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Is the directional distance function a complete generalization of the Farrell approach?

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Abstract

Cost or revenue efficiency measurement based on the approach initiated by Farrell has received great attention from academics and practitioners since the fifties. Farrell's approach decomposes cost efficiency into two different sources, viz. technical efficiency and allocative efficiency. Technical efficiency is estimated by the implementation of the Shephard's input or output distance functions, while allocative efficiency is derived as a residual between cost or revenue efficiency and its corresponding technical efficiency component. The directional distance function (DDF) was introduced later in the literature to complete duality theory with respect to the profit function and as a generalization of the Shephard input and output distance functions. Considering the case of cost efficiency we show that, although the DDF correctly encompasses the technical efficiency component of the Farrell approach, this is not true for the allocative component. Additionally, we show that allocative inefficiency is underestimated when the DDF-additive approach is used for decomposing cost inefficiency unless technical efficiency is assumed.

Keywords: Technical efficiency, Allocative efficiency, Directional Distance Functions.

JEL: C61; D21; D24

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1. Introduction

Cost or revenue efficiency measurement based on the approach initiated by Farrell (1957) has received great attention from academics and practitioners. Since Farrell, researchers have analytically decomposed cost and revenue efficiency into technical efficiency and allocative efficiency. In the spirit of this decomposition, technical efficiency is first of all estimated resorting to radial movements, relating this particular component to both Debreu´s coefficient of resource utilization (1951) and the inverse of Shephard's input or output distance functions (Shephard, 1953). Secondly, in accordance with Farrell's approach, and resorting to duality theory, allocative efficiency is derived as a residual between cost or revenue efficiency and its corresponding technical efficiency component. As a result of this residual nature of the allocative efficiency term, where its technical efficiency counterpart is the driving component, the mathematical formulation of the estimation of the allocative efficiency component has received much less attention in the literature than that associated with technical efficiency.

While Farrell resorted to Shephard's distance function for his decomposition, nowadays there are alternative ways to estimate technical efficiency, e.g., the generalized distance function by Chavas and Cox (1999) and the directional distance function (DDF) by Chambers et al. (1996, 1998). In particular, the precursor of the latter is the benefit function, a notion introduced by Luenberger (1992) in consumer theory as a generalization of the willingness-to-pay concept, measured with respect to an arbitrary bundle of goods. Luenberger (1992) also introduced the shortage function, which is the analog for production theory of the benefit function defined for consumer theory. Later, Chambers et al. (1996, 1998) renamed the shortage function as the Directional Distance Function, DDF, highlighting its properties as distance function and its geometrical interpretation.

Focusing on the input side, Chambers et al. (1996) introduced the directional input distance function, which measures the amount that one can translate an 'observed' input vector from itself to the frontier of the technology in a pre-assigned reference direction vector. Additionally, when the reference direction is considered equivalent to the assessed input vector, the directional input distance function encompasses the Shephard input distance function. In this way, the directional input distance function constitutes the analytical tool that allows breaking the straight jacket represented by the classical and restrictive framework of the radial based decomposition of cost efficiency (Farrell, 1957), and extends it to the case of considering any possible direction in the input space as a reference. Specifically, Färe and Grosskopf (2003) show how cost inefficiency may be additively decomposed into technical inefficiency plus allocative inefficiency by using the directional input distance function. These features, among others, have yielded an increasing interest of researchers in the DDF in the last two decades.

In this paper we show that, in contrast to what is commonly assumed, Farrell's classical decomposition of cost efficiency is not completely encompassed by the approach based on the directional input distance function, except for the set of technically efficient observations. We also show that allocative inefficiency is underestimated when the DDF approach is used to decompose cost inefficiency for technically inefficient firms, which may result in wrong interpretations and lead to faulty managerial prescriptions if researchers rely on it to assess economic efficiency.

2. Preliminary notions, results and notation

In this section, we formalize some key notions about the technology and distance functions and recall how cost efficiency may be decomposed.

A technology is a set $T \subset R_{+}^{m} \times R_{+}^{s}$ satisfying several axioms; the most usual are the following (Färe and Primont, 1995): (A1) T is closed, (A2) Inputs and outputs are freely disposable, i.e. $(x,y) \in T$, $(x',y') \in R_{+}^{m} \times R_{+}^{s}$ and $x' \geq x$, $y' \leq y$ imply $(x',y') \in T$, (A3) There is no free lunch, i.e. $(0,y) \in T$ implies $y = 0$, (A4) Doing nothing is feasible, i.e. $(0,0) \in T$ and (A5) T is convex.

For the sake of brevity, we state our discussion in the input space, defining the input requirement set $L(y)$ as the set of inputs that can produce output $y \in R_{+}^{s}$, formally $L(y) = \{x \in R_{+}^{m} : (x,y) \in T\}$, and the isoquant of $L(y)$: $IsoqL(y) = \{x \in L(y) : \varepsilon < 1 \Rightarrow \varepsilon x \notin L(y)\}$.

Let us also denote by $C(y,w)$ the minimum cost of producing the output level y given the input market price vector $w = (w_1,...,w_m) \in R_{++}^{m}$: $C(y,w) = \min\left\{\sum_{i=1}^{m} w_i x_i : x \in L(y)\right\}$.

The standard (multiplicative) Farrell approach (Farrell, 1957) views cost efficiency (CE) as originating from technical efficiency (TE) and allocative efficiency (AE). Specifically, Farrell quantified, and therefore defined each of these terms as follows:

$$\underbrace{\frac{C(y,w)}{C(x)}}_{CE} = \underbrace{\frac{1}{D_i(y,x)}}_{TE} \cdot AE, \tag{1}$$

where $C(x) = \sum_{i=1}^{m} w_i x_i$ is the cost at x, $D_i(y,x) = \sup\{\delta > 0 : x/\delta \in L(y)\}$ is the Shephard input distance function (Shephard, 1953) and *AE* is defined residually as *AE=CE/TE*.

After Farrell's work, and particularly during the last two decades, part of the literature has focused on duality theory and distance functions, Chambers et al. (1996) with their directional input distance function being a good example. Let $g = (g_1,...,g_m) \in R_+^m$ be a vector such that $g \neq 0$, then the directional input distance function is defined as $\vec{D}_i(x,y;g) = \sup\{\beta : x - \beta g \in L(y)\}$. It is well-known that if $g = x$ then $\vec{D}_i(x,y;x) = 1 - 1/D_i(x,y)$, and from this relationship and the flexibility of g, the directional input distance function, which measures technical inefficiency, encompasses the Shephard input distance function in the input space. Moreover, Färe and Grosskopf (2003, p. 25, expression 1.44) showed that cost inefficiency (CI) may be additively decomposed into technical inefficiency (TI) plus allocative inefficiency (AI) relying on the directional input distance function and duality results:

$$\underbrace{\frac{\sum_{i=1}^{m} w_i x_i - C(y,w)}{\sum_{i=1}^{m} w_i g_i}}_{CI} = \underbrace{\vec{D}_i(x,y;g)}_{TI} + AI, \tag{2}$$

where *AI* is defined residually as *AI=CI –TI*.

3. The results

Since Färe and Grosskopf (1997), it is commonly accepted that the directional distance function encompasses the Farrell approach for decomposing cost efficiency.

However, both approaches are not completely equivalent since the allocative components in (1) and (2) do not have the appropriate mathematical relationship, something that does hold in the case of cost and technical terms. To show that, let us explicitly rewrite *TI* in (2) as $\vec{D}_i(x,y;g) = \left(\sum_{i=1}^{m} w_i \vec{D}_i(x,y;g) g_i\right) \Big/ \sum_{i=1}^{m} w_i g_i$. Then, from the relationship *AI*=*CI*–*TI*, *AI* can be expressed as $\left[\sum_{i=1}^{m} w_i\left(x_i - \vec{D}_i(x,y;g) g_i\right) - C(y,w)\right] \Big/ \sum_{i=1}^{m} w_i g_i$. Now, considering $g = x$ we have that

$$CI = \frac{\sum_{i=1}^{m} w_i x_i - C(y,w)}{\sum_{i=1}^{m} w_i x_i} = 1 - \frac{C(y,w)}{C(x)} = 1 - CE , \tag{3}$$

and

$$TI = \vec{D}_i(x,y;x) = 1 - 1/D_i(x,y) = 1 - TE . \tag{4}$$

However, if $g = x$, regarding the allocative inefficiency term in (2), we have that $AI = \frac{\sum_{i=1}^{m} w_i\left(x_i - \vec{D}_i(x,y;x) x_i\right) - C(y,w)}{\sum_{i=1}^{m} w_i x_i} = \frac{\sum_{i=1}^{m} w_i\left(x_i - \left(1 - 1/D_i(y,x)\right) x_i\right) - C(y,w)}{\sum_{i=1}^{m} w_i x_i} =$ $\frac{1}{D_i(y,x)} - \frac{C(y,w)}{C(x)} = \frac{1}{D_i(y,x)}(1 - AE)$, since by (1) $1 - AE = 1 - D_i(y,x)\left(\frac{C(y,w)}{C(x)}\right)$. Therefore, $AI \neq 1 - AE$ except for the case in which $D_i(y,x) = 1$. In other words, the allocative inefficiency estimated from (2) does not present the desired relationship: one minus the allocative efficiency estimated from (1), unless $(x,y) \in IsoqL(y)$. This fact has important implications when cost efficiency or inefficiency must be decomposed for technically inefficient firms by resorting to the traditional Farrell approach or to the directional input distance function approach, even in the case of selecting $g = x$. We illustrate this situation through Figure 1.

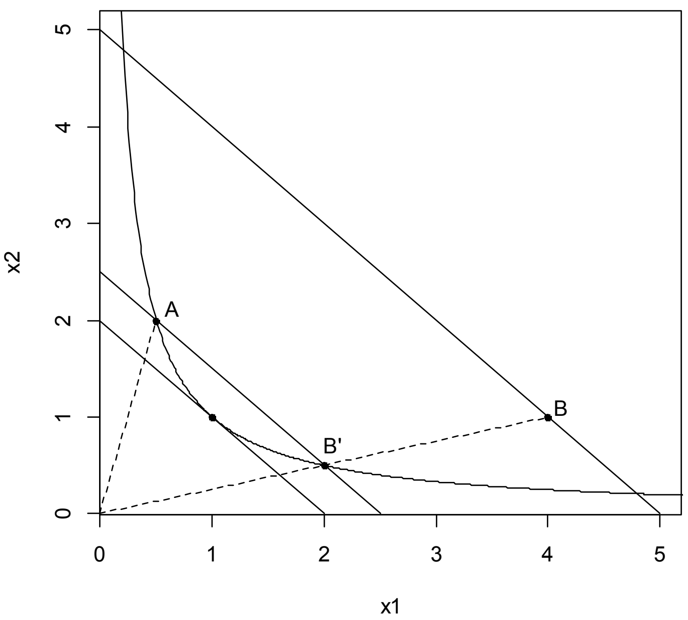

Figure 1. Graphical example.

Let us assume that $L(y)=\left\{x\in R_{+}^{2}:y=x_{1}^{1/2}x_{2}^{1/2}\right\}$ with $y\in R_{+}$. We also suppose that we have to evaluate the cost efficiency and its sources corresponding to points A=(0.5,2,1) and B=(4,1,1). Additionally, we assume that $w_1=w_2=1$. In this context, the Shephard input distance functions for A and B are $D_i(A)=1$ and $D_i(B)=2$, respectively. Furthermore by applying (1) we have that $AE_A=AE_B=2/2.5$, i.e., A and B present the same allocative efficiency level. In contrast, if we resort to the directional input distance function, we have that $AI_A=\frac{0.5}{2.5}>\frac{0.5}{5}=AI_B$. In this way, we show that the directional input distance function approach does not always preserve the order relationship determined by the Farrell decomposition for the allocative efficiency component.

In the situation described in Figure 1, Farrell's approach indicate that A and B have the same allocative efficiency. However, the DDF approach does not yield the same result. In the following proposition we characterize when the allocative inefficiency estimated by the DDF is coherent with the allocative efficiency determined by the Farrell method of decomposition.

Proposition 1. Let $x_A, x_B \in L(y)$ and $AE_A = AE_B$. Then, $AI_A = AI_B$ is equivalent to $C(x_A) = C(x_B)$.

Proof. (i) If $AI_A = AI_B$, by hypothesis and from $AI_A = \frac{1}{D_i(y, x_A)}(1 - AE_A)$ and $AI_B = \frac{1}{D_i(y, x_B)}(1 - AE_B)$, we have that $D_i(y, x_A) = D_i(y, x_B)$. Then, by (1), $C(x_A) = C(x_B)$.
(ii) If $C(x_A) = C(x_B)$, by hypothesis and using (1) we get that $D_i(y, x_A) = D_i(y, x_B)$. Finally, taking into account that $AI = \frac{1}{D_i(y, x)}(1 - AE)$, we have that $AI_A = AI_B$. ■

It is worth mentioning that in the numerical example associated with Figure 1, points A and B do not belong to the same isocost line, and therefore by propostion 1 the allocative inefficiencies estimated by the DDF approach cannot be the same, even in the case of $AE_A = AE_B$..

Finally, considering the traditional Farrell approach as the reference model to decompose cost efficiency, we highlight that the allocative component utilized by the directional input distance function approach always underestimates allocative inefficiency for technically inefficient firms, since $AI = \frac{1}{D_i(y, x)}(1 - AE)$ and $\frac{1}{D_i(y, x)} \in (0, 1)$ for these types of firms. This result questions the interpretation of AI as a consistent definition of an allocative inefficiency term, and by extension, the complete decomposition of cost efficiency as in (2). Indeed, the above results shows that there is a trade-off between technical efficiency TE, and allocative inefficiency, AI, as when technical efficiency increases

4. Conclusions

It can be concluded from our analysis that practitioners must keep in mind that in contrast to what has been commonly accepted, the choice between the traditional Farrell approach and the directional input distance function approach with $g = x$ for decomposing cost efficiency is not irrelevant, as both approaches are not equivalent. Additionally, regarding the bias associated to the DDF approach if the Farrell model is considered as benchmark, we have shown that the allocative inefficiency determined by applying the DDF is underestimated unless the evaluated firm is technically efficient.

The following three questions come to light. First, can we be sure that allocative inefficiency is not underestimated when one uses the directional input distance function approach for decomposing cost inefficiency with $g \neq x$? Second, this question extends to the decomposition of profit inefficiency resorting to the directional distance function (Chambers et al., 1998) in the input-output space: Is allocative inefficiency underestimated in this general context? Third and finally, it is still an open problem in the economic theory literature whether there exists a distance function that really generalizes the Farrell approach allowing, at the same time, measuring graph technical efficiency in the input-output space and presenting a natural dual relationship with the profit function or, alternatively, whether there is a way of correcting the directional distance function in order to provide a full generalization of the Farrell approach.

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