &\geq T_{\mu,p}(x,r) = \frac{\mu(B_r)}{\mu(B(x,|x|+r))}\left(\frac{\mu(B_{|x|})}{\mu(B_r)}\right)^{1/p} \\ &= \frac{\mu(B_{|x|})}{\mu(B(x,|x|+r))}\left(\frac{\mu(B_r)}{\mu(B_{|x|})}\right)^{1-1/p}.\end{aligned}$$

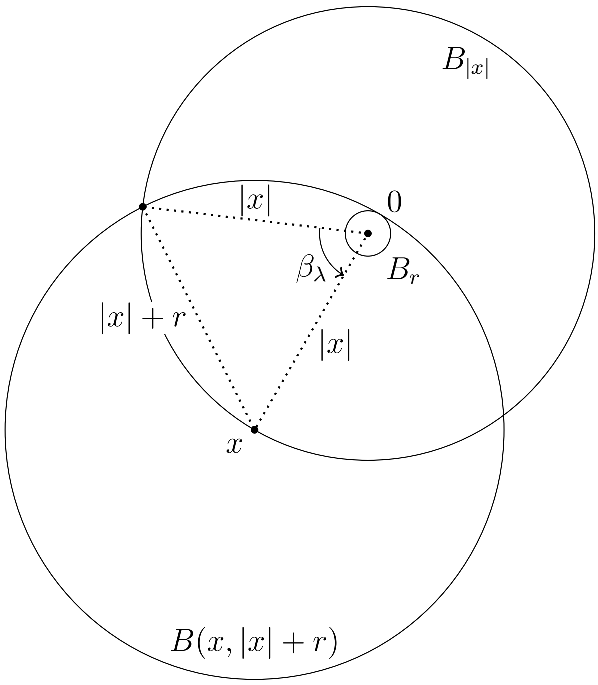
By hypothesis, we can write $d\mu(y) = f(|y|)\,dy$, where f is a decreasing function along $[0,\infty)$. Hence, setting $\lambda := r/|x|$,

$$\begin{aligned}\frac{\mu(B_{|x|})}{\mu(B_r)} &= \frac{\int_0^{|x|} f(s)s^{n-1}\,ds}{\int_0^r f(s)s^{n-1}\,ds} = 1 + \frac{\int_r^{|x|} f(s)s^{n-1}\,ds}{\int_0^r f(s)s^{n-1}\,ds} \leq 1 + \frac{\int_r^{|x|} f(r)s^{n-1}\,ds}{\int_0^r f(r)s^{n-1}\,ds} \\ &= 1 + \frac{|x|^n - r^n}{r^n} = \lambda^{-n}. \qquad (3.7)\end{aligned}$$

Now we compare the μ measures of $B(x,|x|+r)$ and $B_{|x|}$. For this, we split $B(x,|x|+r)$ into two disjoint pieces

$$\begin{aligned} D &:= B(x,|x|+r)\cap B_{|x|}, \\ E &:= B(x,|x|+r)\setminus B_{|x|},\end{aligned}$$

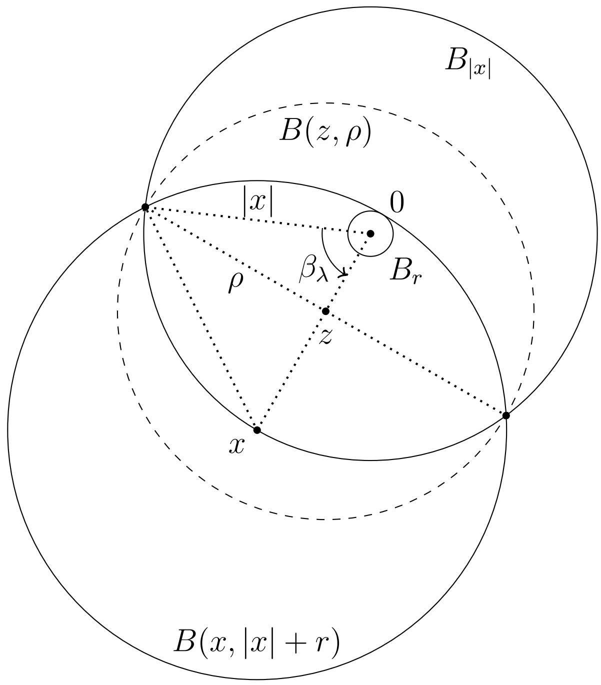
and work on them separately.



We denote by β_λ, the angle between the segment that connects the origin with x and the one that connects the origin with any point in $\partial B(x, |x| + r) \cap \delta B_{|x|}$. The cosine law applied to the triangle whose vertices are the origin, x and a point in $\partial B(x, |x| + r) \cap \partial B_{|x|}$ yields

$$\cos \beta_\lambda = 1 - \frac{(1+\lambda)^2}{2}.$$

The notation β_λ is very convenient because λ (conversely to $|x|$ or r) will be an invariant respect to n.



$z = \cos\beta_\lambda\, x,$

$\rho = \sin\beta_\lambda\, |x|.$

Observe that the maximal diameter of D is $|x|\sin\beta_\lambda$ and that D is contained in $B(x\cos\beta_\lambda, |x|\sin\beta_\lambda)$. Then, using that μ is radially decreasing we have

$$\mu(D) \le \mu(B(x\cos\beta_\lambda, |x|\sin\beta_\lambda)) \le \mu(B_{|x|\sin\beta_\lambda}). \tag{3.8}$$

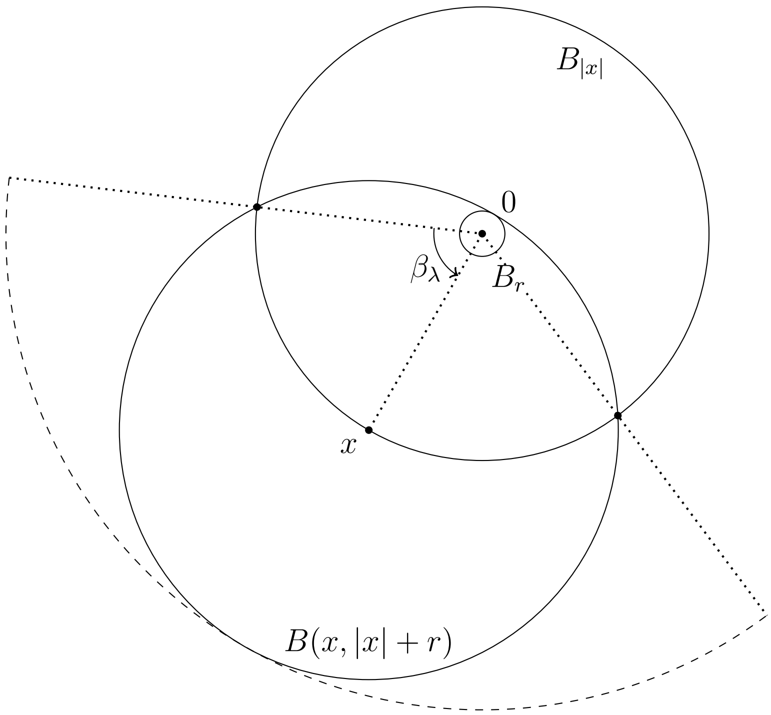
The function $u : s \mapsto \mu(B_{s\sin\beta_\lambda})/\mu(B_s)$ tends to 1 as s tends to infinity and by Lebesgue differentiation Theorem

$$\begin{aligned}\lim_{s\to 0} u(s) &= \lim_{s\to 0} \frac{\int_{B_{s\sin\beta_\lambda}} f(y)\,dy}{|B_{as\sin\beta_\lambda}|} \frac{|B_{s\sin\beta_\lambda}|}{|B_s|} \frac{|B_s|}{\int_{B_s} f(y)\,dy} \\ &= f(0)(\sin\beta_\lambda)^n \frac{1}{f(0)} = (\sin\beta_\lambda)^n.\end{aligned}$$

As u is continuous, it is possible to find an s so that $u(s) = (\sin\beta_\lambda)^{kn}$ for a $k \in (0,1)$ that will be determined later. So there exists an $x \in \mathbb{R}^n$ such that

$$\mu(B_{|x|\sin\beta_\lambda}) = (\sin\beta_\lambda)^{kn}\mu(B_{|x|}). \tag{3.9}$$

For E we proceed as follows. We impose $\lambda < \sqrt{2} - 1$ in order to have $\cos\beta_\lambda > 0$. We define the cone $F = \{y \in \mathbb{R}^n : \langle x, y\rangle \ge \cos\beta_\lambda\, |x||y|\} \supset E$.



Integrating in spherical coordinates

$$\begin{aligned}
\mu(E) &\leq \mu(F \cap B_{2|x|+r} \setminus B_{|x|}) = \int_{|x|}^{2|x|+r} \Big| F \cap \partial B_s \Big|_{n-1} f(s)\, ds \\
&= \int_{|x|}^{2|x|+r} \omega_{n-2} \int_0^{\beta_\lambda} (s \sin\theta)^{n-2} s\, d\theta f(s)\, ds \\
&\leq \frac{\omega_{n-2}}{\cos\beta_\lambda} \int_{|x|}^{2|x|+r} \int_0^{\beta_\lambda} (\sin\theta)^{n-2} \cos\theta\, d\theta\ f(s) s^{n-1}\, ds \\
&= \frac{\omega_{n-2}}{\cos\beta_\lambda} \frac{(\sin\beta_\lambda)^{n-1}}{n-1} \int_{|x|}^{2|x|+r} f(s) s^{n-1}\, ds.
\end{aligned}$$

Using inequality (7) the last quantity above is less than or equal to

$$\frac{1}{\sqrt{2\pi(n-2)}} \frac{(\sin\beta_\lambda)^n}{\sin\beta_\lambda \cos\beta_\lambda} \mu(B_{2|x|+r} \setminus B_{|x|}).$$

As $0 < \sin\beta_\lambda < 1$, for

$$\gamma = -\frac{\log(2+\lambda)}{\log\sin\beta_\lambda},$$

we have $(\sin\beta_\lambda)^{-\gamma}|x| = 2|x| + r$. We can assume that $|x|$ is the maximal radius for which (3.9) holds, and then writing $\ell = \lceil \gamma \rceil$

$$\mu(B_{2|x|+r}) = \mu(B_{|x|(\sin\beta_\lambda)^{-\gamma}}) \leq \mu(B_{|x|(\sin\beta_\lambda)^{-\ell}}) \leq (\sin\beta_\lambda)^{-\ell k n}\mu(B_{|x|}).$$

Summarising, we have obtained

$$\mu(E) \leq \frac{1}{\sqrt{2\pi(n-1)}} \frac{(\sin\beta_\lambda)^{n(1-\ell k)}}{\sin\beta_\lambda \cos\beta_\lambda}\mu(B_{|x|}),$$

which together with (3.8) and (3.9) gives

$$\frac{\mu(B_{|x|})}{\mu(B_{2|x|+r})} \leq \frac{\mu(B_{|x|})}{\mu(D)+\mu(E)} \leq \frac{1}{Q(\sin\beta_\lambda)^{n(1-\ell k)} + (\sin\beta_\lambda)^{nk}},$$

where $Q = (\sqrt{2\pi(n-1)}\sin\beta_\lambda\cos\beta_\lambda)^{-1} \overset{n\to\infty}{\longrightarrow} 0$. The last quantity in the previous inequality attains its maximal growth with respect to n when the two terms in the denominator are of the same exponential size, that is when $1 - \ell k = k$. This fixes $k = 1/(1+\ell)$, and consequently $|x|$ is fixed too by the condition of being the maximal radius for which equality (3.9) is verified. Now

$$\begin{aligned} c_{\mu,p} &\geq T_{\mu,p}(|x|, r) = \frac{\mu(B_{|x|})}{\mu(B(x, |x|+r))}\left(\frac{\mu(B_r)}{\mu(B_{|x|})}\right)^{1/p'} \\ &\geq \frac{1}{1+Q}\left(\frac{\lambda^{1/p'}}{(\sin\beta_\lambda)^k}\right)^n. \end{aligned}$$

Observe that, although $|x|$ (and consequently r) can change with the dimension (see remark 3.9) and with μ, it is possible to choose a universal λ so that none of k, ℓ or β_λ depend on n. The constant $c_{\mu,p}$ will grow exponentially with the dimension if

$$\frac{\lambda^{1/p'}}{(\sin\beta_\lambda)^k} > 1,$$

and this is equivalent to

$$p' > \frac{\log\lambda}{k\log\sin\beta_\lambda}.$$

Thus, we are done by taking p_0 so that

$$p_0' = \inf_{0<\lambda<\sqrt{2}-1} \frac{\log\lambda}{k\log\sin\beta_\lambda}.$$

□

The analytic computation of p_0 is rather complicated. By a numerical estimate via MATLAB we obtained $p_0 \approx 1.005274$.

Remark 3.8. *Note that in the previous proof the finiteness of μ was only used to ensure that $\lim_{s\to\infty} u(s) = 1$, and that the boundedness of the density f was only necessary to have $\lim_{s\to 0} u(s) = (\sin\beta_\lambda)^n$. It is natural to think that these assumptions can be replaced by milder ones with the same effect. This was mentioned in [2], and used explicitly in [4]. We will comment in more detail the results of this last paper at the end of this section and in the following ones.*

Remark 3.9. *It is interesting to make the following observations about the radii chosen above. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a decreasing function such that all the measures $d\mu_n = f(|x|)\,dx$ are finite on $\mathbb{R}^n$. Fixing λ and taking β_λ as in the previous proof, we take $x_n \in \mathbb{R}^n$ as one of the vectors x with maximal modulus for which (3.9) holds for μ_n in $\mathbb{R}^n$. The observation is that this modulus does not shrink to 0 as the dimension grows. In fact, given $R > 0$ such that $f(R) > 0$, using that f is decreasing*

$$\frac{\mu(B_{\sin\beta_\lambda} R)}{\mu(B_R)} = \frac{\int_0^{\sin\beta_\lambda} f(s)s^{n-1}\,ds}{\int_0^R f(s)s^{n-1}\,ds} \leq \frac{f(0)\int_0^{R\sin\beta_\lambda} s^{n-1}\,ds}{f(R)\int_0^R f(s)s^{n-1}\,ds} = \frac{f(0)}{f(R)}(\sin\beta_\lambda)^n,$$

and this last amount is smaller than $(\sin\beta_\lambda)^{nk}$ for large n. Thus, for (3.9) to hold in $\mathbb{R}^n$ for such n, it is necessary to take $|x_n| > R$. In particular, if f has compact support, then

$$\liminf_{n\to\infty} |x_n| \geq \max_{R\in supp f} R;$$

and, if the support of f is unbounded, then $\lim_{n\to\infty} |x_n| = \infty$.

Another observation is that $f(|x_n|)$ decreases exponentially. To see this, observe that by the definition of x_n and the decreasing property of f

$$(\sin\beta_\lambda)^{nk} = \frac{\mu(B_{\sin\beta_\lambda |x_n|})}{\mu(B_{|x_n|})} \leq \frac{f(0)}{f(|x_n|)}(\sin\beta_\lambda)^n.$$

Hence, we obtain $f(|x_n|) \leq f(0)(\sin\beta_\lambda)^{n(1-k)}$ as wanted.

Remark 3.10. *Note that since the chosen test functions are of the form χ_r, we have proved indeed that, for $1 \leq p < p_0$, the best constants in the weak-type*

$L^p(\mu)$ are unbounded even when restricting the action to radially decreasing functions. Recall that the centred maximal operators on Euclidean balls associated to Lebesgue measure admits dimension-free weak L^1 estimates when acting on radial functions, as was shown by M.T. Menárguez and F. Soria in [54] (see Chapter 1). With the arguments of [54], A. Infante [44] has shown that M_μ has the same behaviour on radial functions when μ is given by a radial non-decreasing density.

3.3 Concrete examples. The scope of the method.

The bound of p_0 obtained ($p_0 \geq 1.005274$) in the proof of Theorem 3.4 for general measures is rather disappointing. One would expect that better estimates may hold for concrete families of measures. This will be exploited in the next Chapter. Before doing that, it is worth pointing out that with the arguments reflected in Theorem 3.4 there is little room for improvement.

In order to see this, recall that the starting point in our reasoning was the bound

$$c_{\mu,p} \geq T_{\mu,p}(x,r).$$

Then for $p \in [1, p_0)$ and each $n \in \mathbb{N}$ we can find $x_n \in \mathbb{R}^n$ and r_n so that $T_{\mu,p}(x_n, r_n)$ grows to infinity as $n \to \infty$. We will show, in the two examples that follow in this section, that is not true in general for slightly larger values of p.

3.3.1 The Gaussian measure.

Consider γ_n as the Gaussian measure over $\mathbb{R}^n$, given by the density $\gamma_n(x) = \mathrm{e}^{-\pi|x|^2}$.

Proposition 3.11. *Set $T_n := \sqrt{(n-1)/2\pi}$. Then, there exist exponents $p_1 > p_0 > 1$, with approximate values $p_0 \approx 1.011871$ and $p_1 \approx 1.049427$ such that*

(i) for each $p \in [1, p_0)$ there exists an $a_p > 1$ only depending on p and sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{r_n\}_{n\in\mathbb{N}}$ such that $c_{\gamma_n,p} \geq T_{\gamma_n,p}(x_n, r_n) \geq a_p^n$.

(ii) for each $p > p_1$, given any choice of $x_n \in \mathbb{R}^n$ and $r_n > 0$, one has $T_{\gamma_n,p}(x_n, r_n) \leq C_p$ for some $C_p > 0$ depending only on p.

Part (i) of the proposition shows a better estimate of the exponent p for γ_n with regards to the one obtained in Theorem 3.4. Part (ii), on the other hand, shows that the method used so far has a strong limitation. This will be clarified in Chapter 4.

Before proving this result we will comment on some properties of the Gaussian measure that will be used here and in the next chapter.

By an integration in polar coordinates the Gaussian measure of a centred ball B_R can be written as

$$\gamma_n(B_R) = \omega_{n-1} \int_0^R \mathrm{e}^{-\pi s^2} s^{n-1}\, ds.$$

An elementary study of the function $G(s) := \mathrm{e}^{-\pi s^2} s^{n-1}$ yields that G is increasing in the interval $(0, T_n)$ and decreasing in (T_n, ∞), where $T = \sqrt{(n-1)/2\pi}$.

Lemma 3.12. *Letting* $0 < R < T_n$, *we have*

$$\mathrm{e}^{-\pi R^2}|B_R| \leq \gamma_n(B_R) \leq n\mathrm{e}^{-\pi R^2}|B_R|.$$

Proof. The first inequality is very easy

$$\gamma_n(B_R) = \int_{B_R} \mathrm{e}^{-\pi|y|^2}\, dy \geq \int_{B_R} \mathrm{e}^{-\pi R^2}\, dy = \mathrm{e}^{-\pi R^2}|B_R|.$$

For the second one we write the γ_n measure of B_R as the integral of G and use that G is decreasing in $(0, T_n)$

$$\begin{aligned} \gamma_n(B_R) &= \omega_{n-1} \int_0^R G(s)\, ds \leq \omega_{n-1} \int_0^R G(R)\, ds \\ &= \omega_{n-1}\mathrm{e}^{-\pi R^2} R^n = n\mathrm{e}^{-\pi R^2}|B_R|. \end{aligned}$$

□

Moreover it is also easy to see that G is concave in the interval (T_n^-, T_n^+) and convex in $(0, T_n^-)$ and in (T_n^+, ∞), where

$$T_n^{\pm} = \sqrt{\frac{2n - 1 \pm \sqrt{8n-7}}{4\pi}}.$$

An essential part of the mass of γ_n is concentrated around the sphere of radius T_n. Indeed we have

Lemma 3.13. *For n large enough, we have that $\gamma_n(B_{T_n} \setminus B_{T_n^-}) > 1/5$.*

Proof. Since $G(s) = \mathrm{e}^{-\pi s^2} s^{n-1}$ is decreasing in $(0, T_n)$

$$\begin{aligned}\gamma_n(B_{T_n} \setminus B_{T_n^-}) &= \omega_{n-1} \int_{T_n^-}^{T_n} G(s)\, ds \geq \omega_{n-1} \int_{T_n^-}^{T_n} G(T_n^-)\, ds \\ &= \omega_{n-1} G(T_n^-)\, (T_n - T_n^-).\end{aligned}$$

It is enough to see that

$$\lim_{n\to\infty} \omega_{n-1} G(T_n^-)(T_n - T_n^-) \geq 1/5.$$

We can calculate

$$T_n - T_n^- = \frac{T_n^2 - (T_n^-)^2}{T_n + T_n^-} = \frac{\dfrac{1+\sqrt{8n-7}}{4\pi}}{\sqrt{\dfrac{n-1}{2\pi}} + \sqrt{\dfrac{2n - 1\sqrt{8n-7}}{4\pi}}} \overset{n \longrightarrow \infty}{\longrightarrow} \frac{1}{2\sqrt{\pi}}.$$

Using the Stirling Formula

$$\Gamma(t) = \sqrt{\frac{2\pi}{t}} \left(\frac{t}{e}\right)^t (1 + \mathcal{O}(1/t)),$$

it follows that

$$\omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)} = (1 + \mathcal{O}(1/n))\sqrt{2e}\, \frac{e^{-(n-1)/2}}{\left(\dfrac{2\pi}{n}\right)^{(n-1)/2}}.$$

Note that

$$e^{-(n-1)/2}\, e^{-\pi(T_n^-)^2} = e^{(-1+\sqrt{8n-7})/4}.$$

We write

$$\frac{(T_n^-)^{n-1}}{\left(\dfrac{n}{2\pi}\right)^{(n-1)/2}} = \left(1 - \frac{1+\sqrt{8n-7}}{2n}\right)^{(n-1)/2} = (1-h)^{n/2-1/2},$$

where

$$h = \frac{1+\sqrt{8n-7}}{2n}.$$

Recall that if $h > 0$, then

$$e^{1+h/2} \leq (1-h)^{-1/h} \leq e^{1+h/2+h^2/(2-2h)}.$$

If $0 < h < 1/2$ this implies that $(1-h)^{1/h} \leq e^{-1-h/2-h^2}$. Hence, for large enough n we have

$$(1-h)^{n/2} \geq \exp\left(-\frac{1+\sqrt{8n-7}}{4} - \frac{(1+\sqrt{8n-7})^2}{16n} - \frac{(1+\sqrt{8n-7})^3}{16n^2}\right).$$

Thus,

$$\lim_{n\to\infty} \omega_{n-1} G(T_n^-) \geq \left(\frac{2}{e}\right)^{1/2}.$$

and putting everything together we have

$$\lim_{n\to\infty} \omega_{n-1} G(T_n^-)\,(T_n - T_n^-) \geq (2e\pi)^{-1/2} > 1/5.$$

□

Now we are prepared to demonstrate Proposition 3.11.

Proof of Proposition 3.11. First, let us show why the only case that we have to study is the one that $0 < r_n < |x_n| \leq T_n$ and $\lambda = r_n/|x_n| < \sqrt{2}-1$. In view of Remark 3.7 there is no point in considering $r_n \geq |x_n|$. Moreover, taking $|x_n| > T_n$ implies no worthy improvement compared with the choice $|x_n| = T_n$. Assume that $|x_n| > T_n$ and call $y_n = T_n x_n/|x_n|$. In view of Lemma 3.13, for large n we have $\gamma_n(B_{|y_n|}) \leq 1 \leq 5\gamma_n(B_{T_n})$. This and the inclusion $B(x_n, |x_n| + r_n) \supset B(y_n, |y_n| + r_n)$ imply

$$\begin{aligned} T_{\gamma_n,p}(x_n, r_n) &= \frac{\gamma_n(B_{r_n})}{\gamma_n(B(x_n, |x_n| + r_n))}\left(\frac{\gamma_n(B_{|x_n|})}{\gamma_n(B_{r_n})}\right)^{1/p} \\ &\leq 5^{1/p}\frac{\gamma_n(B_{r_n})}{\gamma_n(B(y_n, |y_n| + r_n))}\left(\frac{\gamma_n(B_{|y_n|})}{\gamma_n(B_{r_n})}\right)^{1/p} \\ &= 5^{1/p} T_{\gamma_n,p}(y_n, r_n). \end{aligned}$$

If $\lambda \geq \sqrt{2}-1$, then $\beta_\lambda \geq \pi/2$. This means that $B(x_n, |x_n| + r_n)$ contains more than half of $B_{|x_n|}$, and so $\gamma_n(B(x_n, |x_n| + r_n)) \geq \gamma_n(B_{|x_n|})/2$, which together with $B_{r_n} \subset B_{|x_n|}$ gives

$$T_{\gamma_n,p}(x_n, r_n) = \frac{\gamma_n(B_{|x_n|})}{\gamma_n(B(x_n, |x_n| + r_n))}\left(\frac{\gamma_n(B_{r_n})}{\gamma_n(B_{|x_n|})}\right)^{1/p'} \leq 2.$$

Hence, we assume that $|x_n| = \xi T_n$ and $r_n = rT_n$ for certain $0 < r < \xi \geq 1$ satisfying $\lambda = r/\xi < \sqrt{2} - 1$. We begin showing (ii). By Lemma 3.12, fixing λ, we have

$$\begin{aligned} \frac{\gamma_n(B_{r_n})}{\gamma_n(B_{|x_n|})} &\leq \frac{n\,\mathrm{e}^{-\pi r_n^2}|B_{r_n}|}{\mathrm{e}^{-\pi|x_n|^2}|B_{|x_n|}|} = n\mathrm{e}^{(\xi^2-r^2)(n-1)/2}\lambda^n \\ &\leq Cn\left(\mathrm{e}^{(\xi^2-r^2)/2}\lambda\right)^n \leq Cn\left(\mathrm{e}^{(1-\lambda^2)/2}\lambda\right)^n \end{aligned} \tag{3.10}$$

We call F the cone in $\mathbb{R}^n$ with axis direction x_n and angle β_λ, i.e. $F = \{y \in \mathbb{R}^n : \langle x_n, y\rangle \geq \cos\beta_\lambda |x_n||y|\}$. We have $B_{|x_n|} \cap F \subset B(x_n, |x_n| + r_n)$. Integrating in spherical caps where the density is constant

$$\begin{aligned} \gamma_n(B(x_n, |x_n| + r_n)) &\geq \gamma_n(B_{|x_n|} \cap F) \\ &= \int_0^{|x_n|} \mathrm{e}^{-\pi s^2} \int_0^{\beta_\lambda} \omega_{n-2}(s\ \sin\theta)^{n-2} s\, d\theta\, ds \\ &\geq \omega_{n-2} \int_0^{|x_n|} \int_0^{\beta_\lambda} (\sin\theta)^{n-2} \cos\theta\, d\theta\ \mathrm{e}^{-\pi s^2} s^{n-1}\, ds \\ &= \frac{\omega_{n-2}}{n-1} \frac{(\sin\beta_\lambda)^n}{\sin\beta_\lambda} \int_0^{|x_n|} \mathrm{e}^{-\pi s^2} s^{n-1}\, ds \\ &\geq \frac{1}{\sqrt{2\pi(n-1)}} \frac{(\sin\beta_\lambda)^n}{\sin\beta_\lambda} \gamma_n(B_{|x_n|}), \end{aligned} \tag{3.11}$$

where for the last inequality (7) was used. Now using (3.10) and (3.11) we have

$$\begin{aligned} T_{\gamma_n,p}(x_n, r_n) &= \frac{\gamma_n(B_{|x_n|})}{\gamma_n(B(x_n, |x_n| + r_n))} \left(\frac{\gamma_n(B_{r_n})}{\gamma_n(B_{|x_n|})}\right)^{1/p'} \\ &\leq \sqrt{2\pi(n+1)}\sin\beta_\lambda n^{1/p'} \left(\frac{(\mathrm{e}^{(1-\lambda^2)/2}\lambda)^{1/p'}}{\sin\beta_\lambda}\right)^n. \end{aligned}$$

We can ensure the uniform boundedness of this last term for a fixed $\lambda \in (0, \sqrt{2} - 1)$ if we had

$$\frac{(\mathrm{e}^{(1-\lambda^2)/2}\lambda)^{1/p'}}{\sin\beta_\lambda} < 1.$$

This condition is equivalent to

$$p' < \frac{(1-\lambda^2)/2 + \log\lambda}{\log\sin\beta_\lambda}.$$

Then for every $p > p_1$, where p_1 is such that

$$p_1' = \inf_{0<\lambda<\sqrt{2}-1} \frac{(1-\lambda^2)/2 + \log\lambda}{\log\sin\beta_\lambda},$$

there exists a constant $C_p > 0$ so that $T_{\gamma_n,p}(x_n, r_n) < C_p$. A numerical estimate using MATLAB yields $p_1 \approx 1.04942$.

Now we turn to the proof of (i). We will bound from below

$$T_{\gamma_n,p}(x_n, r_n) = \frac{\gamma_n(B_{|x_n|})}{\gamma_n(B(x_n, |x_n| + r_n))} \left(\frac{\gamma_n(B_{r_n})}{\gamma_n(B_{|x_n|})}\right)^{1/p'} .$$

By Lemma 3.12 we have

$$\frac{\gamma_n(B_{r_n})}{\gamma_n(B_{|x_n|})} \geq \frac{\mathrm{e}^{-\pi r_n^2} r_n^n}{n\,\mathrm{e}^{-\pi|x_n|^2}|x_n|^n} = \mathrm{e}^{(\xi^2-r^2)(n-1)/2}\lambda^n \geq \frac{C}{n}\left(\mathrm{e}^{(\xi^2-r^2)/2}\lambda\right)^n. \quad (3.12)$$

For $B(x_n, |x_n| + r_n)$ we consider the same partition as we did in the proof of Theorem 3.4:

$$\begin{aligned} D &= B(x_n, |x_n| + r_n) \cap B_{|x_n|}, \\ E &= B(x_n, |x_n| + r_n) \setminus B_{|x_n|}. \end{aligned}$$

With the same reasoning that led to (3.8) together with Lemma 3.12 we obtain that

$$\begin{aligned} \gamma_n(D) &\leq \gamma_n(B_{|x_n|\sin\beta_\lambda}) \leq n\,e^{-\pi(|x_n|\sin\beta_\lambda)^2}(|x_n|\sin\beta_\lambda)^n|B_1| \\ &= \omega_{n-1}\,e^{(\xi\sin\beta_\lambda)^2/2}\left(T_n\,\mathrm{e}^{-(\xi\sin\beta_\lambda)^2/2}\,\xi\sin\beta_\lambda\right)^n. \quad (3.13) \end{aligned}$$

Recall that for the cone F defined before

$$E \subset B_{2|x_n|+r_n} \setminus B_{|x_n|} \cap F.$$

Integrating in spherical caps

$$\gamma_n(E) \leq \gamma_n(F \cap B_{2|x_n|+r_n} \setminus B_{|x_n|}) = \int_{|x_n|}^{2|x_n|+r_n} |\partial B_s \cap F|_{n-1}\, e^{-\pi s^2}\, ds.$$

We have

$$\begin{aligned} |\partial B_s \cap F|_{n-1} &= s^{n-1}\left|\mathbb{S}^{n-1} \cap F\right|_{n-1} = s^{n-1}\int_0^{\beta_\lambda}\left|\sin\theta\mathbb{S}^{n-2}\right|_{n-2}\, d\theta \\ &= \omega_{n-2}s^{n-1}\int_0^{beta_\lambda}(\sin\theta)^{n-2}\, d\theta \\ &\leq \frac{\omega_{n-2}s^{n-1}}{\cos\beta_\lambda}\int_0^{beta_\lambda}(\sin\theta)^{n-2}\,\cos\theta\, d\theta \\ &= \frac{\omega_{n-2}s^{n-1}}{\cos\beta_\lambda}\,\frac{(\sin\beta_\lambda)^{n-1}}{n-1} = \frac{\omega_{n-1}(s\sin\beta_\lambda)^{n-1}}{\sqrt{2\pi(n-1)}\sin 2\beta_\lambda}. \end{aligned}$$

Then, using that $G(s) = e^{-\pi s^2} s^{n-1}$ attains its maximum at the point T_n

$$\begin{aligned} \gamma_n(E) &\leq \frac{\omega_{n-1}(\sin\beta_\lambda)^{n-1}}{\sqrt{2\pi(n-1)}\sin 2\beta_\lambda} \int_{|x_n|}^{2|x_n|+r_n} e^{-\pi s^2} s^{n-1}\, ds \\ &\leq \frac{\omega_{n-1}(\sin\beta_\lambda)^{n-1}}{\sqrt{2\pi(n-1)}\sin 2\beta_\lambda} (|x_n| + |r_n|) G(T_n) \\ &= \frac{e^{1/2}(\xi + R)\omega_{n-1}(e^{-1/2}T_n \sin\beta_\lambda)^{n-1}}{\sqrt{2\pi(n-1)}\sin 2\beta_\lambda} \end{aligned} \tag{3.14}$$

The value of ξ for which the quantities raised to the power n in (3.13) and in (3.14) are the same is the solution to the transcendental equation $\xi\, e^{-(\xi \sin\beta_\lambda)^2/2} = e^{-1/2}$. We use the approximate solution $\xi = e^{-(\cos\beta_\lambda)^2/2}$. Since $\xi < 1$ we have

$$\xi\, e^{-(\xi \sin\beta_\lambda)^2/2} = e^{-(\cos\beta_\lambda)^2/2}\, e^{-(\xi \sin\beta_\lambda)^2/2} \geq e^{-1/2}.$$

This means that with our choice of ξ, for large enough dimensions $\gamma_n(E)$ will be much smaller than $\gamma_n(D)$. As a consequence

$$\gamma_n(B(x_n, |x_n| + r_n)) \leq 2\gamma(E) \leq \frac{C}{\sqrt{n-1}}\, \omega_{n-1}\, \left(e^{-(\xi \sin\beta_\lambda)^2/2} \xi T_n \sin\beta_\lambda\right)^n. \tag{3.15}$$

By Lemma 3.12

$$\gamma_n(B_{|x_n|}) \geq e^{-\pi(\xi T_n)^2} (\xi T_n)^n |B_1| = e^{\xi^2/2} (e^{-\xi^2/2} \xi T_n)^n \frac{\omega_{n-1}}{n},$$

and we have then

$$\frac{\gamma_n(B_{|x_n|})}{\gamma_n(B(x_n, |x_n| + r_n))} \geq \frac{C}{\sqrt{n}} \frac{\mathrm{e}^{-\xi^2/2}}{\sin\beta_\lambda \mathrm{e}^{-(\xi \sin\beta_\lambda)^2/2}} = \frac{C}{\sqrt{n}} \frac{e^{-(\xi \cos\beta_\lambda)^2/2}}{\sin\beta_\lambda}.$$

Now we can estimate $T_{\gamma_n,p}(x, r)$ using all the previous calculations

$$T_{\gamma_n,p}(x, r) \geq \frac{C}{n^{3/2}} \frac{e^{-(\xi \cos\beta_\lambda)^2/2}}{(\sin\beta_\lambda)^n} \left(e^{\xi^2(1-\lambda^2)/2} \lambda\right)^{(p-1)/p},$$

which will grow to infinity with the dimension only if

$$\frac{e^{-(\xi \cos\beta_\lambda)^2/2}}{\sin\beta_\lambda} \left(e^{\xi^2(1-\lambda^2)/2} \lambda\right)^{1/p'} > 1.$$

This is equivalent to

$$p' > \frac{\xi^2(1-\lambda^2)/2 + \log\lambda}{(\xi \cos\beta_\lambda)^2/2 + \log\sin\beta_\lambda}.$$

So we can take

$$p_0' := \inf_{0<\lambda<\sqrt{2}-1} \frac{\xi^2(1-\lambda^2)/2 + \log\lambda}{(\xi\cos\beta_\lambda)^2/2 + \log\sin\beta_\lambda}.$$

A numerical estimation via Matlab yields the approximative value $p_0 \approx 1.011871$.

□

3.3.2 Lebesgue measure restricted to the unit ball.

If we consider the radial measure ν_n, given by Lebesgue measure restricted to the unit ball of $\mathbb{R}^n$, that is $d\nu_n(x) = \chi_1(x)\,dx$, there is a more direct way to estimate $T_{\nu_n,p}(x,r)$. As in the case of the Gaussian measure, we will obtain unboundedness of $c_{\nu_n,p}$ with respect to the dimension for slightly larger values of p than in Theorem 3.4. We can show that the method is optimal, in the sense that no bigger exponent p can be reached. This is the content of next proposition.

Proposition 3.14. *There exists a $p_0 > 1$ with approximate value $p_0 \approx 1.03946$ such that*

(i) for each $p \in [1, p_0)$ there exist a constant $a_p > 1$ only depending on p such that $T_{\nu_n,p}(x,r) \geq a_p^n$, for some $0 < r < \sqrt{2}-1$, and where x is any unit vector in $\mathbb{R}^n$.

(ii) for each $p > p_0$, there exist a constant $C_p > 0$ such that for any choice of $x \in \mathbb{R}^n$ and $r > 0$ one has $T_{\nu_n,p}(x,r) < C_p$.

Proof. Let us begin by showing that in order to study the boundedness of $T_{\nu_n,p}(x,r)$ it is enough to concentrate on the case when $|x| = 1$ and $0 < r < \sqrt{2}-1$. First,in view of Remark 3.7 we discard the choice $r \geq |x|$. We also can remove the case $r > 1$, which implies $T_{\nu_n,p}(x,r) = 1$. Taking $|x| > 1$ is no better than $|x| = 1$. To see this, just note that increasing $|x|$ over 1 will only make $\nu_n(B(x,|x|+r))$ bigger, and that makes $T_{\nu_n,p}(x,r)$ decrease. If $|x| < 1$, calling $y = x/|x|$ and $\lambda = r/|x|$ as usual

$$\begin{aligned}
T_{\nu_n,p}(x,r) &= \frac{|B_{|x|}|}{|B(x,|x|+r)\cap B_1|}\left(\frac{|B_r|}{|B_{|x|}|}\right)^{1/p'} \\
&\leq \frac{|B_{|x|}|}{|B(x,|x|+r)\cap B_{|x|}|}\left(\frac{|B_r|}{|B_{|x|}|}\right)^{1/p'} \\
&= \frac{|B_1|}{|B(y,1+\lambda)\cap B_1|}\left(\frac{|B_\lambda|}{|B_1|}\right)^{1/p'} = T_{\nu_n,p}(y,\lambda),
\end{aligned}$$

where the equality before the last one was obtained by homogeneity, by applying the dilation of factor $1/|x|$ to all the balls appearing on the left hand side. Thus, we can assume from now on that x is a unit vector and that $\lambda = r$. There is only left to show that we can restrict ourselves to $r = \lambda < \sqrt{2}-1$. This can be done exactly in the same way as we did in the proof of Proposition 3.11.

Consider a unit vector $x \in \mathbb{R}^n$, an exponent $p \geq 1$ and a radius $r \in (0, \sqrt{2}-1]$, we shall estimate

$$T_{\nu_n,p}(x,r) = \frac{|B_1|}{|B(x,1+r)\cap B_1|}\left(\frac{|B_r|}{|B_1|}\right)^{1/p'} = \frac{|B_1|}{|B(x,1+r)\cap B_1|}\left(\lambda^{1/p'}\right)^n,$$

where we recall that $\lambda = r/|x| = r$. Denote by F the cone with aperture β_λ whose ax direction is given by x, that is $F = \{y \in \mathbb{R}^n : \langle x, y\rangle \geq \cos\beta_\lambda |y|\}$. Since we have the inclusion $B(x,1+r)\cap B_1 \supset F \cap B_1$ we can integrate in spherical caps to obtain

$$\begin{aligned}
|B(x,1+r)\cap B_1| &\geq |F\cap B_1| = \int_0^1\int_0^{\beta_\lambda}\omega_{n-2}(s\sin\theta)^{n-2}s\,d\theta\,ds \\
&\geq \int_0^{\beta_\lambda}(\sin\theta)^{n-2}\cos\theta\,d\theta\sqrt{\frac{n-2}{2\pi}}\omega_{n-1}\int_0^1 s^{n-1}ds \\
&= \sqrt{\frac{n-2}{2\pi}}\frac{1}{(n-1)\sin\beta_\lambda}(\sin\beta_\lambda)^n|B_1| \\
&\geq \frac{C}{\sqrt{n}}(\sin\beta_\lambda)^n|B_1|,
\end{aligned}$$

since $\sin\beta_\lambda \geq \sin\beta_0 = \sqrt{3}/2$. On the other hand, $B(x,1+r)\cap B_1$ is contained in $B(\cos\beta_\lambda x, \sin\beta_\lambda)$ and this provides us the upper bound

$$|B(x,1+r)\cap B_1| \leq |B(\cos\beta_\lambda x,\sin\beta_\lambda)| = (\sin\beta_\lambda)^n|B_1|.$$

Summarising we have

$$\frac{C}{\sqrt{n}}(\sin\beta_\lambda)^n|B_1| \leq |B(x,1+r)\cap B_1| \leq (\sin\beta_\lambda)^n|B_1|,$$

whence

$$\left(\frac{\lambda^{1/p'}}{\sin\beta_\lambda}\right)^n \leq T_{\nu_n,p}(x,r) \leq C\sqrt{n}\left(\frac{\lambda^{1/p'}}{\sin\beta_\lambda}\right)^n.$$

These three quantities will tend to infinity as $n\to\infty$ if $a_p := \lambda^{1/p'}/\sin\beta_\lambda > 1$. This is equivalent to $p' < \log\lambda/\log\sin\beta_\lambda$. So, for $p > p_0$ with

$$p_0' = \inf_{0<\lambda<\sqrt{2}-1}\frac{\log\lambda}{\log\sin\beta_\lambda},$$

we have an $a_p > 1$ such that $T_{\nu_n,p}(x,r) > a_p^n$. This proves (i). By a numerical estimation using MATLAB one can obtain $p_0 \approx 1.03946$.

To prove (ii), note that for a fixed $\lambda \in (0, \sqrt{2}-1)$ we have $T_{\nu_n,p}(x,r)$ bounded independently of n if $\lambda^{1/p'}/\sin\beta_\lambda < 1-\varepsilon$ for some $\varepsilon > 0$. This condition can be rewritten as

$$p' < \frac{\log\lambda}{\log((1-\varepsilon)\sin\beta_\lambda)}.$$

Calling

$$p'_\varepsilon = \inf_{0<\lambda<\sqrt{2}-1} \frac{\log\lambda}{\log((1-\varepsilon)\sin\beta_\lambda)},$$

for each $p > p_\varepsilon$ we have a C_p such that $T_{\nu_n,p}(x,r) < C_p$. And we are done because, by continuity,

$$\inf_{0<\varepsilon<1/4} p_\varepsilon = p_0.$$

□

3.4 Final remark.

Most of the results in this chapter appeared in the publication [21]. When preparing the manuscript, we learned from J.M. Aldaz that he and J. Pérez Lázaro were working on the same problem, the extension to L^p of the result in [2]. Their approach is presented in [4]. In this last work they obtained, among other results, a better estimate for p_0 ($p_0 \geq 1.0378$). Moreover they showed that the finiteness of μ and the boundedness of its density are not strictly necessary. The precise statement of the result is the following (see Theorem 3.4 in [4]):

Theorem 3.15. *Fix $n \in \mathbb{N}$ and let μ be a radial measure over $\mathbb{R}^n$ with a radially decreasing density. Let $u = \sqrt{2/3}$, if μ is such that*

$$\sup_{R>0} \frac{\mu(B_r)}{\mu(B_{ur})} \geq u^{-\left(\frac{6\log 2-\log 55}{3\log 3-3\log 2}\right)n} = \left(\frac{64}{55}\right)^{n/6} \geq \limsup_{R\to\infty} \frac{\mu(B_r)}{\mu(B_{ur})},$$

then for each p such that $1 \leq p < 6\log 2/\log 55 \approx 1.0378$, we have

$$c_{\mu,p} \geq \frac{1}{4+C/\sqrt{n}}\left(\frac{2^{1/p}}{55^{1/6}}\right)^n,$$

with $2^{1/p}/55^{1/6} > 1$.

Chapter 4

Two examples of non-doubling measures and their associated maximal functions.

In this chapter we further study the two families of measures treated in Chapter 3. Here we give a more precise answer to the problem of finding bounds independent of the dimension for the associated maximal operators. For ν_n, Lebesgue measure restricted to the unit ball of $\mathbb{R}^n$, we show that $c_{\nu_n,p}$, the best constants in the weak $L^p(\nu_n)$ bounds for M_{ν_n}, tend to infinity with the dimension for all $1 \leq p < 2$, no matter if we restrict the action just to radially decreasing functions. The situation, however, is completely different for $p \geq 2$. As we will see, this maximal operator admits a dimension-free $L^2(\nu_n)$ bound on radially decreasing functions. We will also study the relation between M_{ν_n} and a modified Hardy operator, which gives better information on $L^p(\nu_n)$ bounds of M_{ν_n} when $p > 2$.

On the other hand, for γ_n, the Gaussian measure in $\mathbb{R}^n$, we will prove that M_{γ_n} does not admit weak $L^p(\gamma_n)$ bounds independent of the dimension for any $p \in [1, \infty)$. The counterexamples are given again by radially decreasing functions. The same will be proved for measures with exponential decay and double exponential decay.

4.1 Lebesgue measure restricted to the unit ball

Our first result asserts that the best constants in the weak $L^p(\nu_n)$ inequalities for M_{ν_n} grow exponentially with the dimension if $1 \leq p < 2$.

Theorem 4.1. *If $1 \leq p < 2$, there exist constants $a_p > 1$ and $c_p > 0$ such that for all n*

$$c_{\nu_n,p} > c_p\, a_p^n.$$

As a consequence, this shows that the operator norms of M_{ν_n} are not uniformly bounded with respect to the dimension on $L^p(\nu_n)$, for any $p < 2$.

To obtain this result, we will look at the action of the maximal operators on characteristic functions of balls centred at the origin. However, for $p \geq 2$ these functions cannot be used to find a counterexample; indeed, we will show that the action of M_{ν_n} on them has weak $L^p(\nu_n)$ bounds independent of the dimension:

Proposition 4.2. *Let $p \geq 2$ and $r > 0$. Then*

$$\|M_{\nu_n}\chi_r\|^*_{L^{p,\infty}(\nu_n)} \leq 2^{2/p}\|\chi_r\|_{L^p(\nu_n)}.$$

Before going on, let us recall some basic facts about Lorentz spaces $L^{p,q}$ and their norms and quasi-norms. Let μ be a Radon measure in $\mathbb{R}^n$. Given a measurable function f, we denote by f^* its non-increasing rearrangement with respect to μ. Let $1 \leq p < \infty$. The quasi-norm of f in the Lorentz space $L^{p,q}(\mu)$ with $1 \leq q < \infty$ is defined by

$$\|f\|^*_{L^{p,q}(\mu)} = \left(\frac{q}{p}\int_0^\infty [s^{1/p}f^*(s)]^q\,\frac{ds}{s}\right)^{1/q},$$

and for $q = \infty$ by

$$\|f\|^*_{L^{p,\infty}(\mu)} = \sup_{s>0} s^{1/p}f^*(s) = \sup_{\lambda>0} \lambda\,\mu(\{|f| > \lambda\})^{1/p},$$

with the usual agreement that $\|f\|^*_{L^{\infty,\infty}(\mu)} = \|f^*\|_{L^\infty(\mathbb{R})} = \|f\|_{L^\infty(\mu)}$. In most of the cases this is not a norm, since the triangle inequality may fail. However, the spaces $L^{p,q}(\mu)$ admit a norm denoted $\|\cdot\|_{L^{p,q}(\mu)}$ for $1 < p \leq \infty$ and $1 \leq q \leq \infty$. Denoting $m_f(s) = \frac{1}{s}\int_0^s f^*(t)\,dt$, the norm of f in the Lorentz space $L^{p,q}(\mu)$ with $1 < p < \infty$ and $1 \leq q < \infty$ is defined by

$$\|f\|_{L^{p,q}(\mu)} = \left(\frac{q}{p}\int_0^\infty [s^{1/p}m_f(s)]^q\,\frac{ds}{s}\right)^{1/q},$$

and for $q = \infty$ by

$$\|f\|_{L^{p,\infty}(\mu)} = \sup_{s>0} s^{1/p}m_f(s).$$

As before $\|f\|_{L^{\infty,\infty}(\mu)} = \|m_f\|_{L^\infty(\mathbb{R})} = \|f^*\|_{L^\infty(\mathbb{R})} = \|f\|_{L^\infty(\mu)}$.

As quasi-norms, $\|\cdot\|^*_{L^{p,q}(\mu)}$ and $\|\cdot\|_{L^{p,q}(\mu)}$ are equivalent in the sense that

$$\|f\|_{L^{p,q}(\mu)} \leq \|f\|^*_{L^{p,q}(\mu)} \leq \frac{p}{p-1}\|f\|_{L^{p,q}(\mu)},$$

for any $1 < p < \infty$ and $1 \leq q \leq \infty$. For more details, see Chapter V, §3 of [75].

As a consequence of Proposition 4.2 we obtain the following inequality for radially decreasing functions.

Theorem 4.3. *Let g be a radial, decreasing function in $\mathbb{R}^n$. Then for $p \geq 2$*

$$\|M_{\nu_n} g\|^*_{L^{p,\infty}(\nu_n)} \leq \frac{p}{p-1} 2^{2/p} \|g\|^*_{L^{p,1}(\nu_n)}.$$

One can check that the proof of the Marcinkiewicz theorem for Lorentz spaces (see [75], page 197) is also valid when restricting the action of the operator to radially decreasing functions. This allows us to interpolate between the case $p = 2$ of Theorem 4.3 and the $L^\infty(\nu_n)$ inequality to obtain

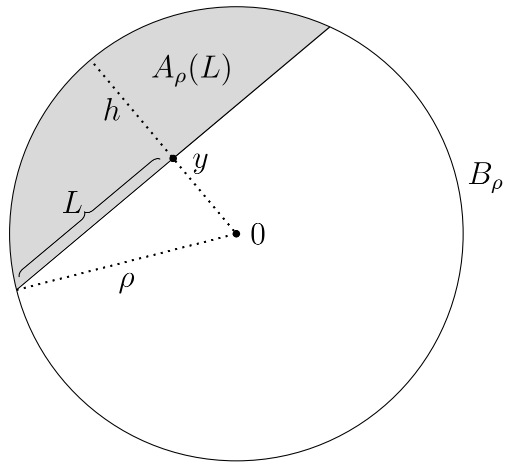
Theorem 4.4. *Let g be a radially decreasing function. One has for $p > 2$*

$$\begin{aligned}\|M_{\nu_n} g\|^*_{L^{p,\infty}(\nu_n)} &\leq 2^{1/p}\,\frac{5p-2}{p-2}\,\|g\|_{L^p(\nu_n)},\\ \|M_{\nu_n} g\|_{L^p(\nu_n)} &\leq 2^{1/p}\,\frac{5p-2}{p-2}\,\|g\|_{L^p(\nu_n)}.\end{aligned}$$

However, as we will show in the next section, there is a more direct way to obtain these estimates.

Before turning to the proofs of the results stated above, we need to introduce solid spherical caps and some properties of them. Given a ball B_ρ and a vector $y \neq 0$ in B_ρ, consider the hyperplane $y + y^\perp$, which divides the ball into two closed sets. We focus on the one of these sets which does not contain the origin. Its diameter is $2L = 2\sqrt{\rho^2 - |y|^2}$. We denote this set by $A_\rho(L)$, and any set congruent with it will be called a solid spherical cap. The height h of this cap is given by the function

$$h(\rho, L) = \rho - \sqrt{\rho^2 - L^2}. \tag{4.1}$$



The Lebesgue measure of the cap $A_\rho(L)$ is

$$|A_\rho(L)| = \int_{|y|}^{\rho} \frac{\omega_{n-2}}{n-1}(\rho^2 - s^2)^{\frac{n-1}{2}}\, ds = \frac{\omega_{n-2}}{n-1}\int_0^L \frac{t^n}{\sqrt{\rho^2 - t^2}}\, dt,$$

where the last equality comes from the change of variables $t = \sqrt{\rho^2 - s^2}$. Since $\sqrt{\rho^2 - L^2} < \sqrt{\rho^2 - t^2} < \rho$, we obtain that

$$\frac{\omega_{n-2}}{n^2-1} L^{n+1}\frac{1}{\rho} \le |A_\rho(L)| \le \frac{\omega_{n-2}}{n^2-1} L^{n+1}\frac{1}{\sqrt{\rho^2 - L^2}}. \tag{4.2}$$

We turn now to the proof of Theorem 4.1.

The following lemma explains why both in this proof and in the one of Proposition 4.2 it is enough to concentrate on the situation where $|x| = 1$.

Lemma 4.5. *For any $r < 1$ and each $x \in B_1$ we have that*

$$M_{\nu_n}\chi_r(x) \le M_{\nu_n}\chi_{r/|x|}\left(\frac{x}{|x|}\right).$$

Proof. For any $R > 0$

$$\frac{|B(x,R)\cap B_r|}{|B(x,R)\cap B_1|} \le \frac{|B(x,R)\cap B_r|}{|B(x,R)\cap B_{|x|}|} = \frac{\left|B\left(\frac{x}{|x|}, \frac{R}{|x|}\right)\cap B_{\frac{r}{|x|}}\right|}{\left|B\left(\frac{x}{|x|}, \frac{R}{|x|}\right)\cap B_1\right|},$$

and, taking the supremum in $R > 0$, the lemma is proved. □

Proof of Theorem 4.1. Let $x \in B_1$ and $0 < r < 1$. Lemma 3.5 asserts that

$$c_{\nu_n,p} \geq M_{\nu_n}\chi_r(x)\left(\frac{\nu_n(B_{|x|})}{\nu_n(B_r)}\right)^{1/p} = M_{\nu_n}\chi_r(x)\left(\frac{|x|}{r}\right)^{n/p}.$$

In view of Lemma 4.5,

$$M_{\nu_n}\chi_r(x)\left(\frac{|x|}{r}\right)^{n/p} \leq M_{\nu_n}\chi_{r/|x|}(x/|x|)\left(\frac{1}{r/|x|}\right)^{n/p},$$

thus, we can assume that x is a unit vector. With this assumption we have

$$c_{\nu_n,p} \geq M_{\nu_n}\chi_r(x)\left(\frac{|x|}{r}\right)^{n/p} \geq \frac{|B(x,R)\cap B_r|}{|B(x,R)\cap B_1|}\left(\frac{1}{r}\right)^{n/p}, \tag{4.3}$$

for $R, r > 0$. With $r < 1$ and $1 - r < R < r$, we shall choose r close to 1 and R small.

We split the intersection of two balls into two solid spherical caps and conclude from (4.2) that

$$\begin{aligned} |B(x,R)\cap B_1| &= |A_R(L)| + |A_1(L)| \\ &\leq \frac{\omega_{n-2}}{n^2-1}L^{n+1}\left(\frac{1}{\sqrt{R^2-L^2}} + \frac{1}{\sqrt{1-L^2}}\right), \end{aligned} \tag{4.4}$$

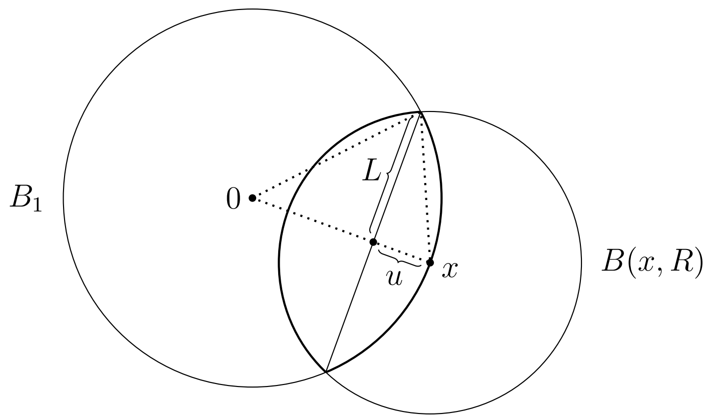
where $L = \sqrt{R^2 - R^4/4}$. The last factor can be bounded as follows

$$\frac{1}{\sqrt{R^2-L^2}} + \frac{1}{\sqrt{1-L^2}} = \frac{2}{R^2} + \frac{1}{1-\frac{R^2}{2}} \leq \frac{2}{R^2} + 2.$$

The calculation of L is elementary from the equations

$$\begin{cases} 1 &= L^2 + (1-u)^2, \\ R^2 &= L^2 + u^2, \end{cases}$$

obtained by orthogonality (see figure below).

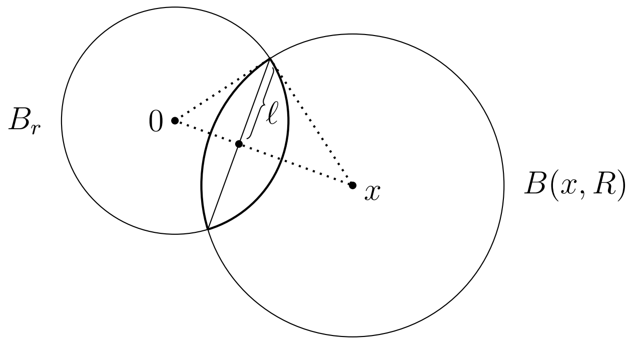


In the same fashion

$$|B(x,R)\cap B_r| = |A_R(\ell)| + |A_r(\ell)| \geq \frac{\omega_{n-2}}{n^2-1}\ell^{n+1}\left(\frac{1}{R}+\frac{1}{r}\right), \tag{4.5}$$

with $\ell = \sqrt{R^2-(R^2-r^2+1)^2/4} = \sqrt{r^2-(r^2-R^2+1)^2/4}$. And as $R < r < 1$ one has

$$\frac{1}{R}+\frac{1}{r} \geq 2.$$



Putting estimates (4.4) and (4.5) together, we obtain

$$\frac{|B(x,R)\cap B_r|}{|B(x,R)\cap B_1|} \geq \beta\left(\frac{\ell}{L}\right)^{n+1},$$

where $\beta = \beta(r,R) > 0$ is independent of n. We set

$$\phi_r(R) = \left(\frac{\ell}{L}\right)^2 = \frac{4R^2-(R^2-r^2+1)^2}{4R^2-R^4}.$$

Inequality (4.3) implies now

$$c_{\nu_n,p} \geq \beta\,\phi_r(R)^{(n+1)/2}\,r^{-n/p}.$$

It is enough to show that r and R can be chosen so that $\phi_r(R)^{1/2}\,r^{-1/p} > 1$. This is equivalent to

$$p < \frac{2\log r}{\log\phi_r(R)}.$$

By setting $t = 1-r^2$, we obtain

$$\phi_r(R) = 1-\frac{2R^2t+t^2}{4R^2-R^4},$$

and with the choice $R = t^{1/4}$ one has

$$\phi_r(t^{1/4}) = 1-\frac{2t-t^{3/2}}{4-t^{1/2}} = 1-\frac{t}{2}+o(t),$$

as $t \longrightarrow 0$. Now Theorem 4.1 is proved, because

$$\lim_{r\to 1^-} \frac{2\log r}{\log \phi_r(t^{1/4})} = \lim_{t\to 0^+} \frac{\log(1-t)}{\log(1-t/2+o(t))} = 2.$$

□

Now we devote ourselves to prove Proposition 4.2. It will be an easy task with the help of the following geometrical result, whose proof we postpone until the end of this section.

Proposition 4.6. *Given $x \neq 0$ in $\mathbb{R}^n$, $0 < r \leq 1$ and $R > 0$, one has*

$$\frac{|B(x,R)\cap B_r|}{|B(x,R)\cap B_1|} \leq 2\left(\frac{|B_r|}{|B_{|x|}|}\right)^{\frac{1}{2}}. \tag{4.6}$$

Equivalently, for each $x \in \mathbb{R}^n$ and $0 < r \leq 1$, one has

$$M_{\nu_n}\chi_r(x) \leq 2\left(\frac{r}{|x|}\right)^{\frac{n}{2}}.$$

Remark 4.7. *It may be the case that the best constant in (4.6) is 1 rather than 2, but we have not succeeded in computing its exact value.*

Proof of Proposition 4.2. The result is trivial when $r \geq 1$. We just have to show that given $0 < r < 1$, for each $x \in \mathbb{R}^n$,

$$M_{\nu_n}\chi_r(x)\,\nu_n(\{y \in \mathbb{R}^n : M_{\nu_n}\chi_r(y) > M_{\nu_n}\chi_r(x)\})^{1/p} \leq 2^{1/p}|B_r|^{1/p}.$$

Since $M_{\nu_n}\chi_r(x) \leq 1$, by Proposition 4.6

$$M_{\nu_n}\chi_r(x) \leq M_{\nu_n}\chi_r(x)^{2/p} \leq 2^{2/p}\left(\frac{r}{|x|}\right)^{n/p}.$$

In view of Lemma 3.6

$$\{y \in \mathbb{R}^n : M_{\nu_n}\chi_r(y) > M_{\nu_n}\chi_r(x)\} \subset B_{|x|}.$$

Hence,

$$\begin{aligned} M_{\nu_n}\chi_r(x)\,\nu_n(\{y \in \mathbb{R}^n : M_{\nu_n}\chi_r(y) > M_{\nu_n}\chi_r(x)\})^{1/p} &\leq 2^{2/p}\left(\frac{r}{|x|}\right)^{n/p}|B_{|x|}|^{1/p} \\ &= 2^{2/p}|B_r|^{1/p}. \end{aligned}$$

□

Before presenting the proof of Proposition 4.6, let us show that Proposition 4.2 implies Theorem 4.3.

Proof of Theorem 4.3. By a density argument it is enough to prove the result for a simple function of the form $g = \sum_{i=1}^{N} c_i \chi_{B_i}$, where $B_1 \supset \cdots \supset B_i \supset \cdots \supset B_N$ are balls centred at the origin and $c_i,\ i = 1, \ldots, N$ are positive real numbers. We can assume that the radius of B_1 is less than or equal to 1. Since M_{ν_n} is a sublinear operator, we have for such a function g

$$\begin{aligned} \|M_{\nu_n} g\|^*_{L^{p,\infty}(\nu_n)} &\leq \|M_{\nu_n} g\|_{L^{p,\infty}(\nu_n)} \leq \sum_{i=1}^{N} c_i \|M_{\nu_n} \chi_{B_i}\|_{L^{p,\infty}(\nu_n)} \\ &\leq \frac{p}{p-1} \sum_{i=1}^{N} c_i \|M_{\nu_n} \chi_{B_i}\|^*_{L^{p,\infty}(\nu_n)}. \end{aligned}$$

By Proposition 4.2

$$\begin{aligned} \sum_{i=1}^{N} c_i \|M_{\nu_n} \chi_{B_i}\|^*_{L^{p,\infty}(\nu_n)} &\leq 2^{2/p} \sum_{i=1}^{N} c_i \, \|\chi_{B_i}\|_{L^p(\nu_n)} = 2^{2/p} \sum_{i=1}^{N} c_i \, \nu_n(B_i)^{1/p} \\ &= 2^{2/p} \|g\|^*_{L^{p,1}(\nu_n)}. \end{aligned}$$

To see this last equality, note that writing $B_{N+1} = \emptyset$, if $|B_{i+1}| < s \leq |B_i|$, then the decreasing rearrangement of g at s is given by $g^*(s) = \sum_{j=1}^{i} c_j$ for $i = 1, \ldots, N$. Thus,

$$\begin{aligned} \|g\|^*_{L^{p,1}(\nu_n)} &= \frac{1}{p} \int_0^\infty s^{1/p} g^*(s) \, \frac{ds}{s} = \frac{1}{p} \sum_{i=1}^{N} \sum_{j=1}^{i} c_j \int_{|B_{i+1}|}^{|B_i|} s^{1/p-1} \, ds \\ &= \sum_{i=1}^{N} \sum_{j=1}^{i} c_j \left(|B_{i+1}|^{1/p} - |B_i|^{1/p} \right) = \sum_{i=1}^{N} c_i \, \nu_n(B_i)^{1/p}. \end{aligned}$$

□

We finish this section by proving Proposition 4.6.

Proof of Proposition 4.6. It is enough to prove the result in the case when $|x| = 1$, because once we have that, for any other x Lemma 4.5 gives

$$M_{\nu_n} \chi_r(x) \leq M_{\nu_n} \chi_{\frac{r}{|x|}}(x/|x|) \leq 2 \left(\frac{r}{|x|} \right)^{n/2}.$$

So, assuming that $|x| = 1$, we want to prove that for each $R > 0$ and $0 < r \leq 1$

$$\frac{|B(x,R) \cap B_r|}{|B(x,R) \cap B_1|} \leq 2r^{n/2}. \tag{4.7}$$

The case where $R \leq 1 - r$ is trivial since then $|B(x,R) \cap B_r| = 0$. From now on we assume $R > 1 - r$. Using the same notation as in the proof of Theorem 4.1, we have that

$$|B(x,R) \cap B_r| = |A_R(\ell)| + |A_r(\ell)|, \tag{4.8}$$
$$|B(x,R) \cap B_1| = |A_R(L)| + |A_1(L)|. \tag{4.9}$$

We first prove the inequality $\ell/L \leq r^{1/2}$. The equivalent statement $\ell^2 \leq L^2 r$ can be rewritten as $-R^4 + 2R^2(1-r) - (1-r)(1+r)^2 \leq 0$. This is a second-degree polynomial in $t = R^2$, whose maximal value, assumed at $t = 1 - r$, is $(1-r)^2 - (1-r)(1+r)^2 \leq 0$ for each r in $[0, 1]$.

We divide the proof of (4.7) into three cases:
CASE 1: $R \leq r$. Using (4.8) and (4.9) we have that

$$|B(x,R) \cap B_r| \leq 2|A_R(\ell)|,$$
$$|B(x,R) \cap B_1| \geq |A_R(L)|.$$

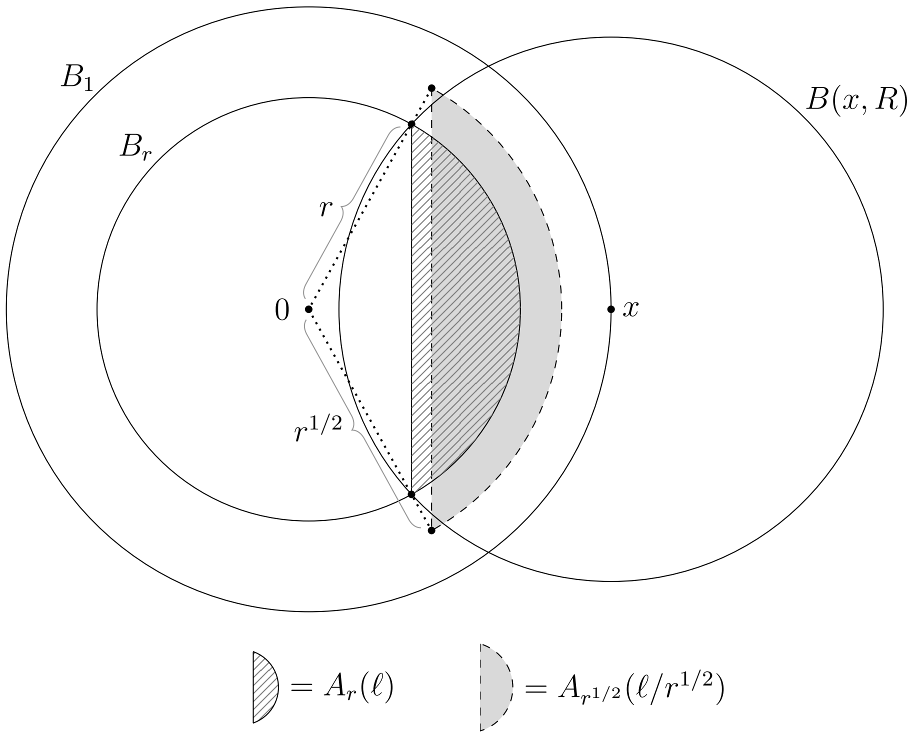
Dilating by the factor $\ell/L \leq 1$, we get $\frac{\ell}{L} A_R(L) = A_{\frac{\ell}{L}R}(\ell)$. This implies that $|A_R(\ell)| \leq |A_{\frac{\ell}{L}R}(\ell)| = (\ell/L)^n |A_R(L)|$. So we have

$$\frac{|B(x,R) \cap B_r|}{|B(x,R) \cap B_1|} \leq \frac{2|A_R(\ell)|}{|A_R(L)|} \leq 2\left(\frac{\ell}{L}\right)^n \leq 2r^{n/2}.$$

CASE 2: $r < R \leq \sqrt{1 + r^2}$. In this situation (4.8) implies

$$|B(x,R) \cap B_r| \leq 2|A_r(\ell)|.$$

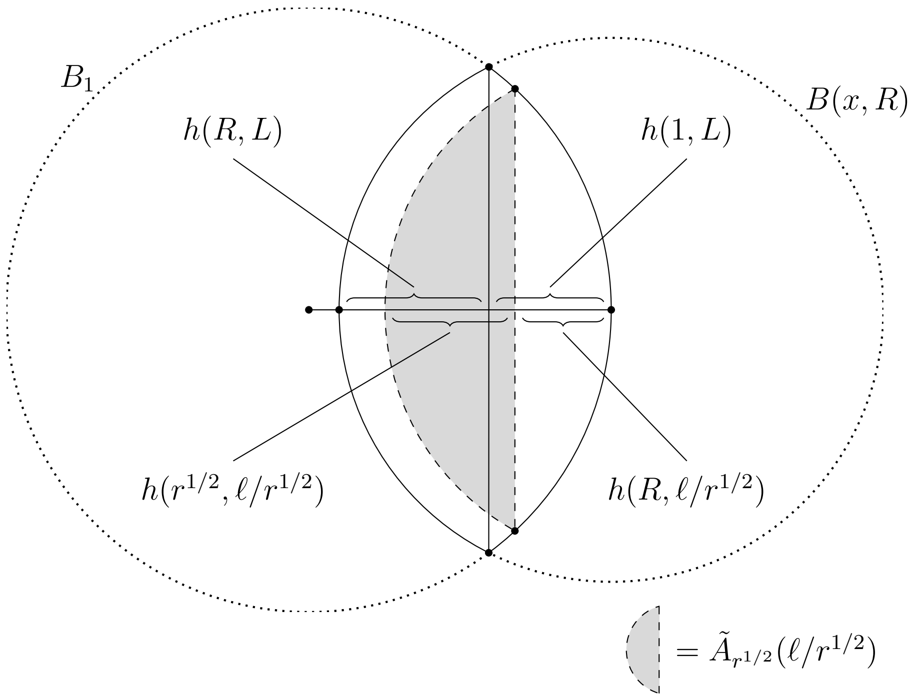
Here we dilate $A_r(\ell)$ by the factor $r^{-1/2}$ instead: $r^{-1/2} A_r(\ell) = A_{r^{1/2}}(\ell/r^{1/2})$.



We claim that a set congruent with $A_{r^{1/2}}(\ell/r^{1/2})$ is contained in $B(x,R)\cap B_1$. This would give the bound $|A_{r^{1/2}}(\ell/r^{1/2})| \leq |B(x,R)\cap B_1|$, and as a consequence

$$\frac{|B(x,R)\cap B_r|}{|B(x,R)\cap B_1|} \leq \frac{2\,|A_r(\ell)|}{|B(x,R)\cap B_1|} = 2\,\frac{r^{n/2}\,|A_{r^{1/2}}(\ell/r^{1/2})|}{|B(x,R)\cap B_1|} \leq 2\,r^{n/2}. \quad (4.10)$$

Thus we only have to justify the claim. For this we regard $B(x,R)\cap B_1$ as the union of two solid spherical caps $\tilde{A}_1(L)$ and $\tilde{A}_R(L)$ congruent with $A_1(L)$ and $A_R(L)$, respectively. Consider the unique hyperplane parallel to the planar boundary of $\tilde{A}_R(L)$ whose intersection with $\tilde{A}_R(L)$ is a circular disc D of radius $\ell/r^{1/2}$. This disc divides $\tilde{A}_R(L)$ into two sets. One of them, $\tilde{A}_R(\ell/r^{1/2})$, is congruent with $A_R(\ell/r^{1/2})$. Call $\tilde{A}_{r^{1/2}}(\ell/r^{1/2})$ the cap congruent with $A_{r^{1/2}}(\ell/r^{1/2})$ such that $\tilde{A}_R(\ell/r^{1/2})\cap\tilde{A}_{r^{1/2}}(\ell/r^{1/2}) = D$.



To see that $\tilde{A}_{r^{1/2}}(\ell/r^{1/2})$ is contained in $\tilde{A}_1(L) \cup \tilde{A}_R(L) \setminus A_R(\ell/r^{1/2})$, and thus in $B(x, R) \cap B_1$, it is enough to compare the heights of four caps and verify that

$$h(r^{1/2}, \ell/r^{1/2}) \leq h(1, L) + h(R, L) - h(R, \ell/r^{1/2}).$$

In view of the definition (4.1) of h, it is not difficult to see that $h(1, L)+h(R, L) = R$, and the above inequality becomes

$$r^{1/2} - \sqrt{r - \ell^2/r} \leq \sqrt{R^2 - \ell^2/r}.$$

We can multiply by $r^{1/2}$ on both sides and use that $\ell^2 = r^2 - ((r^2 - R^2 + 1)/2)^2$ to get the equivalent statement

$$r - \frac{r^2 - R^2 + 1}{2} \leq \sqrt{R^2 r - r^2 + \left(\frac{r^2 - R^2 + 1}{2}\right)^2}.$$

Here the left-hand side is positive since $R > 1 - r$, and one obtains by squaring the equivalent inequality

$$-r(1 - r)^2 \leq 0,$$

which holds since $0 < r \leq 1$. The claim follows.

CASE 3: $R > \sqrt{1+r^2}$. In this case, the ball $B(x,R)$ contains more than half of the ball B_r and we have

$$\frac{|B(x,R)\cap B_r|}{|B(x,R)\cap B_1|} \leq \frac{|B_r|}{|B(x,\sqrt{1+r^2})\cap B_1|} = \frac{2|A_r(r)|}{|B(x,\sqrt{1+r^2})\cap B_1|}.$$

Now we can use (4.10) in the special case that $R = \sqrt{1+r^2}$ and $\ell = r$ to get

$$\frac{2|A_r(r)|}{|B(x,\sqrt{1+r^2})\cap B_1|} \leq 2\, r^{n/2}.$$

□

4.2 Relation of M_{ν_n} with a Hardy type Operator.

This section presents a different method to obtain weak-type L^p bounds for M_{ν_n} acting on radially decreasing functions. The resulting bounds are better than the ones given in Theorem 4.4. The main idea is to control M_{ν_n} by a Hardy type operator. Defining the modified Hardy operator $\mathcal{A}$ for a locally integrable function g in $\mathbb{R}^n$ as

$$\mathcal{H}g(x) = \frac{1}{|B_{|x|}|}\int_{B_{|x|}} |g(y)|\, dy,$$

we have the following estimate.

Proposition 4.8. *Given a radially decreasing function g, one has for $p > 2$*

$$M_{\nu_n} g(x) \leq \frac{p+2}{p-2}\left(\mathcal{H}g^p(x)\right)^{1/p},$$

for each $x \in \mathbb{R}^n$.

This is useful, because we can bound the operator $g \mapsto (\mathcal{H}g^p)^{1/p}$ as follows.

Proposition 4.9. *If g is a radially decreasing function in $\mathbb{R}^n$ and $p \geq 1$, then*

$$\|(\mathcal{H}g^p)^{1/p}\|^*_{L^{p,\infty}(\nu_n)} \leq \|g\|_{L^p(\nu_n)}.$$

As an immediate consequence of Propositions 4.8 and 4.9, we have the following weak $L^p(\nu_n)$ bound for M_{ν_n}, sharper than the one in Theorem 4.4.

Theorem 4.10. *Let $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a radial, decreasing function. For each $p > 2$ one has*

$$\|M_{\nu_n} g\|^*_{L^{p,\infty}(\nu_n)} \le \frac{p+2}{p-2} \|g\|_{L^p(\nu_n)}.$$

The proof of Proposition 4.8 is based on Proposition 4.6.

Proof of Proposition 4.8. We fix $x \in \mathbb{R}^n$. By homogeneity we can assume that the radially decreasing function g satisfies $\mathcal{H}g^p(x) = 1$. Given $t > 0$, the level set $\{y : g(y) > t\}$ is the ball $B_{r(t)}$ for a certain $r(t) > 0$. Thus

$$\begin{aligned} M_{\nu_n} g(x) &= \sup_{R>0} \frac{1}{|B(x,R) \cap B_1|} \int_{B(x,R)\cap B_1} g(y)\, dy \\ &= \frac{1}{|B(x,R) \cap B_1|} \int_0^\infty |\{y \in B(x,R) \cap B_1 : g(y) > t\}|\, dt \\ &= \frac{1}{|B(x,R) \cap B_1|} \int_0^\infty |B(x,R) \cap B_{r(t)} \cap B_1|\, dt \\ &= \int_0^1 \frac{|B(x,R) \cap B_{r(t)} \cap B_1|}{|B(x,R) \cap B_1|}\, dt + \int_1^\infty \frac{|B(x,R) \cap B_{r(t)} \cap B_1|}{|B(x,R) \cap B_1|}\, dt. \end{aligned}$$

The first term on the last line is clearly bounded by 1. For the second one, we can use Proposition 4.6 to get

$$M_{\nu_n} g(x) \le 1 + 2\int_1^\infty \frac{|B_{r(t)}|^{1/2}}{|B_{|x|}|^{1/2}}\, dt \le 1 + \frac{2}{|B_{|x|}|^{1/2}} \int_1^\infty |\{y : g(y) > t\}|^{1/2}\, dt.$$

The hypothesis $\mathcal{H}g^p(x) = 1$ implies $g(x) \le 1$, so for $g(y) > t > 1$ it is necessary that $y \in B_{|x|}$. Then, by the Tchebychev inequality applied to the above expression we have

$$\begin{aligned} M_{\nu_n} g(x) &\le 1 + \frac{2}{|B_{|x|}|^{1/2}} \int_1^\infty \frac{1}{t^{p/2}} \left(\int_{B_{|x|}} g(y)^p\, dy \right)^{1/2} dt \\ &= 1 + \frac{4}{p-2} \left(\frac{1}{|B_{|x|}|} \int_{B_{|x|}} g(y)^p\, dy \right)^{1/2} = 1 + \frac{4}{p-2}. \end{aligned}$$

□

The proof of Proposition 4.9 follows a standard argument. We give the details for the sake of completeness.

Proof of Proposition 4.9. If g is radially decreasing, so is $(\mathcal{H}g^p)^{1/p}$, and its level sets are balls centred at the origin. So given $\lambda > 0$

$$\{y : (\mathcal{H}g^p)^{1/p}(y) > \lambda\} = B_{r(\lambda)},$$

for some $r(\lambda) > 0$. Hence

$$\left(\frac{1}{|B_{r(\lambda)}|}\int_{B_{r(\lambda)}} g(y)^p\,dy\right)^{1/p} \geq \lambda,$$

which we can rearrange as

$$\big|\{y : (\mathcal{H}g^p)^{1/p}(y) > \lambda\}\big| = \big|B_{r(\lambda)}\big| \leq \frac{1}{\lambda^p}\int_{B_{r(\lambda)}} g(y)^p\,dy.$$

□

4.3 The Gaussian measure revisited.

We will now concentrate on the study of the Gaussian measure, given by $d\gamma_n(x) = \mathrm{e}^{-\pi|x|^2}\,dx$. In this case we will prove that the associated maximal function does not admit dimension-free bounds on $L^p(\gamma_n)$ for every $1 \leq p < \infty$:

Theorem 4.11. *There exist absolute constants $a > 1$ and $c > 0$ such that for every p in the range $1 \leq p < \infty$,*

$$c_{\gamma_n,p} \geq c\,a^{n/p}.$$

This result can be extended to the case in which the density is given by $f_\alpha(|x|) = \mathrm{e}^{-|x|^\alpha}$, with $\alpha > 0$. The same is true for the density $e^{-e^{|x|}}$.

Theorem 4.12. *Let $d\gamma_{\alpha,n}$ be the measure given for $\alpha > 0$ by $d\gamma_{\alpha,n}(x) = \mathrm{e}^{-|x|^\alpha}\, dx$. Then, there exist constants $a = a(\alpha) > 1$ and $c_\alpha > 0$ such that the corresponding weak $L^p(\gamma_{\alpha,n})$ constant satisfies*

$$c_{\gamma_{\alpha,n},p} \geq c_\alpha\, a_\alpha^{n/p},$$

for every n and $1 \leq p < \infty$.

Theorem 4.13. *Denote by μ_n the measure on $\mathbb{R}^n$ whose density is $f(x) = e^{-e^{|x|}}$. Then, there exist constants $a > 1$ and $c > 0$ so that*

$$c_{\mu_n,p} \geq c\, a^{n/p},$$

for every n and $1 \leq p < \infty$.

4.3.1 Proof of Theorem 4.11

The idea of the proof of Theorem 4.11 is the following. From Lemma 3.5 we know that for any $x_n \in \mathbb{R}^n$ and $r_n > 0$

$$c_{\gamma_n,p} \geq M_{\gamma_n}\chi_{r_n}(x_n) \left(\frac{\gamma_n(B_{|x_n|})}{\gamma_n(B_{r_n})} \right)^{\frac{1}{p}}. \tag{4.11}$$

Since

$$M_{\gamma_n}\chi_{r_n}(x_n) \geq \frac{\gamma_n(B(x_n, R_n) \cap B_{r_n})}{\gamma_n(B(x_n, R_n))},$$

for each $R_n > 0$, we only need to prove the following lemma.

Lemma 4.14. *There exist sequences $\{x_n\}, \{r_n\}$ and $\{R_n\}$ with $x_n \in \mathbb{R}^n$ and $r_n, R_n > 0$ for $n \in \mathbb{N}$, such that*

$$\frac{\gamma_n(B(x_n, R_n) \cap B_{r_n})}{\gamma_n(B(x_n, R_n))} \geq \frac{c}{\sqrt{n}}, \tag{4.12}$$

$$\frac{\gamma_n(B_{|x_n|})}{\gamma_n(B_{r_n})} \geq c\, a^n, \tag{4.13}$$

for some absolute constants $a > 1$ and $c > 0$.

Proof of Lemma 4.14. We start with statement (4.13). Following Remark 3.7, we will only consider the situation that $|x_n| > r_n$. Taking $|x_n| > T_n$ does not yield any worthy improvement in (4.11) compared with the choice $|x_n| = T_n$, as we saw in the proof of Proposition 3.11.

Let us assume that $x_n = \xi T_n x_n'$, with $0 < \xi \le 1$ and x_n' a unit vector in $\mathbb{R}^n$. In order to make the quotient $|x_n|/r_n$ independent of n, set $r_n = rT_n$, with $0 < r \le \xi$. Lemma 3.12 implies that

$$\frac{\gamma_n(B_{|x_n|})}{\gamma_n(B_{r_n})} \ge \frac{\mathrm{e}^{-\pi\xi^2 T_n^2}|B_{\xi T_n}|}{n\mathrm{e}^{-\pi\xi^2 T_n^2}|B_{rT_n}|} = \frac{\mathrm{e}^{\frac{\xi^2-r^2}{2}}}{n}\left(\mathrm{e}^{\frac{r^2-\xi^2}{2}}\frac{\xi}{r}\right)^n.$$

We want the quantity raised to the power n to be greater than 1. Write $\xi = sr$, with $s > 1$. Then $\mathrm{e}^{\frac{r^2-\xi^2}{2}}\frac{\xi}{r} = \mathrm{e}^{\frac{(1-s^2)}{2}r^2}s =: h(s)$. It is elementary to see that $h(1) = 1$, and that h is increasing in the interval $(1, \frac{1}{r})$. So, for the sake of optimality, we will take $s = \frac{1}{r}$ which means $\xi = 1$. Summarising, we obtained

$$\frac{\gamma_n(B_{T_n})}{\gamma_n(B_{r_n})} \ge \frac{\mathrm{e}^{\frac{1-r^2}{2}}}{n}\left(\mathrm{e}^{\frac{r^2-1}{2}}\frac{1}{r}\right)^n, \tag{4.14}$$

where $\mathrm{e}^{\frac{r^2-1}{2}}\frac{1}{r} > 1$. This proves (4.13).

We now turn to the proof of (4.12). Take $R_n = RT_n$, with $1 - r < R < 1$. To calculate $\gamma_n(B(x_n, R_n))$, we will integrate over spherical caps where the density $\mathrm{e}^{-\pi|x|^2}$ is constant. We get

$$\begin{aligned}\gamma_n(B(x_n, R_n)) &= \int_{T_n-R_n}^{T_n+R_n} \Big|\partial B_\rho \cap B(x_n, R_n)\Big|_{n-1} \mathrm{e}^{-\pi\rho^2}\,d\rho \\ &= T_n^n \int_{1-R}^{1+R} \Big|\partial B_s \cap B(x_n', R)\Big|_{n-1} \mathrm{e}^{-\frac{n-1}{2}s^2}\,ds,\end{aligned}$$

where $|\cdot|_{n-1}$ denotes $(n-1)$-dimensional Hausdorff measure, and the second equality is justified by the change of variables $\rho = T_n s$. Call β_s the angle determined by the segment that joins the origin with x_n' and the one that connects the origin with any point in $\partial B_s \cap \partial B(x_n, R_n)$. By the cosine law applied to the triangle whose vertices are given by the origin, x_n' and any $y \in \partial B_s \cap \partial B(x_n', R)$, one obtains

$$\cos\beta_s = \frac{1 + s^2 - R^2}{2s},$$

and consequently

$$\sin\beta_s = \left(1 - \left(\frac{1 + s^2 - R^2}{2s}\right)^2\right)^{\frac{1}{2}}.$$

The maximal value of β_s occurs when ∂B_s and $\partial B(x'_n, R)$ meet perpendicularly, and then $\sin\beta_s = R$. Thus one always has $\cos\beta_s \geq \sqrt{1-R^2}$. We compute the surface measure of the spherical caps in the following way

$$\Big|\partial B_s \cap B(x'_n, R)\Big|_{n-1} = \int_0^{\beta_s} \omega_{n-2}(s\sin\theta)^{n-2} s\, d\theta. \tag{4.15}$$

For the last integral we have the bounds

$$\int_0^{\beta_s} \sin^{n-2}\theta\, d\theta \leq \int_0^{\beta_s} \frac{\cos\theta}{\cos\beta_s}\sin^{n-2}\theta\, d\theta = \frac{1}{\sqrt{1-R^2}}\frac{\sin^{n-1}\beta_s}{n-1}.$$

and

$$\int_0^{\beta_s} \sin^{n-2}\theta\, d\theta \geq \int_0^{\beta_s} \cos\theta\sin^{n-2}\theta\, d\theta = \frac{\sin^{n-1}\beta_s}{n-1}, \tag{4.16}$$

We start with the upper bound for $\gamma_n(B(x_n, R_n))$. Calling $F(s^2) = \sin^2\beta_s s^2 \mathrm{e}^{-s^2}$, we have

$$\gamma_n(B(x_n, R_n)) \leq \frac{\omega_{n-2}\, T_n^n}{(n-1)\sqrt{1-R^2}} \int_{1-R}^{1+R} \frac{1}{\cos\beta_s} F(s^2)^{\frac{n-1}{2}}\, ds.$$

With the change $s^2 = t$ we get

$$\gamma_n(B(x_n, R_n)) \leq \frac{\omega_{n-2}\, T_n^n}{(n-1)\sqrt{1-R^2}} \int_{(1-R)^2}^{(1+R)^2} F(t)^{\frac{n-1}{2}} \frac{dt}{2\sqrt{t}},$$

where $F(t) = \left(t - \left(\frac{1+t-R^2}{2}\right)^2\right)\mathrm{e}^{-t}$.

It is easy to check that $F((1-R)^2) = F((1+R)^2) = 0$ and that F is increasing in the interval $((1-R)^2, t_0)$ and decreasing in $(t_0, (1+R)^2)$, where $t_0 = 2 + R^2 - \sqrt{1+4R^2}$ is the maximum point. So we can estimate

$$\begin{aligned}\gamma_n(B(x_n, R_n)) &\leq \frac{\omega_{n-2}\, T_n^n}{(n-1)\sqrt{1-R^2}} \int_{(1-R)^2}^{(1+R)^2} F(t_0)^{\frac{n-1}{2}} \frac{dt}{2\sqrt{t}} \\ &= \frac{2\,\omega_{n-2}\, T_n^n\, R}{(n-1)\sqrt{1-R^2}} F(t_0)^{\frac{n-1}{2}}.\end{aligned}$$

Next, we obtain a lower bound for $\gamma(B_{r_n} \cap B(x_n, R_n))$. As above we have

$$\gamma(B_{r_n} \cap B(x_n, R_n)) = \int_{T_n - R_n}^{r_n} \Big|(\partial B_\rho \cap B(x_n, R_n)\Big|_{n-1} \mathrm{e}^{-\pi\rho^2}\, d\rho,$$

and by (4.15) and (4.16)

$$\begin{aligned}\gamma(B_{r_n} \cap B(x_n, R_n)) &\geq \frac{\omega_{n-2}}{n-1} T_n^n \int_{1-R}^{r} F(s^2)^{\frac{n-1}{2}}\, ds \\ &= \frac{\omega_{n-2} T_n^n}{n-1} \int_{(1-R)^2}^{r^2} F(t)^{\frac{n-1}{2}} \frac{dt}{2\sqrt{t}}.\end{aligned}$$

Since $R < 1$, we have that $t_0 < 1$, and it will be very convenient to choose $r^2 = t_0$. As F is a smooth function, we can write $F(t) = F(t_0) + \frac{F''(\tau_t)}{2}(t - t_0)^2$, with τ_t a point between t and t_0. We denote by M the maximum value of $|F''|$ in the interval $[(1-R)^2, (1+R)^2]$. So if $0 < \delta < t_0 - (1-R)^2$

$$\begin{aligned}\int_{(1-R)^2}^{t_0} F(t)^{\frac{n-1}{2}} \frac{dt}{2\sqrt{t}} &\geq \int_{t_0-\delta}^{t_0} \left(F(t_0) + \frac{F''(\tau_t)}{2} (t - t_0)^2 \right)^{\frac{n-1}{2}} \frac{dt}{2\sqrt{t}} \\ &\geq F(t_0)^{\frac{n-1}{2}} \int_{t_0-\delta}^{t_0} \left(1 - \frac{M}{2F(t_0)} \delta^2 \right)^{\frac{n-1}{2}} \frac{dt}{2\sqrt{t}},\end{aligned}$$

the last inequality provided δ is small enough to make the last parenthesis positive. Choosing $\delta = \sqrt{\frac{4F(t_0)}{(n-1)M}}$, we will have $(1 - \frac{M}{2F(t_0)}\delta^2)^{\frac{n-1}{2}} > c_0 > 0$ for n large enough. Hence, the last expression is greater than or equal to

$$c_0 \int_{t_0-\delta}^{t_0} \frac{dt}{2\sqrt{t}} F(t_0)^{\frac{n-1}{2}} \geq c_0 F(t_0)^{\frac{n-1}{2}} \frac{\delta}{2\sqrt{t_0}}.$$

Putting together all the estimates, we conclude

$$\frac{\gamma_n(B(x_n, R_n) \cap B_{r_n})}{\gamma_n(B(x_n, R_n))} \geq \frac{c}{\sqrt{n-1}}.$$

where $c > 0$ may depend on R and r, but not on n. Observe finally that r is determined by R via t_0, and that R can be chosen arbitrarily in $(0, 1)$. □

4.3.2 Proof of Theorem 4.12

The proof of Theorem 4.12 follows the same scheme as the previous one, so we just hint the main steps. It is enough to show the following analogue of Lemma 4.14:

Lemma 4.15. *There exist sequences* $\{x_n\}, \{r_n\}$ *and* $\{R_n\}$*, with* $x_n \in \mathbb{R}^n$ *and* $r_n,\ R_n > 0$ *for* $n \in \mathbb{N}$ *such that*

$$\frac{\gamma_{\alpha,n}(B(x_n,R_n)\cap B_{r_n})}{\gamma_{\alpha,n}(B(x_n,R_n))} \geq \frac{c}{\sqrt{n}}, \tag{4.17}$$

and

$$\frac{\gamma_{\alpha,n}(B_{|x_n|})}{\gamma_{\alpha,n}(B_{r_n})} \geq c\, a^n, \tag{4.18}$$

for some $a = a(\alpha) > 1$.

Proof. We first deal with the proof of (4.18). The measure of a centred ball is $\gamma_{\alpha,n}(B_\rho) = \omega_{n-1}\int_0^\rho f_{\alpha,n}(t)\,dt$, where $f_{\alpha,n}(t) = \mathrm{e}^{-t^\alpha} t^{n-1}$. This function attains its maximum at the radius $T_{\alpha,n} = ((n-1)/\alpha)^{1/\alpha}$, around which an essential part of the mass is concentrated. For $\rho < T_{\alpha,n}$ we have as well that

$$\mathrm{e}^{-\rho^\alpha}|B_\rho| \leq \gamma_\alpha(B_\rho) \leq n\,\mathrm{e}^{-\rho^\alpha}|B_\rho|. \tag{4.19}$$

Take $r_n = rT_{\alpha,n}$ and $x_n = T_{\alpha,n}x'_n$ with $r < 1$ and x'_n a unit vector. Inequalities (4.19) imply that

$$\frac{\gamma_{\alpha,n}(B_{|x_n|})}{\gamma_{\alpha,n}(B_{r_n})} \geq \frac{\mathrm{e}^{-T^\alpha_{\alpha,n}}\,|B_{T_{\alpha,n}}|}{n\,\mathrm{e}^{-r^\alpha T^\alpha_{\alpha,n}}\,r^n\,|B_{T_{\alpha,n}}|} = \frac{\mathrm{e}^{(1-r^\alpha)/\alpha}}{n}\left(\mathrm{e}^{(r^\alpha-1)/\alpha}\,\frac{1}{r}\right)^n.$$

It is easy to see that $\mathrm{e}^{(r^\alpha-1)/\alpha}/r > 1$ by applying the inequality $e^x > 1+x$ to $e^{r^\alpha-1}$.

To prove (4.17) take $R_n = RT_{\alpha,n}$ with $1-r < R < r$. Following the steps of the proof of (4.12), we have

$$\begin{aligned}\frac{\omega_{n-2}\,T^n_{\alpha,n}}{n-1}\int_{(1-R)^2}^{(1+R)^2} F_\alpha(t)^{\frac{n-1}{2}}\,\frac{dt}{2\sqrt{t}} &\leq \gamma_{\alpha,n}(B(x_n,R_n)) \\ &\leq \frac{\omega_{n-2}T^n_{\alpha,n}}{(n-1)\sqrt{1-R^2}}\int_{(1-R)^2}^{(1+R)^2} F_\alpha(t)^{\frac{n-1}{2}}\,\frac{dt}{2\sqrt{t}},\end{aligned}$$

where $F_\alpha(t) = \left(t - \left(\frac{1+t-R^2}{2}\right)^2\right)\mathrm{e}^{-2t^{\alpha/2}/\alpha}$. This function attains its maximum at a point $t_\alpha < 1$. This is a consequence of the following facts: $F_\alpha((1-R)^2) = F_\alpha((1+R)^2) = 0$, $F_\alpha(t) > 0$ for $(1-R)^2 < t < (1+R)^2$, and $F'_\alpha(t) < 0$

whenever $1 \le t < (1+R)^2$. To see the last assertion, write the derivative of F_α as

$$\begin{aligned}\frac{\partial}{\partial t}F_\alpha(t) &= \left\{1-\frac{1+t-R^2}{2}-t^{\alpha/2-1}\left(t-\left(\frac{1+t-R^2}{2}\right)^2\right)\right\}\mathrm{e}^{-2t^{\alpha/2}/\alpha}\\ &=: G_\alpha(t)\mathrm{e}^{-2t^{\alpha/2}/\alpha}.\end{aligned}$$

Now it is clear that for $\alpha > 0$ and $1 < t < 1 + R^2$

$$G_\alpha(t) < G_0(t) = t^{-1}\left(\frac{1+t-R^2}{2}\right)^2 - \frac{1+t-R^2}{2} < 0.$$

All this was done to justify that we can take $r = \sqrt{t_\alpha} < 1$. Now we just follow the same steps as in the proof of Lemma 4.15 to estimate

$$\frac{\gamma_{\alpha,n}(B(x_n,R_n)\cap B_{r_n})}{\gamma_{\alpha,n}(B(x_n,R_n))} \ge \frac{c}{\sqrt{n-1}},$$

where the constant c may depend on r, R and α but not on n. □

4.3.3 Proof of Theorem 4.13

Consider μ_n to be the measure on $\mathbb{R}^n$ given by $d\mu_n(x) = e^{-e^{|x|}}\,dx$. Similarly as we did before it is possible to show for each $p \ge 1$ that $c_{\mu_n,p}$ grows exponentially with n.

Given $r > 0$ and $x \in \mathbb{R}^n$, Lemma 3.5 gives

$$c_{\mu_n,p} \ge M_{\mu_n}\chi_r(x)\left(\frac{\mu_n(B_{|x|})}{\mu_n(B_r)}\right)^{1/p}.$$

So we only need to prove the following

Lemma 4.16. *There exist sequences $\{x_n\}$, $\{r_n\}$ and $\{R_n\}$ with $x_n \in \mathbb{R}^n$ and $r_n, R_n > 0$ such that*

$$M_{\mu_n}\chi_{r_n}(x_n) \ge \frac{\mu_n(B(x_n,R_n)\cap B_{r_n})}{\mu_n(B(x_n,r_n))} \ge \frac{c}{\sqrt{n-1}}, \tag{4.20}$$

and

$$\frac{\mu_n(B_{|x_n|})}{\mu_n(B_{r_n})} \ge c\,a^n, \tag{4.21}$$

for some $a > 1$.

Before proving the lemma we will study how the measure μ_n acts on centred balls. For $R > 0$

$$\mu_n(B_R) = \int_{B_R} e^{-e^{|x|}}\,dx \geq \int_{B_R} e^{-e^R}\,dx = e^{-e^R}|B_R|.$$

Also integrating radially gives

$$\mu_n(B_R) = \int_{B_R} e^{-e^{|x|}}\,dx = \omega_{n-1}\int_0^R e^{-e^s}s^{n-1}\,ds.$$

Differentiating one sees that there exists a $T_n > 0$ such that the function $s \mapsto e^{-e^s}s^{n-1}$ grows for $s \in (0, T_n)$ and decreases for $s \in (T_n, \infty)$ and that T_n is the solution of the transcendental equation $se^s = n-1$. So if additionally $R \leq T_n$ one also has

$$\begin{aligned}\mu_n(B_R) &= \int_{B_R} e^{-e^{|x|}}\,dx = \omega_{n-1}\int_0^R e^{-e^s}s^{n-1}\,ds \\ &\leq \omega_{n-1}\int_0^R e^{-e^R}s^{n-1}\,ds = ne^{-e^R}|B_R|.\end{aligned}$$

We have proved:

Lemma 4.17. *If* $0 \leq R \leq T_n$ *one has*

$$e^{-e^R}|B_R| \leq \mu_n(B_R) \leq ne^{-e^R}|B_R|.$$

We give a possible way of approximating T_n. Applying logarithms to the equation defining T_n we obtain that $T_n + \log T_n = \log(n-1)$ which implies $T_n \leq \log(n-1)$. This upper bound yields a lower one, since $T_n = \log(n-1) - \log T_n \geq \log(n-1) - \log\log(n-1)$. Thus, T_n behaves asymptotically like $\log(n-1)$ in the sense that $T_n/\log(n-1) \longrightarrow 1$ as $n \to \infty$.

Proof of Lemma 4.16. As previously, scaling by T_n will simplify calculations. Since it is not possible to compute T_n exactly we will scale by $\log(n-1)$. Let $r, R, x > 0$ and take $r_n = r\log(n-1)$, $R_n = R\log(n-1)$ and $x_n = x\log(n-1)e_1$. Our (possibly not optimal but sufficient) choice is $R = 1/4$, $x = 1/2$, but most of the times we will not explicit it in order to preserve the formal analogy with Lemmas 4.14 and 4.15. The value of r is to be choosen later.

To prove (4.21) we use Lemma 4.17 to get

$$\frac{\mu_n(B_{|x_n|})}{\mu_n(B_{r_n})} \geq \frac{e^{-(n-1)e^x}|B_{|x_n|}|}{ne^{-(n-1)e^r}|B_{r_n}|} = \frac{e^{e^x-e^r}}{n}\left(e^{e^r-e^x}\frac{x}{r}\right)^n.$$

Now it would to check that with our choice of r and x one has.

$$e^{e^r-e^x}\frac{x}{r} > 1. \tag{4.22}$$

We turn to (4.20). Similarly as in the proofs of Lemmas 4.14 and 4.15 we find that

$$\mu_n(B(x_n, R_n)) \leq \frac{\omega_{n-2}}{\sqrt{1-\frac{R^2}{x^2}}}\frac{(\log(n-1))^n}{n-1}\int_{x-R}^{x+R} F(t)^{(n-1)/2}\,dt.$$

where

$$F(t) = \left(t^2 - \left(\frac{x^2+t^2-R^2}{2x}\right)^2\right)e^{-2e^t}.$$

The function F is continuous, $F(x-R) = F(x+R) = 0$ and $F(t) > 0$ if $x-R < t < x+R$. Then F attains its maximum on $[x-R, x+R]$ at an interior point t_0 of the interval and we have

$$\mu_n(B(x_n, R_n)) \leq c\omega_{n-2}\frac{(\log(n-1))^n}{n-1}F(t_0)^{(n-1)/2}.$$

In an analogous way

$$\mu_n(B(x_n, R_n)\cap B_{r_n}) \geq \omega_{n-2}\frac{(\log(n-1))^n}{n-1}\int_{x-R}^{r} F(t)^{(n-1)/2}\,dt.$$

By Taylor's expansion if $t \in (x-R, x+R)$ and $|t-t_0| < \delta$

$$F(t) = F(t_0) + \frac{F''(\tau)}{2}(t-t_0)^2 \geq F(t_0) - \frac{M}{2}\delta^2,$$

where $M = \max_{\tau\in[x-R,x+R]}|F''(\tau)|$. At the end we will see that $t_0 < x$, hence we can take $r = t_0$ and $\delta = (2F(t_0)/M(n-1))^{1/2}$ for a large to have

$$\begin{aligned}\int_{x-R}^{r} F(t)^{(n-1)/2}\,dt &\geq \int_{t_0-\delta}^{t_0} F(t)^{(n-1)/2}\,dt \\ &\geq \int_{t_0-\delta}^{t_0}\left(F(t_0) - \frac{M}{2}\delta^2\right)^{(n-1)/2} dt \\ &= \delta\left(1-\frac{2}{n-1}\right)^{(n-1)/2} F(t_0)^{(n-1)/2} \\ &\geq \frac{c}{\sqrt{n-1}}F(t_0)^{(n-1)/2}.\end{aligned}$$

Thus we have reached the desired bound

$$\frac{\mu_n(B(x_n, R_n) \cap B_{r_n})}{\mu_n(B(x_n, R_n))} \geq \frac{c}{\sqrt{n-1}}.$$

In order to finish the proof of (4.20) we have to justify that $t_0 < x$. It is enough to see that in $F' < 0$ in the interval $[x, x+R] = [1/2, 3/4]$. Using that $x = 1/2$ and $R = 1/4$ it is easy to see that $F'(t) < 0$ if and only if

$$\left(\frac{5}{4}t - 4t^3\right) - \left(\frac{5}{4}t^2 - 2t^4 - \frac{9}{128}\right)e^t < 0.$$

For $t > 0$ one has $e^t > 1 + t$, then we just need to show that

$$P(t) = \left(\frac{5}{4}t - 4t^3\right) - \left(\frac{5}{4}t^2 - 2t^4 - \frac{9}{128}\right)(1+t) < 0,$$

for $t \in [1/2, 3/4]$. This can be done by checking that $P(1/2) = -13/256 < 0$ and that $P'(t) < 0$ for $t \in [1/2, 3/4]$.

To finish the proof of (4.6) there is left to check that (4.22) holds for our choice of r and x. Observe that it is equivalent to

$$xe^{-e^x} > re^{-e^r},$$

But this follows from the fact that the function

$$s \mapsto se^{-e^s},$$

increases in the interval $[0, 1/2]$. □

Chapter 5

Doubling measures.

In this last chapter we will deal with maximal operators associated to measures satisfying certain doubling or regularity conditions. We will see that some of the known results for the Lebesgue measure can be carried out to this general situation. Although our model example will be that of the power weights (the measures with densities of the form $|x|^\alpha$) over $\mathbb{R}^n$, we will give some results that are generalizable to a more abstract setting.

Recently, A. Naor and T. Tao [63] extended Theorem 1.16 (the "$n \log n$" result of Stein and Strömberg) to the context of strong n-microdoubling metric measure spaces. They proved that this result could be obtained as a corollary of a general localisation principle. The techniques used by Naor and Tao are of a probabilistic nature and include martingale theory and Doob's maximal theorem. We will present here a different approach to this principle using geometric arguments and covering lemmas. These arguments are similar to those justifying Theorem 1.16 in the alternative proof given by M.T. Menárguez and F. Soria in [54]. In addition, we will prove too that Theorem 1.9 holds for radial, uniformly weakly regular measures in $\mathbb{R}^n$ (see Theorem 5.5). As we will see, these measures are all doubling, at least for large n. This is in a clear contrast with the situation of the measures considered in Chapters 3 and 4.

The other main result in this section is concerned with the extension of Stein's Theorem 1.13. We will show that the same result on uniform L^p bounds holds for the maximal operators associated to certain radial measures, including those given by the power weights.

5.1 Micro-doubling and weakly regular measures.

In this section we deal with maximal operators defined in a more abstract setting, that we explain now. We say that (X, d, μ) is a metric measure space if (X, d) is a separable metric space and μ a Radon measure on it. We denote by $B(x, r)$ the ball centred at x with radius r with respect to the metric d, that is

$$B(x, r) = \{y \in X : d(x, y) \leq r\}.$$

We will assume that the measure μ is not degenerate, that means that any ball with positive radius has positive and finite measure. Given $T \subset (0, \infty)$, for any locally integrable function f over X we can define the following maximal operator

$$M_T f(x) = \sup_{r \in T} \frac{1}{\mu(B(x, r))} \int_{B(x,r)} |f(y)| \, d\mu(y).$$

When $T = (0, \infty)$, M_T is the Hardy-Littlewood maximal operator.

We will require the measure μ to satisfy a doubling property, i.e. that the measure of a ball is comparable to the measure of certain dilation of this ball. In the classical doubling property the dilation factor is 2. That is, we say that μ is doubling if there exists a constant $K > 0$ so that for each $x \in X$ and $R > 0$ one has $\mu(B(x, 2R)) \leq K\mu(B(x, R))$. We will use instead what is called the n-micro-doubling condition, in which the dilation factor is $(1 + 1/n)$. Given a natural number n we will say that μ is n-micro-doubling if there exists a constant $K > 0$ such that for each $x \in X$ and $R > 0$ one has

$$\mu(B((x, (1 + 1/n)R)) \leq K\mu(B(x, R)).$$

We will refer to K as the micro-doubling constant. It is trivial that Lebesgue measure over $\mathbb{R}^n$ is n-micro-doubling, and moreover since

$$|B((x, (1 + 1/n)R)| = \left(1 + \frac{1}{n}\right)^n |B(x, R)| \leq e|B(x, R)|,$$

we can take $K = e$ for all dimensions.

We will also need a regularity condition. A measure μ is regular if the measure of two balls with the same radius is comparable. We will only require weak regularity. That will mean that the measure of two intersecting balls with the same radius is comparable. That is, there exists a constant $K > 0$ such that for each $x \in X$, $R > 0$ and $y \in B(x, R)$ one has

$$\mu(B(y, R)) \leq K\mu(B(x, R)).$$

Following the terminology of A. Naor and T. Tao in [63], we will say that a measure μ is strong n-micro-doubling if it is n-micro-doubling and weakly regular. Alternatively we can say that μ is strong n-micro-doubling if there exist a constant $K > 0$ so that for each $x \in X$, $R > 0$, and $y \in B(x, R)$ one has

$$\mu(B(y, (1 + 1/n)R)) \leq K\mu(B(x, R)).$$

It is easy to see the equivalence of these two definitions.

A measure μ over $\mathbb{R}^n$ is called a power weight if its density is of the form $|x|^\alpha$ for some α, that is, $d\mu(x) = |x|^\alpha\, dx$. Since we will make n tend to infinity there is no problem in assuming $\alpha > -n$, which is necessary for $|x|^\alpha$ to be locally integrable. We finish this section by proving that power weights are strong n-micro-doubling measures over $\mathbb{R}^n$ equipped with the Euclidean distance. This will be easy, once the following result is established.

Lemma 5.1. *Let μ be the measure over $\mathbb{R}^n$ with density $w(|x|)$, where $w : [0, \infty) \to [0, \infty]$ satisfies the following properties.*

i) *$w(|\cdot|)$ is locally integrable,*

ii) *w is essentially constant on dyadic intervals, i.e. there exist constants $\beta > 0$ and a_k, for $k \in \mathbb{Z}$, such that for each $k \in \mathbb{Z}$, if $2^k \leq |x| \leq 2^{k+1}$, then*

$$\frac{1}{\beta} a_k \leq w(|x|) \leq \beta\, a_k,$$

and

$$\frac{1}{\beta} a_k \leq a_{k+1} \leq \beta\, a_k.$$

iii) *there exists $\eta > 0$ such that if $|x| \leq 4R$ one has*

$$\frac{1}{\eta} w(R)|B_R| \leq \mu(B(x, R)) \leq \eta\, w(R)|B_R|.$$

Then μ is strong n-micro-doubling with a constant that only depends on β and η.

Now we are going to see that power weights fulfil the hypothesis of the Lemma. Indeed these hypothesis are satisfied for all w so that

i') $w(|\cdot|)$ is locally integrable

ii') w is essentially constant on dyadic intervals with constant β, iii') w is monotone,

iv') w satisfies the following "Hardy condition": there exists $\eta > 0$ such that for every $R > 0$ one has

$$\frac{1}{\eta} w(R)|B_R| \le \mu(B_R) \le \eta\, w(R)|B_R|.$$

Let us show that iii') and iv') imply condition $iii)$ of the Lemma. We will do it for a decreasing w, the proof for an increasing w is analogous. Assume that $|x| \le 4R$. Since w is decreasing then we have that

$$\mu(B(x,R)) \le \mu(B(0,R)) \le \eta\, w(R)\, |B_R|$$

and also that

$$\mu(B(x,R)) \ge \mu(B(4Rx',R)) \ge w(3R)\, |B(x,R)|.$$

Then we are done because w is essentially constant on dyadic intervals and if $2^k \le R \le 2^{k+1}$, then

$$w(3R) \ge w(4R) \ge \frac{1}{\beta} a_{k+2} \ge \frac{1}{\beta^3} a_k \ge \frac{1}{\beta^4} w(R).$$

Now it is elementary to check that if $w(t) = t^\alpha$ then condition $ii')$ holds with $\beta = \max(2^\alpha, 2^{-\alpha})$. Also, one has in this case $\mu(B_R) = n/(n+\alpha)w(R)|B_R|$. Therefore, condition $iv")$ holds for both, α positive and negative, with $\eta = 2$ if we take, say, $n \ge 2|\alpha|$.

Proof of Lemma 5.1. Let $x \in \mathbb{R}^n$, $R > 0$ and $y \in B(x,R)$, we have to prove that

$$\mu(B(y,(1+1/n)R)) \le K\mu(B(x,R)),$$

with $K > 0$ only depending on β and η. We distinguish the situation that $|x| \ge 3R$ and the one that $|x| \le 3R$.

Let $|x| \ge 3R$, then

$$\frac{2}{3}|x| \le |x| - R \le |y| \le |x| + R \le \frac{4}{3}|x|.$$

Therefore $|y|$ is at most one dyadic interval away from $|x|$ and we have $w(|y|) \ge \beta^{-3} w(|x|)$. As a consequence,

$$\mu(B(x,R)) = \int_{B(x,R)} w(|y|)\, dy \ge \frac{1}{\beta^3} w(|x|)\, |B_R|.$$

If $z \in \mu(B(y,(1+1/n)R))$, assuming $n \geq 2$ we have

$$\frac{1}{6}|x| \leq |y| - (1+1/n)R \leq |z| \leq |y| + (1+1/n)R \leq \frac{11}{6}|x|.$$

Hence, $|z|$ lies at most 3 dyadic intervals away from $|x|$, which implies that $w(|z|) \leq \beta^5 w(|x|)$. Thus,

$$\begin{aligned} \mu(B(y,(1+1/n)R)) &= \int_{B(z,(1+1/n)R)} w(|z|)\,dz \\ &\leq \beta^5 w(|x|)\,|B(z,(1+1/n)R)| \\ &\leq e\beta^5 w(|x|)\,|B_R|, \end{aligned}$$

and we finished with this case.

Assume now that $|x| \leq 3R$, by $iii)$ we obtain

$$\mu(B(x,R)) \geq \frac{1}{\eta} w(R)\ |B_R|.$$

In this situation $|y| \leq |x| + R \leq 4R \leq 4(1+1/n)R$. Then we can finish by applying $iii)$ again to get

$$\mu(B(y,(1+1/n)R)) \leq \eta\, w((1+1/n)R)|B_{(1+1/n)R}| \leq e\eta\beta^3 w(R)|B_R|.$$

□

5.2 Weak type estimates and a localisation principle.

The main result in this section is that Theorem 1.16 is valid in a strong n-micro-doubling measure metric space. We will be interested in the maximal operators associated to the following subsets of $(0,\infty)$. A sequence $\{a_k\}_{k\in\mathbb{Z}} \subset (0,\infty)$ is lacunary if there exist $a > 1$ so that $a_{k+1} > a a_k$. If $a = n$ in this setting of n-micro-doubling measures, we will say explicitly that the sequence is n-lacunary.

Theorem 5.2. *Let (X,d,μ) be a measure metric space, with μ a strong n-micro-doubling measure with constant K. Then we have the following weak type estimates* *i) if $T \subset (0,\infty)$ is an n-lacunary sequence, then*

$$\mu(\{x \in X : M_T f(x) > \lambda\}) \leq \frac{C_K}{\lambda} \|f\|_{L^1(\mu)},$$

ii) if $T \subset (0,\infty)$ is a lacunary sequence with constant a, then

$$\mu(\{x \in X : M_T f(x) > \lambda\}) \leq \frac{C_K \log n}{\lambda} \|f\|_{L^1(\mu)},$$

iii) if $T = (0,\infty)$ so that $M = M_T$ is the Hardy-Littlewood maximal operator, then

$$\mu(\{x \in X : M f(x) > \lambda\}) \leq \frac{C_K n \log n}{\lambda} \|f\|_{L^1(\mu)},$$

where C_k are constants only depending on K and, additionally in ii), on a.

Note that part iii) is a generalization of Theorem 1.16 by E.M. Stein and J.O. Strömberg [74]. This result was proved by M.T. Menárguez and F. Soria in [54] in the n-dimensional Euclidean space case and, with the generality presented here, by A. Naor and T. Tao in [63].

A metric measure space (X, d, μ) is said to be Ahlfors-David n-regular if there exists a constant $K > 0$ so that

$$\frac{1}{K} R^n \leq \mu(B(x,R) \leq K R^n,$$

for all $x \in X$ and $R > 0$. It is easy to see that this property implies the strong n-micro-doubling condition. A. Naor and T. Tao showed in [63] that the $\mathcal{O}(n \log n)$ bound is optimal even in this setting, by constructing a sequence of Ahlfors-David n-regular spaces (X_n, d_n, μ_n) so that

$$\|M_{\mu_n}\|_{L^1(\mu_n) \to L^{1,\infty}(\mu_n)} \geq C n \log n.$$

In [63] Theorem 5.2 is shown as a corollary of the following localisation principle for maximal operators.

Theorem 5.3. *Let (X, d, μ) be a metric measure space and let μ be n-micro-doubling with constant K. If $T \subset (0,\infty)$ and $p \geq 1$, then we have the following localisation property for the weak L^p operator norms of the associated maximal operator*

$$\|M_T\|_{L^p \to L^{p,\infty}} \leq C_K + C_K \sup_{k \in \mathbb{Z}} \|M_k\|_{L^p \to L^{p,\infty}},$$

where $M_k = M_{T \cap (n^k, n^{k+1}]}$ and C_K is a constant only depending on K.

The proof given by Naor and Tao of this result is probabilistic and relies on random martingales and Doob-type maximal inequalities. We present here (see Theorem 5.4 below) a more geometrical proof based on covering lemmas and selection processes that is closer in spirit to the arguments in [74]. Before doing that, we show hoy to obtain Theorem 5.2 from Theorem 5.3 through the following chain of implications: Theorem 5.3 $\Rightarrow$ $i)$ $\Rightarrow$ $ii)$ $\Rightarrow$ $iii)$. We do each step separately.

Proof that Theorem 5.3 $\Rightarrow$ $i)$. If $T = \{a_k\}_{k\in\mathbb{Z}}$ is an n-lacunary sequence then each interval of the form $[n^k, n^{k+1}]$ contains at most one element of T. Theorem 5.3 implies then that

$$\|M_T\|_{L^1\to L^{1,\infty}} \leq C_K \sup_{k\in\mathbb{Z}} \|M_{\{a_k\}}\|_{L^1\to L^{1,\infty}}.$$

We only have to show now that $M_{\{a_k\}}$ is bounded independently of k. Since

$$M_{\{a_k\}}f(x) = \frac{1}{\mu(B(x,a_k))}\int_{B(x,a_k)} |f(y)|\, d\mu(y),$$

by the weak regularity of μ and Fubini Theorem we have

$$\begin{aligned}
\|M_{\{a_k\}}f\|_{L^1} &= \int_X \frac{1}{\mu(B(x,a_k))}\int_X \chi_{B(x,a_k)}(y)|f(y)|\, d\mu(y)\, d\mu(x) \\
&\leq \int_X\int_X \frac{K}{\mu(B(y,a_k))}\chi_{B(y,a_k)}(x)|f(y)|\, d\mu(y)\, d\mu(x) \\
&= \int_X \frac{K}{\mu(B(y,a_k))}\int_{B(y,a_k)} d\mu(x)\, |f(y)|\, d\mu(y) = K\|f\|_{L^1}.
\end{aligned}$$

This says, in particular, that each $M_{\{a_k\}}$ is bounded on L^1 with operator norm bounded by K. $\square$

Now we are going to see that $i) \Rightarrow ii)$ and that $ii) \Rightarrow iii)$. The idea is to show that a lacunary operator is controlled by $\mathcal{O}(\log n)$ operators, each associated to an n-lacunary sequence, and that the Hardy-Littlewood maximal function is controlled by n operators, each associated to a lacunary sequence.

Proof that $i) \Rightarrow ii)$. Let $\alpha = \{\alpha_k\}_{k\in\mathbb{Z}}$ be a lacunary sequence, and let $a > 1$ be such that $\alpha_{k+1} \geq a\alpha_k$ for each $k > 1$. Fix $J \in \mathbb{N}$, for $j = 1, \ldots, J$ we define the sequences $\beta^j = \{\beta^j_k\}_{k\in\mathbb{Z}}$ by $\beta^j_k = \alpha_{Jk+j}$. Note that

$$\beta^j_{k+1} = \alpha_{J(k+1)+j} \geq a^J \alpha_{Jk+j} = a^J \beta^j_k.$$

Hence if $a^J > n$ (which can be achieved taking $J = 1 + \lfloor \log n / \log a \rfloor$) each of the sequences β^j is n-lacunary. Given a locally integrable function then we have that

$$M_\alpha f(x) = \sup_{j=1,\ldots,J} M_{\beta^j} f(x).$$

This is all we needed to prove, since each of the operators M_{β^j} satisfies a weak type estimate with constants only depending on K. In this case, the constant may also depend linearly on the value of $1/\log a$. □

Proof that $ii) \Rightarrow iii)$. Consider the lacunary sequence $\alpha = \{2^k\}_{k\in\mathbb{Z}}$. For $j = 0, \ldots, n-1$ the dilates $\alpha^j = \{2^{k+j/n}\}_{k\in\mathbb{Z}}$ are also lacunary sequences.

We consider the ball $B(x, R)$ and assume that k and j are such that

$$\alpha_k^j = 2^{k+j/n} \leq R \leq 2^{k+(j+1)/n} = \alpha_k^{j+1}.$$

Note that $\alpha_k^{j+1}/\alpha_k^j = 2^{1/n} \leq 1 + 1/n$. Then by the micro-doubling condition we have

$$\begin{aligned} \mu(B(x, \alpha_k^{j+1})) &= \mu(B(x, 2^{1/n}\alpha_k^j)) \leq \mu(B(x, (1+1/n)\alpha_k^j)) \\ &\leq K\mu(B(x, \alpha_k^j)) \leq K\mu(B(x, R)). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{\mu(B(x,R))}\int_{B(x,R)} |f(y)|\, d\mu(y) &\leq \frac{K}{\mu(B(x,\alpha_k^{j+1}))}\int_{B(x,\alpha_k^{j+1})} |f(y)|\, d\mu(y) \\ &\leq KM_{\alpha^{j+1}} f(x). \end{aligned}$$

Hence, we have that

$$M_\alpha f(x) = \sup_{j=0,\ldots,n-1} M_{\alpha^j} f(x).$$

This finishes the proof, because for each $j = 0, \ldots, n-1$ the sequence α^j is lacunary and by $ii)$ the weak $L^1(\mu)$ operator norm of M_{α^j} is controlled by $C_K \log n$. □

The remaining part of this section is devoted to proving Theorem 5.3. Our proof relies on yet a new localisation theorem and avoids technical arguments from probability theory. The statement is the following:

Theorem 5.4. *Let (X, d, μ) be a metric measure space and let μ be n-micro-doubling with constant K. If $T \subset (0,\infty)$ and $p \geq 1$, for each locally integrable function f over X and each $\lambda > 0$ one can find a disjoint collection of measurable sets $\{A_k\}_{k\in\mathbb{Z}}$ such that*

$$\begin{aligned}&\mu(\{x \in X : \lambda < M_T f(x) \leq 2\lambda\})\\ &\qquad \leq \; C_1 \left(\frac{1}{\lambda^p}\|f\|^p_{L^p(\mu)} + \sum_{k\in\mathbb{Z}} \mu(\{x \in X : M_k(f\chi_{A_k})(x) > C_2\lambda\}) \right),\end{aligned}$$

where C_1 and C_2 are constants that only depend on K.

Before proving it, let us see that this result implies Theorem 5.3.

Proof that Theorem 5.4 implies Theorem 5.3. Given a locally integrable f and $\lambda > 0$ we write

$$\begin{aligned}E_\lambda &= \{x \in X : M_T f(x) > \lambda\},\\ F_\lambda &= \{x \in X : \lambda < M_T f(x) \leq 2\lambda\}.\end{aligned}$$

We have to prove that

$$\mu(E_\lambda) \leq \frac{C_K}{\lambda^p}\left(1 + \sup_{k\in\mathbb{Z}} \|M_k\|^p_{L^p \to L^{p,\infty}}\right)\|f\|^p_{L^p(\mu)}.$$

Note that by the disjointness of the collection $\{A_k\}_{k\in\mathbb{Z}}$ we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \mu(\{x \in X : M_k(f\chi_{A_k})(x) > C_2\lambda\}) &\leq \frac{C_k}{\lambda^p}\sum_{k\in\mathbb{Z}} \|M_k\|^p_{L^p\to L^{p,\infty}} \int_{A_k} |f|^p \, d\mu\\ &\leq \frac{C_k}{\lambda^p}\sup_{k\in\mathbb{Z}} \|M_k\|^p_{L^p\to L^{p,\infty}} \int_X |f|^p \, d\mu.\end{aligned}$$

Then, by Theorem 5.4

$$\mu(F_\lambda) \leq \frac{C_K}{\lambda^p}\left(1 + \sup_{k\in\mathbb{Z}} \|M_k\|^p_{L^p \to L^{p,\infty}}\right)\|f\|^p_{L^p(\mu)}.$$

This implies

$$\begin{aligned}\mu(E_\lambda) &= \mu\left(\bigcup_{j=0}^{\infty} F_{2^j\lambda}\right) = \sum_{j=0}^{\infty}\mu(F_{2^j\lambda})\\ &\leq \sum_{j=0}^{\infty}\frac{C_K}{(2^j\lambda)^p}\left(1 + \sup_{k\in\mathbb{Z}} \|M_k\|^p_{L^p \to L^{p,\infty}}\right)\|f\|^p_{L^p(\mu)}\\ &= \frac{C_K}{\lambda^p}\left(1 + \sup_{k\in\mathbb{Z}} \|M_k\|^p_{L^p \to L^{p,\infty}}\right)\|f\|^p_{L^p(\mu)}.\end{aligned}$$

□

We finish this section proving Theorem 5.4.

Proof of Theorem 5.4. Given a locally integrable f and $\lambda > 0$ we write

$$F_\lambda = \{x \in X : \lambda < M_T f(x) \leq 2\lambda\}.$$

Once we determine what the sets A_j are, we will have to prove that

$$\mu(F_\lambda) \leq C_1 V^f_{\lambda,p},$$

where we have used the notation

$$V^f_{\lambda,p} = \frac{1}{\lambda^p}\|f\|^p_{L^p(\mu)} + \sum_{k\in\mathbb{Z}} \mu(\{x \in X : M_k(f\chi_{A_k})(x) > C_2\lambda\}).$$

Note first that it is enough to prove the result for $p = 1$. To see this, given a function f in $L^p(\mu)$ we consider $f = f^\lambda + f_\lambda$ where

$$\begin{aligned} f^\lambda &= f\chi_{\{|f|>\lambda\}} \in L^1(\mu), \\ f_\lambda &= f\chi_{\{|f|\leq\lambda\}} \in L^\infty(\mu). \end{aligned}$$

Since

$$M_T f(x) \leq M_T f^\lambda(x) + M_T f_\lambda(x) \leq M_T f^\lambda(x) + \lambda,$$

for $M_T f(x) > 2\lambda$ to hold it is necessary that $M_T f^\lambda(x) > \lambda$. Hence,

$$F_{2\lambda} \subset \{x \in X : \lambda < M_T f^\lambda(x) \leq 4\lambda\} = G_1 \cup G_2,$$

with

$$\begin{aligned} G_1 &= \{x \in X : \lambda < M_T f^\lambda(x) \leq 2\lambda\}, \\ G_2 &= \{x \in X : 2\lambda < M_T f^\lambda(x) \leq 4\lambda\}. \end{aligned}$$

Then since $f^\lambda \in L^1(\mu)$ we apply the result for $p = 1$ to G_1 and G_2 to obtain

$$\begin{aligned} \mu(F_{2\lambda}) &\leq 2\max(\mu(G_1), \mu(G_2)) \\ &\leq C\left(C_1 \int_X \frac{|f^\lambda|}{\lambda}\, d\mu + \sum_{k\in\mathbb{Z}} \mu(\{x \in X : M_k(f^\lambda\chi_{A_k})(x) > C_2\lambda\})\right) \\ &\leq C\left(C_1 \int_X \frac{|f^\lambda|^p}{\lambda^p}\, d\mu + \sum_{k\in\mathbb{Z}} \mu(\{x \in X : M_k(f\chi_{A_k})(x) > C_2\lambda\})\right), \end{aligned}$$

for each $p \geq 1$. Since λ is arbitrary the result is proved for any $p \geq 1$.

Now we prove the result in the case that $p = 1$, that is

$$\mu(F_\lambda) \leq C_1 V^f_{\lambda,1}$$

We consider the following collections of balls

$$\begin{aligned} \mathcal{B} &= \{B(x,R) : x \in X,\ R \in T\}, \\ \mathcal{B}_k &= \{B(x,R) : x \in X,\ R \in T \cap [n^k, n^{k+1})\}. \end{aligned}$$

Given a ball $B \in \mathcal{B}$ we denote by z_B its centre and by R_B its radius. We define the collection

$$\mathcal{A} = \left\{ B \in \mathcal{B} : \lambda < \frac{1}{\mu(B)} \int_B |f|\, d\mu \leq 2\lambda \right\}.$$

For each $B \in \mathcal{A}$ one can find a concentric ball $\tilde{B}$ so that $R_B/(1+1/n) \leq R_{\tilde{B}} < R_B$ and

$$\frac{1}{\mu(B)} \int_{\tilde{B}} |f|\, d\mu > \lambda.$$

We will write $B^0 = B(z_B, R_B - R_{\tilde{B}})$. Since

$$F_\lambda \subset \bigcup_{B \in \mathcal{A}} B^0,$$

it suffices to prove that

$$\mu\left(\bigcup_{B \in \mathcal{A}} B^0 \right) \leq C_1 V^f_{\lambda,1}.$$

It is not difficult to see (see the argument preceding (5.5) below) that $\cup_{B \in \mathcal{A}} B^0$ is contained in a level set of M_R and, as a consequence, has finite μ-measure. Therefore, by monotonicity there exists $\mathcal{A}' \subset \mathcal{A}$ with $\sharp \mathcal{A}_0 < \infty$ such that

$$\mu\left(\bigcup_{B \in \mathcal{A}'} B^0 \right) \geq \frac{1}{2} \mu\left(\bigcup_{B \in \mathcal{A}} B^0 \right).$$

Hence, it is enough to show that

$$\mu\left(\bigcup_{B \in \mathcal{A}'} B^0 \right) \leq C_1 V^f_{\lambda,1},$$

with C independent of $\mathcal{A}'$. Writing

$$\mathcal{A}_k := \{B \in \mathcal{A}' : B \in \mathcal{B}_k\},$$

we have that $\mathcal{A}' = \mathcal{A}^1 \cup \mathcal{A}^2$ with

$$\mathcal{A}^1 = \bigcup_{k \text{ odd}} \mathcal{A}_k,$$
$$\mathcal{A}^2 = \bigcup_{k \text{ even}} \mathcal{A}_k.$$

Assume that $\mu(\bigcup_{B\in\mathcal{A}^1} B^0) \geq \mu(\bigcup_{B\in\mathcal{A}^2} B^0)$ (if not, interchange the names), then $\mu(\bigcup_{B\in\mathcal{A}^1} B^0) \geq \frac{1}{2}\mu(\bigcup_{B\in\mathcal{A}'} B^0)$ and we just need to prove that

$$\mu\left(\bigcup_{B\in\mathcal{A}^1} B^0\right) \leq C_1 V^f_{\lambda,1}.$$

For the sake of simplicity in the notation we rename $\mathcal{A} = \mathcal{A}^1$.

Now we want to write the set $\bigcup_{B\in\mathcal{A}} B^0$ as the union of disjoint sets. Observe that $\mathcal{A} = \{B_j\}_{j=1,\cdots,J}$ for certain J. We define $D_{B_1} = B_1^0$, and if $D_{B_1}, \cdots, D_{B_m}$ are already defined, we take

$$D_{B_{m+1}} = B_{m+1} \setminus \bigcup_{j=1}^{m} B_j^0.$$

Then, we have

$$\mu\left(\bigcup_{B\in\mathcal{A}} B^0\right) = \mu\left(\bigcup_{B\in\mathcal{A}} D_B\right).$$

We also define the functions

$$g_B(x) = \frac{\mu(D_B)}{\mu(B)}\chi_{\tilde{B}}(x)$$

for $B \in \mathcal{A}$ and

$$G_k(x) = \sum_{B\in\mathcal{A}_k} g_B(x).$$

We start a selection process. Take k_1 as the largest $k \in \mathbb{Z}$ with $\mathcal{A}_k \neq \emptyset$, then we take $\tilde{G}_{k_1} = G_{k_1}$ and $\tilde{\mathcal{A}}_{k_1} = \mathcal{A}_{k_1}$. Once $\tilde{G}_{k_1}, \cdots, \tilde{G}_{k_m}$ are determined we take k_{m+1} as the largest $k < k_m$ so that $\mathcal{A}_k \neq \emptyset$. We say that $B \in \tilde{\mathcal{A}}_{k_{m+1}}$ if $B \in \mathcal{A}_{k_{m+1}}$ and

$$\sum_{j=1}^{m} \tilde{G}_{k_j} \leq 1$$

on B. Then we define

$$\tilde{G}_{k_{m+1}}(x) = \sum_{B \in \mathcal{A}_{k_{m+1}}} g_B(x).$$

Since $\mathcal{A}$ is finite this process ends in a finite number of steps, and we have obtained $\tilde{G}_{k_1}, \cdots, \tilde{G}_{k_M}$ for certain M. We call

$$\tilde{\mathcal{A}} = \bigcup_{m=1}^{M} \tilde{\mathcal{A}}_{k_m},$$

and claim that

$$\mu\left(\bigcup_{B \in \mathcal{A}} D_B\right) \leq (1+K) \sum_{B \in \tilde{\mathcal{A}}} \mu(D_B). \tag{5.1}$$

To prove this claim note that if $A \in \mathcal{A}_{k_m} \setminus \tilde{\mathcal{A}}_{k_m}$, then there exists $z \in A$ such that

$$\sum_{j=1}^{m-1} \tilde{G}_{k_j}(z) > 1.$$

Note that if $A \cap \tilde{B} \neq \emptyset$ for some $B \in \tilde{\mathcal{A}}_{k_j}$ with $j < m$, then $A \subset B^\star := B(x_B, (1+1/n)R_B)$. Thus for all $z \in A$ we have

$$\sum_{j=1}^{m-1} G^*_{k_j}(z) := \sum_{j=1}^{m-1} \sum_{B \in \tilde{\mathcal{A}}_{k_j}} \frac{\mu(D_B)}{\mu(B)} \chi_{B^*}(z) \geq 1.$$

Then, by Tchebychev inequality

$$\begin{aligned}
\mu\left(\bigcup_{B \in \mathcal{A} \setminus \tilde{\mathcal{A}}} D_B\right) &\leq \mu\left(\{z \in X : \sum_{j=1}^{M} G^*_{k_j}(z) > 1\}\right) \\
&\leq \sum_{j=1}^{M} \int_X G^*_{k_j}(z)\, d\mu(z) = \sum_{B \in \tilde{\mathcal{A}}} \frac{\mu(D_B)}{\mu(B)} \mu(B^*) \\
&\leq K \sum_{B \in \tilde{\mathcal{A}}} \mu(D_B).
\end{aligned}$$

Now that the claim is justified we only need to prove

$$\sum_{B \in \tilde{\mathcal{A}}} \mu(D_B) \leq C_1 V^f_{\lambda,1}.$$

By the definition of $\tilde{B}$ we have

$$\sum_{B\in\tilde{\mathcal{A}}} \mu(D_b) \leq \frac{1}{\lambda} \sum_{B\in\tilde{\mathcal{A}}} \frac{1}{\mu(B)} \int_{\tilde{B}} |f|\, d\mu\, \mu(D_b) = \frac{1}{\lambda} \int_X |f| \left(\sum_{j=1}^{M} \tilde{G}_{k_j} \right) d\mu. \quad (5.2)$$

For each $j = 1, \cdots, M$ we define $\tilde{A}_j := \operatorname{supp} \tilde{G}_{k_j}$. We now take $A_M = \tilde{A}_M$ and

$$A_j = \tilde{A}_j \setminus \bigcup_{\ell=j+1}^{M} \tilde{A}_\ell.$$

If $z \in A_m$ we have $\tilde{G}_{k_j}(z) = 0$ if $j > m$, and then, by the way we selected $\tilde{G}_{k_m}$

$$\sum_{j=1}^{M} \tilde{G}_{k_j}(z) = \tilde{G}_{k_m}(z) + \sum_{j=1}^{m-1} \tilde{G}_{k_j}(z) \leq \tilde{G}_{k_m}(z) + 1.$$

Then we have that

$$\sum_{j=1}^{M} \tilde{G}_{k_j} \leq \sum_{j=1}^{M} \tilde{G}_{k_j} \chi_{A_j} + 1,$$

which combined with (5.1) and (5.2) yields

$$\sum_{B\in\tilde{\mathcal{A}}} \mu(D_B) \leq \frac{2K}{\lambda} \left(\int_X |f|\, d\mu + \sum_{j=1}^{M} \int_X |f| \chi_{A_j} \tilde{G}_{k_j}\, d\mu \right). \quad (5.3)$$

In order to bound the last sum note that

$$\frac{2K}{\lambda} \sum_{j=1}^{M} \int_X f \chi_{A_j} \tilde{G}_{k_j}\, d\mu = \frac{2K}{\lambda} \sum_{j=1}^{M} \sum_{B\in\tilde{\mathcal{A}}_{k_j}} \frac{1}{\mu(B)} \int_{\tilde{B}} |f| \chi_{A_j}\, d\mu\, \mu(D_B)$$

Given $B \in \tilde{\mathcal{A}}_{k_j}$, we say that $B \in \tilde{\mathcal{A}}^\star_{k_j}$ if

$$\frac{1}{\mu(B)} \int_{\tilde{B}} |f| \chi_{A_j}\, d\mu \leq \frac{\lambda}{4K}.$$

Hence, we have

$$\frac{2K}{\lambda} \sum_{j=1}^{M} \sum_{B\in\tilde{\mathcal{A}}^\star_{k_j}} \frac{1}{\mu(B)} \int_{\tilde{B}} |f| \chi_{A_j}\, d\mu\, \mu(D_B) \leq \frac{1}{2} \sum_{B\in\mathcal{A}} \mu(D_B)$$

and this term can be absorbed in the left hand side of (5.3).

On the other hand note that if $B \in \mathcal{A}$

$$\frac{1}{\mu(B)} \int_{\tilde{B}} |f| \chi_{A_j} \, d\mu \leq \frac{1}{\mu(B)} \int_B |f| \chi_{A_j} \, d\mu \leq 2\lambda. \tag{5.4}$$

We claim that if $z \in B^0$, then $\tilde{B} \subset B(z, R_B) \subset B^\star$. The n-micro-doubling condition implies then that $\mu(B) \geq \mu(B^\star)/K \geq \mu(B(z, R_B))/K$ and consequently if $B \in \tilde{\mathcal{A}}_{k_j} \setminus \tilde{\mathcal{A}}_{k_j}^\star$

$$\begin{aligned} \frac{\lambda}{4K} &< \frac{1}{\mu(B)} \int_{\tilde{B}} |f| \chi_{A_j} \, d\mu \leq \frac{K}{\mu(B(z, R_B))} \int_{B(z, R_B)} |f| \chi_{A_j} \, d\mu \\ &\leq K \, M_{k_j}(f \chi_{A_j})(z). \end{aligned}$$

Thus, for such B,

$$B^0 \subset \left\{ x \in X : M_{k_j}(f \chi_{A_j})(x) > \frac{\lambda}{4K^2} \right\}. \tag{5.5}$$

Therefore, using (5.4) we have

$$\begin{aligned} &\frac{2K}{\lambda} \sum_{j=1}^{M} \sum_{B \in \tilde{\mathcal{A}}_{k_j} \setminus \tilde{\mathcal{A}}_{k_j}^\star} \frac{1}{\mu(B)} \int_{\tilde{B}} |f| \chi_{A_j} \, d\mu \, \mu(D_B) \\ &\qquad \leq 4K \sum_{j=1}^{M} \sum_{B \in \tilde{\mathcal{A}}_{k_j}} \mu(D_B) \\ &\qquad \leq 4K \sum_{j=1}^{M} \mu \left(\left\{ x \in X : M_{k_j}(f \chi_{A_j})(x) > \frac{\lambda}{4K^2} \right\} \right). \end{aligned}$$

This finishes the proof, provided we justify the last claim.

In order to do so, suppose that $y \in \tilde{B}$, then

$$|y - z| \leq |y - z_B| + |z_B - z| \leq R_{\tilde{B}} + (R_B - R_{\tilde{B}}) = R_B,$$

which means that $y \in B(z, R_B)$. Assume now that $y \in B(z, R_B)$

$$|y - z_B| \leq |y - z| + |z - z_B| \leq R_B + R_B/n = (1 + 1/n) R_B,$$

hence $y \in B^\star$. The claim is proved. $\square$

5.3 Radial functions and the weak type L^1

In this section we will focus on the Euclidean space equipped with a weakly regular, rotation invariant measure. In this setting we have the following extension of Theorem 1.9.

Theorem 5.5. *Let μ be a rotation invariant measure over $\mathbb{R}^n$ that is absolutely continuous with respect to Lebesgue measure. If μ is weakly regular with constant K and f is a radial function we have*

$$\mu(\{x \in X : Mf(x) > \lambda\}) \leq \frac{4K}{\lambda} \|f\|_{L^1(\mu)}.$$

For the proof, we will mostly follow [54]. Next lemma, from which Theorem 1.9 derives easily, uses the following key ingredient: associated to a weight v on $\mathbb{R}$ we define the non-centred maximal function

$$\tilde{M}_v F(x) = \sup_{a \leq x \leq b} \frac{1}{v([a,b])} \int_{[a,b]} |F(t)| v(t)\, dt,$$

for each locally integrable F (with respect to v). Then, independently of the weight v one has

$$v(\{r \geq 0 \,:\, \tilde{M}_v F(r) > \lambda\}) \leq \frac{2}{\lambda} \int_{\mathbb{R}} |F(r)| v(r)\, dr. \tag{5.6}$$

A simple proof of this well-known fact can be found in [60], [38] or [50].

Lemma 5.6. *Let μ be a radial measure over $\mathbb{R}^n$ as above, with density $w(|x|)$. Given a radial function f over $\mathbb{R}^n$, that can be written as $f(x) = F(|x|)$ for certain $F : [0, \infty) \longrightarrow \mathbb{R}$, one has*

$$Mf(x) \leq (1 + K)\tilde{M}_v F(|x|),$$

where $v(t) = w(t)t^{n-1}$.

Proof of Theorem 1.9. Writing $f(x) = F(|x|)$ and using Lemma 5.6 we have

$$\{x \in \mathbb{R}^n \,:\, Mf(x) > \lambda\} \subset \left\{x \in \mathbb{R}^n \,:\, \tilde{M}_v F(|x|) > \frac{\lambda}{1+K}\right\}.$$

Integrating in polar coordinates we have

$$\begin{aligned}\mu\Big(\Big\{x \in \mathbb{R}^n : \tilde{M}_v F(|x|) \;>\; \frac{\lambda}{1+K}\Big\}\Big) \\ &= \omega_{n-1} \int_{\{r\geq 0\,:\,\tilde{M}_v F(r)>\lambda/(1+K)\}} v(t)\,dt \\ &= \omega_{n-1}\, v\left(\left\{r \geq 0 \;:\; \tilde{M}_v F(r) > \frac{\lambda}{1+K}\right\}\right).\end{aligned}$$

By (5.6) the last is bounded by

$$\frac{2(1+K)}{\lambda}\omega_{n-1}\int_0^\infty |F(t)|\,v(t)\,dt = \frac{2(K+1)}{\lambda}\int_{\mathbb{R}^n} |f(x)|\,d\mu(x).$$

□

In order to prove Lemma 5.6 we need the following geometric definition. Given a measurable set $E \in \mathbb{R}^n$ we define its projection onto the sphere $\mathbb{S}^{n-1}$ by

$$\Sigma_E = \{\theta \in \mathbb{S}^{n-1} \;:\; r\theta \in E \text{ for some } r > 0\}.$$

We claim that

Lemma 5.7. *For each ball $B(x,R) \in \mathbb{R}^n$ there exists a set D such that*

(i) $B(x,R) \subset D$,

(ii) $\Sigma_D = \Sigma_{B(x,R)}$,

(iii) for each $\theta \in \Sigma_D$ there exist $0 \leq a_\theta \leq b_\theta$ such that $r\theta \in D$ if and only if $a_\theta \leq r \leq b_\theta$,

(iv) $|x|\theta \in D$ for each $\theta \in \sigma_D$ (this means $a_\theta \leq |x| \leq b_\theta$),

(v) $\mu(D) \leq (1+K)\mu(B(x,R))$.

Using this, we prove Lemma 5.6

Proof of Lemma 5.6. By conditions (i) and (v)

$$\begin{aligned}\frac{1}{\mu(B(x,R))}\int_{B(x,R)} |f(y)|\,d\mu(y) &\leq \frac{\mu(D)}{\mu(B(x,R))}\frac{1}{\mu(D)}\int_D |f(y)|\,d\mu(y) \\ &\leq \frac{1+K}{\mu(D)}\int_D |f(y)|\,d\mu(y).\end{aligned}$$

Now we integrate along each ray coming from the origin and use conditions (ii)–(iv)

$$\begin{aligned}\int_D |f(y)|\,d\mu(y) &= \int_{\Sigma_D}\int_{a_\theta}^{b_\theta} |F(t)|v(t)\,dt\,d\sigma_n(\theta)\\ &= \int_{\Sigma_D} \frac{v([a_\theta,b_\theta])}{v([a_\theta,b_\theta])}\int_{a_\theta}^{b_\theta} |F(t)|v(t)\,dt\,d\sigma_n(\theta)\\ &\le \int_{\Sigma_D} v([a_\theta,b_\theta])\,\tilde{M}_vF(|x|)\,d\sigma_n(\theta).\end{aligned}$$

Note that

$$\int_{\Sigma_D} v([a_\theta,b_\theta])\,d\sigma_n(\theta) = \int_{\Sigma_D}\int_{a_\theta}^{b_\theta} v(t)\,dt\,d\sigma_n(\theta) = \mu(D),$$

Hence we have proved that

$$\frac{1}{\mu(B(x,R))}\int_{B(x,R)} |f(y)|\,d\mu(y) \le (1+K)\tilde{M}_vF(|x|),$$

and since R is arbitrary the lemma is proved. □

It remains to give the proof of Lemma 5.7.

Proof of Lemma 5.7. If the origin is contained in $B(x,R)$, then $|x| \le R$ and $\Sigma_{B(x,R)} = \mathbb{S}^{n-1}$. We can take $D = B(x,r)\cup B_{|x|}$, which obviously fulfils conditions (i)–(iv). For condition (v) note that by the weak regularity

$$\mu(B_{|x|}) \le \mu(B_R) \le K\mu(B(x,R)).$$

If $|x| > R$, the origin is not contained in $B(x,R)$ and $\Sigma_{B(x,R)} \ne \mathbb{S}^{n-1}$. For each $\theta\in\Sigma_{B(x,R)}$ let

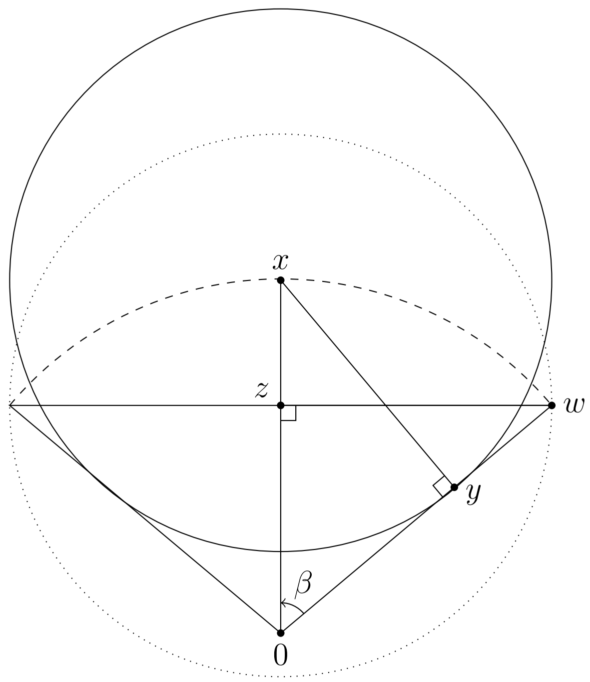
$$a_\theta = \inf\{r>0 \;:\; r\theta\in B(x,R)\}.$$

It easy to see that $a_\theta < |x|$. Note that the maximal value of a_θ occurs when the line directed by θ is tangent to $B(x,R)$. If the tangency point is y, then $a_\theta = |y|$. Looking at the rectangle triangle whose vertices are x, y and the origin it is obvious that $|x| > |y| = a_\theta$.

Then we take

$$D = B(x,R)\cup\{r\theta \;:\; a_\theta \le r \le |x| \text{ and } \theta\in\Sigma_{B(x,R)}\},$$

that is easily checked to satisfy conditions (i)–(iv). To see that condition (v) is fulfilled, by the weak regularity, it is enough to see that there exist a point $z \in B(x,R)$ so that $D \subset B(x,R) \cup B(z,R)$. For this it will suffice to prove that $D \setminus B(x,R) \subset B(z,R)$.



Let $\theta \in \mathbb{S}^{n-1}$ be such that the line directed by θ is tangent to $B(x,R)$, and let y be the tangency point as before. We take z as the orthogonal projection of y onto the line passing through x and the origin. Note that the maximal distance from a point in $D \setminus B(x,R)$ to z is attained at $w = |x|\theta \in D$. Therefore it is enough to prove that $w \in B(z,R)$, i.e. that $|w-z| \leq R$. For this, note that the triangle whose vertices are x, y and the origin is congruent with the one with vertices and z, w and the origin, therefore $|w-z| = |x-y| = R$. □

Remark 5.8. *Note that the weak regularity of μ has been used only for proving part (v) of Lemma 5.7. Note also that if μ is a radial and radially increasing measure in the sense that $\mu(B(x,R)) \geq \mu(B(y,R))$ whenever $|x| \geq |y|$, the proof of Lemma 5.7 is also valid with $K = 1$. This was observed by A. Infante in [44].*

5.4 Measures with dimension free estimates on L^p, $p > 1$.

In this section we will use some of the previous ideas to prove L^p bounds independent of the dimension for maximal operators associated to certain families of radial and doubling measures that include the power weights. The main result is the following one.

Theorem 5.9. *Let μ be a radial measure over $\mathbb{R}^n$ of the form $d\mu(x) = w(|x|)\,dx$. Assume that w is such that*

i) $w(|\cdot|)$ *is locally integrable,*

ii) w *is essentially constant on dyadic intervals with constant* β*,*

iii) there exists $\eta > 0$ *such that if* $|x| \leq 2R$ *one has*

$$\frac{1}{\eta} w(R)|B_R| \leq \mu(B(x,R)) \leq \eta\, w(R)|B_R|.$$

iv) We have, for the usual Hardy-Littlewood maximal operator, the weighted inequalities

$$\begin{aligned} \|Mf\|_{L^p(w(|.|))} &\leq A_1 \quad \|f\|_{L^p(w(|\cdot|))}, \\ \|Mf\|_{L^p(w(|.|)^{1-p})} &\leq A_2 \quad \|f\|_{L^p(w(|\cdot|)^{1-p})}. \end{aligned}$$

Then one has

$$\|M_\mu f\|_{L^p(\mu)} \leq C\|f\|_{L^p(\mu)},$$

where the constant C only depends on β, η, A_1 and A_2.

Theorem 5.9 follows from the following lemma.

Lemma 5.10. *Let μ be a radial measure over $\mathbb{R}^n$ of the form $d\mu(x) = w(|x|)\,dx$. Assume that w is such that*

i) $w(|\cdot|)$ *is locally integrable,*

ii) w *is essentially constant on dyadic intervals with constant* β*,* *iii) there exists $\eta > 0$ such that if $|x| \le 2R$ one has*

$$\frac{1}{\eta} w(R)|B_R| \le \mu(B(x,R)) \le \eta\, w(R)|B_R|.$$

Then we have the following pointwise inequality

$$M_\mu f(x) \le C \left(Mf(x) + \frac{1}{w(|x|)} M(fw(|\cdot|))(x) + \mathcal{H}_\mu f(x) \right),$$

where C depends only on β and η and

$$\mathcal{H}_\mu f(x) = \sup_{R \ge |x|} \frac{1}{\mu(B_R)} \int_{B_R} |f(y)|\, d\mu(y).$$

Proof of Theorem 5.9. By the Lemma

$$\|M_\mu f\|_{L^p(\mu)} \le C \left(\|Mf\|_{L^p(\mu)} + \left\| \frac{M(fw(|\cdot|))}{w(|\cdot|)} \right\|_{L^p(\mu)} + \|\mathcal{H}_\mu f\|_{L^p(\mu)} \right),$$

where the constant C depends only on β and η.

By assumption $iv)$ of the Theorem we have

$$\|Mf\|_{L^p(\mu)} \le A_1 \|f\|_{L^p(\mu)}.$$

We also would like to have

$$\left\| \frac{1}{w(|\cdot|)} M(fw(|\cdot|)) \right\|_{L^p(\mu)} \le C \|f\|_{L^p(\mu)}.$$

Taking $g = fw(|\cdot|)$ this is equivalent to

$$\|Mg\|_{L^p(w(|.|)^{1-p})} \le C \|g\|_{L^p(w(|\cdot|)^{1-p})},$$

which we know true by assumption $iv)$, with $C = A_2$.

For the last term we use the standard argument for Hardy type operators. It is obvious that $\mathcal{H}_\mu$ is bounded on $L^\infty(\mu)$ with constant 1. We will show that it is also weakly bounded on $L^1(\mu)$ with operator norm 1. Then by real interpolation it is bounded on $L^p(\mu)$ with operator norm controlled by an absolute constant.

To see the weak type inequality take $\lambda > 0$ and consider $E_\lambda = \{x \in \mathbb{R}^n : \mathcal{H}_\mu f(x) \geq \lambda\}$. If $x \in E_\lambda$ there exists $R_x > |x|$ so that

$$\frac{1}{\mu(B_{R_x})} \int_{B_{R_x}} |f(y)| \, d\mu(y) \geq \lambda.$$

Note that then $B_{R_x} \subset E_\lambda$, and that

$$E_\lambda = \bigcup_{x \in E_\lambda} B_{R_x}.$$

Then $E_\lambda = B_R$ for certain $R > 0$ and by monotonicity

$$\begin{aligned} \mu(E_\lambda) &= \mu(B_R) \leq \sup_{x \in E_\lambda} \mu(B_{R_x}) \leq \sup_{x \in E_\lambda} \frac{1}{\lambda} \int_{B_{R_x}} |f(y)| \, d\mu(y) \\ &\leq \frac{1}{\lambda} \int_{\mathbb{R}^n} |f(y)| \, d\mu(y). \end{aligned}$$

□

Proof of Lemma 5.10. We will bound the mean value over certain ball $B(x, R)$. Fixing x and R, we consider different cases.

If $|x| \geq 2R$ and $y \in B(x, R)$ then

$$\frac{1}{2}|x| \leq |x| - R \leq |y| \leq |x| + R \leq \frac{3}{2}|x|,$$

and since w is essentially constant in dyadic intervals $\beta^{-3} w(|x|) \leq w(|y|) \leq \beta^3 w(|x|)$. Hence

$$\begin{aligned} &\frac{1}{\mu(B(x,R))} \int_{B(x,R)} |f(y)| \, w(|y|) \, dy \\ &\qquad \leq \frac{\beta^6}{w(|x|) \, \mu(B(x,R))} \int_{B(x,R)} |f(y)| \, w(|x|) \, dy \\ &\qquad \leq \beta^6 M f(x). \end{aligned}$$

In the case that $R \leq |x| \leq 2R$ we use assumptions $iii)$ and $ii)$ to see that

$$\mu(B(x, R)) \geq \frac{1}{\eta} \, w(R) \, |B(x, R)| \geq \frac{1}{\eta \, \beta^3} \, w(|x|) \, |B(x, R)|.$$

Therefore we have

$$\begin{aligned}\frac{1}{\mu(B(x,R))}\int_{B(x,R)} &|f(y)|\,w(|y|)\,dy \\ &\leq \frac{\eta\,\beta^3}{w(|x|)|B(x,R)}\int_{B(x,R)}|f(y)|\,w(|y|)\,dy \\ &\leq \frac{\eta\,\beta^3}{w(|x|)}M(fw(|\cdot|))(x).\end{aligned}$$

Last, we consider $0 \leq |x| \leq R$. By assumption $iii)$ we have

$$\mu(B(x,R)) \geq \frac{1}{\eta}\,w(R)\,|B_R| \geq \frac{1}{\eta^2}\mu(B_R).$$

We split the ball $B(x,R)$ into two disjoint pieces

$$\begin{aligned}E &= B(x,R)\cap B_R,\\ F &= B(x,R)\setminus B_R,\end{aligned}$$

and integrate over then separately. For the first one

$$\frac{1}{\mu(B(x,R))}\int_E |f(y)|\,d\mu(y) \;\leq\; \frac{\eta^2}{\mu(B_R)}\int_{B_R}|f(y)|\,d\mu(y) \leq \eta^2\mathcal{H}_\mu f(x).$$

For the second one note that if $y \in F$ then $R \leq |y| \leq 2R$ and then $w(|y|) \leq \beta^3 w(R)$. Hence

$$\begin{aligned}\frac{1}{\mu(B(x,R))}\int_F |f(y)|\,d\mu(y) &\leq \frac{\eta\,\beta^3}{w(R)\,|B_R|}\int_{B(x,R)}|f(y)|\,w(R)\,dy \\ &\leq \eta\,\beta^3 Mf(x).\end{aligned}$$

□

Remark 5.11. *Both, Theorem 5.9 and Lemma 5.10 do not require μ to be radial. For the applications, however, this requirement is the most natural in order to define μ and M_μ simultaneously in all dimensions and to study whether or not there are uniform bounds as $n \longrightarrow \infty$.*

With respect to Theorem 5.9, we already know that there are some families of measures, for example power weights with a fixed exponent, for which β and

η can be bounded independently of the dimension. Now we want to see if the same is possible for A_1 and A_2 in hypothesis $iv)$. The following result by J. Duoandikoetxea and L. Vega which appeared in [29], shows that if hypothesis $iv)$ is true in $\mathbb{R}^k$ then it holds in $\mathbb{R}^n$ for each $n \geq k$ with the same values of A_1 and A_2.

Proposition 5.12. *Let w be a nonnegative function on $[0, \infty)$, so that $w(|\cdot|) \in A_p(\mathbb{R}^k)$, then $w(|\cdot|) \in A_p(\mathbb{R}^n)$ for all $n \geq k$ and moreover*

$$\|Mf\|_{L^p(w(|\cdot|))} \leq C\|f\|_{L^p(w(|\cdot|))},$$

where the constant C might depend on p and w but not on n.

For our model example, the power weights, we know that $|\cdot|^\alpha \in A_p(\mathbb{R}^n)$ if and only if $-n < \alpha < p(n-1)$ (see for example [36]). Then, given $\alpha \in \mathbb{R}$ and $p \geq 1$ for condition $iv)$ to hold we need both

$$\begin{array}{ccccc} -n & < & \alpha & < & p(n-1), \\ -n & < & \alpha(1-p) & < & p(n-1). \end{array}$$

As a consequence of all this, we obtain the following result

Theorem 5.13. *Let $d\mu(x)$ denote the measure $|x|^\alpha\, dx$ in Euclidean space $\mathbb{R}^n$, with $n > -\alpha$. Then for $p > 1$ there exists a constant $C_{\alpha,p}$, that depends only on α and p, so that*

$$\|M_\mu f\|_{L^p(\mu)} \leq C_{\alpha,p}\|f\|_{L^p(\mu)},$$

uniformly in dimension n and on $f \in L^p(\mu)$.

5.5 Families of measures changing with the dimension

As we have seen in the preceding section, maximal operators associated to power weights have L^p operator norms bounded with respect to the dimension. An interesting observation by J.M. Aldaz and J. Pérez Lázaro in [4] showed that given an exponent p (as large as wanted) there exist families of power weights such that the L^p bounds of the associated maximal operators grow to infinity as $n \longrightarrow \infty$. The twist here is that the powers change from one dimension to another. To be more precise they considered measures $\nu_{\alpha,n}$ given by the densities $|x|^{-\alpha n}$ over $\mathbb{R}^n$ with $0 < \alpha < 1$. Their result is the the following (see Theorem 3.12 in [4]).

Theorem 5.14. *Given $p_0 \in [1, \infty)$, there exist $\alpha_0 \in (0, 1)$ and $a > 1$ such that for all $p \in [1, p_0]$ and all $\alpha \in [\alpha_0, 1)$ one has*

$$c_{\nu_{\alpha,n},p} \geq \frac{a^{(1-\alpha)n}}{6}.$$

It is implicit in the proof given in [4] that $\alpha_0 \longrightarrow 1$ as $p_0 \longrightarrow \infty$. This leads to the question of whether, fixing α, $M_{\nu_{\alpha,n}}$ may satisfy a uniform L^p bound for large p. We can apply the method used in Theorem 4.11 for the Gaussian measure to show that this is not the case when $\alpha > 1/2$.

Theorem 5.15. *For each $\alpha \in (1/2, 1)$ there exists a constant $a > 1$ such that for all $p \in [1, \infty)$*

$$c_{\nu_{\alpha,n},p} \geq ca^{n/p},$$

even if the action is restricted to radially decreasing functions.

Another consequence of this result is that the constants for the n-micro-doubling and the weakly regularity conditions grow to infinity with the dimension in these families of measures (see Theorems 5.2, 5.5 and 5.9).

Thus, the maximal operators associated to families of power weights may have different behaviour depending on the choice of the family. The conclusion is that in order to get uniform bounds for the associated maximal functions in a family of power weights of the form $|x|^{\gamma_n}$ for $n \in \mathbb{N}$, then we must take γ_n bounded below away from $-n$.

We finish with the proof of Theorem 5.15. In view of Lemma 3.5 we only need to prove the following

Lemma 5.16. *Given $\alpha \in (1/2, 1)$, there exist $\xi, r, R > 0$ such that if $x \in \mathbb{R}^n$ is such that $|x| = \xi$ one has*

$$M_{\nu_{\alpha,n}}\chi_r(x) \geq \frac{\nu_{\alpha,n}(B(x,R) \cap B_r)}{\nu_{\alpha,n}(B(x,r))} \geq \frac{c}{\sqrt{n}}, \tag{5.7}$$

and

$$\frac{\nu_{\alpha,n}(B_\xi)}{\nu_{\alpha,n}(B_r)} \geq a^n, \tag{5.8}$$

for some $a > 1$ that might depend on α.

Proof. It is easy to prove (5.8). Given $\rho > 0$ one has

$$\nu_{\alpha,n}(B_\rho) = \omega_{n-1}\int_0^\rho s^{-\alpha n} s^{n-1}\,ds = \frac{\omega_{n-1}}{(1-\alpha)n}\rho^{(1-\alpha)n}.$$

Then it is obvious that for any $\xi > r > 0$ one has

$$\frac{\nu_{\alpha,n}(B_\xi)}{\nu_{\alpha,n}(B_r)} = \left(\left(\frac{\xi}{r}\right)^{1-\alpha}\right)^n,$$

where $(\xi/r)^{1-\alpha} > 1$.

As for the proof of (5.7), let $\xi > r > 0$ and $\xi - r < R < \xi$, then

$$\nu_{\alpha,n}(B(x,R) = \int_{\xi-R}^{\xi+R} |\partial B_s \cap B(x,R)|_{n-1}\, s^{-\alpha n}\,ds.$$

Denote by β_s the angle determined at a point of $\partial B_s \cap \partial B(x,R)$ by the two segments joining this point with origin and with x. We have

$$\begin{aligned} |\partial B_s \cap B(x,R)|_{n-1} &= \int_0^{\beta_s} \left|s\sin\theta\mathbb{S}^{n-2}\right|_{n-2} s\,d\theta \\ &= \omega_{n-2}s^{n-1}\int_0^{\beta_s} \sin^{n-2}\theta\,d\theta. \end{aligned}$$

We can estimate this last integral by

$$\int_0^{\beta_s} \sin^{n-2}\theta\,d\theta \le \frac{1}{\cos\beta_s}\int_0^{\beta_s} \sin^{n-2}\theta\,\cos\theta\,d\theta = \frac{1}{\cos\beta_s}\frac{\sin^{n-1}\beta_s}{(n-1)}.$$

By the cosine law we can calculate

$$\cos\beta_s = \frac{s^2+\xi^2-R^2}{2\xi s}.$$

It is easy to see that this quantity is minimal when ∂B_s and $\partial B(x,R)$ meet perpendicularly, and hence by orthogonality $R^2 = s^2+\xi^2$. Then for any $s \in (\xi-R, \xi+R)$

$$\cos\beta_s \ge \cos\beta_{\sqrt{\xi^2-R^2}} = \frac{\sqrt{\xi^2-R^2}}{\xi}.$$

Summarising we obtained

$$\begin{aligned} \nu_{\alpha,n}(B(x,R)) &\le \frac{\omega_{n-2}}{n-1}\frac{\xi}{\sqrt{\xi^2-R^2}}\int_{\xi-R}^{\xi+R} (s^{1-\alpha}\sin\beta_s)^{n-1}\frac{ds}{s^\alpha} \\ &= \frac{\omega_{n-2}}{n-1}\frac{\xi}{\sqrt{\xi^2-R^2}}\int_{\xi-R}^{\xi+R} F(s)^{(n-1)/2}\frac{ds}{s^\alpha}, \end{aligned}$$

where

$$\begin{aligned} F(s) &= s^{2-2\alpha} \sin^2 \beta_s = s^{2-2\alpha} = s^{2-2\alpha} \left(1 - \left(\frac{s^2 + \xi^2 - R^2}{2\xi s}\right)^2\right) \\ &= s^{-2\alpha} \left(s^2 - \frac{(s^2 + \xi^2 - R^2)^2}{2\xi}\right). \end{aligned}$$

Following the same reasoning for the intersection with B_r we have

$$\begin{aligned} \nu_{\alpha,n}(B(x,R) \cap B_r) &= \int_{\xi-R}^{r} |\partial B_s \cap B(x,R)|_{n-1} \, s^{-\alpha n} \, ds \\ &= \omega_{n-2} \int_{\xi-R}^{r} \int_0^{\beta_s} \sin^{n-2} \theta \, d\theta s^{(1-\alpha)(n-1)} \, \frac{ds}{s^\alpha}. \end{aligned}$$

Since

$$\int_0^{\beta_s} \sin^{n-2} \theta \, d\theta \geq \int_0^{\beta_s} \sin^{n-2} \theta \, \cos \theta \, d\theta = \frac{\sin^{n-1} \beta_s}{(n-1)},$$

we have

$$\nu_{\alpha,n}(B(x,R) \cap B_r) \geq \frac{\omega_{n-2}}{n-1} \int_{\xi-R}^{r} F(s)^{(n-1)/2} \, \frac{ds}{s^\alpha}.$$

We claim that if $\alpha > 1/2$ there exist an appropriate choice for ξ and R so that F attains its maximum at a point $s = s_0 < \xi$. Then we have that

$$\begin{aligned} \nu_{\alpha,n}(B(x,R)) &\leq \frac{\omega_{n-2}}{n-1} \frac{\xi}{\sqrt{\xi^2 - R^2}} \int_{\xi-R}^{\xi+R} \frac{ds}{s^\alpha} \, F(s_0)^{(n-1)/2} \\ &= \frac{\omega_{n-2}}{n-1} \frac{\xi}{\sqrt{\xi^2 - R^2}} \frac{2R}{(\xi-R)^\alpha} \, F(s_0)^{(n-1)/2}. \end{aligned}$$

On the other hand for a small $\delta > 0$, if $s \in (s_0 - \delta, s_0 + \delta)$ we have that

$$F(s) = F(s_0) + \frac{F''(\tau)}{2}(s - s_0)^2,$$

for certain τ between s_0 and s. Writing $M = \max_{s \in [\xi-R, \xi+R]} |F''(s)|$ one has

$$\begin{aligned} F(s)^{(n-1)/2} &\geq \left(F(s_0) - \frac{M}{2}\delta^2\right)^{(n-1)/2} \\ &= F(s_0)^{(n-1)/2} \left(1 - \frac{M\delta^2}{2F(s_0)}\right)^{(n-1)/2}. \end{aligned}$$

If we choose

$$\delta \leq \sqrt{\frac{2F(s_0)}{(n-1)M}},$$

then
$$\left(1-\frac{M\delta^2}{2F(s_0)}\right)^{(n-1)/2} \geq \left(1-\frac{2}{n-1}\right)^{(n-1)/2} \geq 3^{-3/2} > 0,$$
for $n \geq 4$. Our claim that $s_0 < \xi$ allows us to take $r = s_0$ and then we have

$$\begin{aligned} \nu_{\alpha,n}(B(x,R)\cap B_r) &\geq \frac{\omega_{n-2}}{n-1}\int_{s_0-\delta}^{s_0} F(s)^{(n-1)/2}\,\frac{ds}{s^\alpha} \\ &\geq 3^{-3/2}\frac{\omega_{n-2}}{n-1}\int_{s_0-\delta}^{s_0} \frac{ds}{s^\alpha} F(s_0)^{(n-1)/2} \\ &\geq 3^{-3/2}\frac{\omega_{n-2}}{n-1}\frac{\delta}{s_0^\alpha}\,F(s_0)^{(n-1)/2}. \end{aligned} \tag{5.9}$$

Hence, recalling that by its definition $\delta \leq C/\sqrt{n-1}$ we have

$$\frac{\nu_{\alpha,n}(B(x,R)\cap B_r)}{\nu_{\alpha,n}(B(x,R))} \geq C\delta \geq \frac{C}{\sqrt{n-1}},$$

where C depends on ξ, R and α but not on n.

To finish we have to justify the claim that for certain ξ and R we have $s_0 < \xi$ if $\alpha > 1/2$. To do so, notice that it is equivalent and easier to calculate the point t_0 where
$$G(t) = F(\sqrt{t}) = t^{-\alpha}\left(t - \frac{(t+\xi^2-R^2)^2}{2\xi}\right),$$
attains its maximum over $[(\xi-R)^2, (\xi+R)^2]$ and check that $t_0 < \xi^2$. An elementary study of G yields that the maximum in the mentioned interval occurs at the point

$$t_0 = \frac{(1-\alpha)(\xi^2+R^2) + \sqrt{(1-\alpha)^2(\xi^2+R^2)^2 + \alpha(2-\alpha)(\xi^2-R^2)^2}}{2-\alpha}.$$

Now it is easy to see that the inequality $t_0 < \xi^2$ holds if and only if

$$\frac{2\xi^2}{4\xi^2 - R^2} < \alpha < 2.$$

For each $\alpha \in (1/2, 1)$ there are adequate choices for ξ and R so that this inequality holds. $\square$

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