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Aggregating Judgments**



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A NON-PROPOSITION-WISE VARIANT OF MAJORITY VOTING FOR AGGREGATING JUDGMENTS

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ABSTRACT

Majority voting is commonly used in aggregating judgments. The literature to date on judgment aggregation (JA) has focused primarily on proposition-wise majority voting (PMV). Given a set of issues on which a group is trying to make collective judgments, PMV aggregates individual judgments issue by issue, and satisfies a salient property of JA rules—independence. This paper introduces a variant of majority voting called *holistic majority voting* (HMV). This new variant also meets the condition of independence. However, instead of aggregating judgments issue by issue, it aggregates individual judgments *en bloc*. A salient and straightforward feature of HMV is that it guarantees the logical consistency of the propositions expressing collective judgments, provided that the individual points of view are consistent. This feature contrasts with the known inability of PMV to guarantee the consistency of the collective outcome. Analogously, while PMV may present a set of judgments that have been rejected by everyone in the group as collectively accepted, the collective judgments returned by HMV have been accepted by a majority of individuals in the group and, therefore, rejected by a minority of them at most. In addition, HMV satisfies a large set of appealing properties, as PMV also does. However, HMV may not return any complete proposition expressing the judgments of the group on all the issues at stake, even in cases where PMV does. Moreover, demanding completeness from HMV leads to impossibility results similar to the known impossibilities on PMV and on proposition-wise JA rules in general.

Key-words: judgment aggregation, judgment aggregation correspondences, proposition-wise majority voting, holistic majority voting.

JEL classification: D70; D71

1 Introduction

The literature on judgment aggregation (JA) has highlighted the potentially serious problems that may result from proposition-wise majority voting (PMV). Pettit (2001) refers to the general fact that PMV may generate inconsistent collective judgments as *the discursive dilemma*. This inconsistency has been the main focus of critique lodged against the majority method in the JA literature, and has been the source of impossibility results on proposition-wise JA rules and the motivation for a search for other more suitable rules.¹ Let us illustrate this dilemma with the following, slightly modified variant of the example used in List (2009).

A multi-member government seeks to decide whether spending on health care should or should not be increased (say, whether b or not b). In order to make the decision, the government considers in addition the following two issues: (1) whether a budget deficit can be afforded (whether a or not a) and (2) whether the spending on health care should be increased in the case that a budget deficit can be afforded (whether ‘if a then b ’, or not ‘if a then b ’).

Let us assume first that the ministers’ points of view are as in Table 1.

Table 1

Support	$a?$	if a then $b?$	$b?$
1/3 of the ministers	a	if a then b	b
1/3 of the ministers	a	not ‘if a then b ’	not b
1/3 of the ministers	not a	if a then b	not b
Proposition-wise majority	a	if a then b	not b

The outcome given by PMV is logically inconsistent because the set of propositions $\{a, \text{‘if } a \text{ then } b\}$ entails the proposition b , while the set of the collective judgments includes the negation of proposition b .

In addition, this example illustrates another feature of PMV. This procedure may return a collective outcome that nobody in the group accepts. Moreover, the collective set of judgments $\{a, \text{‘if } a \text{ then } b\}, \text{‘not } b\}$ is logically inconsistent with the set of judgments

¹ The best-known examples of such rules are the premise-based rules, the conclusion-based rules, sequential priority rules, some quota rules (Dietrich and List, 2007) and distance-based rules (see List, 2009, or List and Puppe, 2009). More recently other proposals have been made such as new distance-based rules (Duddy and Piggins 2011), rules approximating the proposition-wise majority judgments (Nehring, Privato and Puppe, 2011), Borda rules (Zwicker 2011, Dietrich 2011) and scoring rules (Dietrich 2011).

accepted by each of the ministers. Thus, if the ministers behave in a logically correct way, any of them rejects that collective outcome.²

Suppose now that the ministers vote, knowing that their collective position will be made public in a press release. What would their press release be in such a case? Since the PMV outcome is the set of propositions $\{a, \text{'if } a \text{ then } b\}, \text{ not } b\}$, according to this JA method the press release should be a proposition logically equivalent to this set, namely, a proposition such as $\text{'}a, \text{ and if } a \text{ then } b, \text{ but not } b\text{'}$. However, it seems more probable that the press release would merely say $\text{'not } b\text{'}$, perhaps following a rhetorical clause saying that the government has deliberated at length on the question. If this is the case, the outcome will be different from that given by PMV and will coincide with the outcome obtained using the *conclusion-based* procedure.

Imagine now that, instead of the points of view in Table 1, the ministers have the following points of view and that PMV is again used.

Table 2

Support	$a?$	if a then $b?$	$b?$
1/3 of the ministers	a	not 'if a then b '	not b
2/3 of the ministers	not a	if a then b	not b
Proposition-wise majority	not a	if a then b	not b

Given that the majority outcome is consistent, the press release would, in this case, read something like this: the government agrees that, $\text{'if } a \text{ then } b; \text{ but not } a; \text{ and not } b\text{'}$. In other words, and in contrast with the former case, the press release would express the set of propositions given by PMV.

Let us summarize. In some cases, PMV works. In others, however, it gives outcomes that are seriously problematic or even unacceptable, such that it makes sense to use other JA procedures. This may mean that for the same family of judgment aggregation problems, different procedures would be used depending on the individual's judgments.

However, majority voting may be used in other ways as well. When groups arrive at collective judgments, they commonly use the rule of majority voting for texts dealing with diverse issues. For example, the United Nations Security Council proposes resolutions dealing with various issues but approves them based on a single vote. When a parliament passes a law, a similar procedure is used. The passage of a law depends on a single vote on a text previously elaborated according to statutory procedure. A collegial court collects votes on a text that, if agreed, would become the ruling and that includes both the verdict and the reasons under which it should be reached. Analogously, the public declarations of civil associations, whether scientific or cultural,

² Instead of considering this fact as a failure, List and Pettit (2011) view it as a relevant argument in favor of their theory on group agency.

often address a variety of issues. Such a declaration is made public if the group collectively passes the proposed text *en bloc*. In the government example above, majority voting may be used *en bloc* if the government, to determine its collective position and after some deliberation, collects votes on a text that, if collectively agreed, would become the press release. Obviously, it would not be difficult to come up with many other examples. The point is that aggregating judgments *en bloc* by majority voting is a common way of aggregating judgments in practice. Let us refer to this way of using majority voting as *holistic*.

It should be noted that HMV generally leads to a multiplicity of outcomes that are supported by some majority of individuals, so that it becomes necessary to choose one (or several) of those outcomes to express the collective point of view.

An additional and imaginary example that will be used again in Section 5 may help to clarify this and some other points. Each year in the city of Pamplona there are celebrations for the day of Saint Fermín. The most internationally famous element of these celebrations is the bullfighting. With this in mind, a civil association known as *Politeia* called a meeting to propose and approve a declaration about such celebrations. They discussed six issues:

- 1) ‘Bullfights are unjustified torture of the bulls’ (proposition *a* or, if not, proposition ‘not *a*’)
- 2) ‘Every unjustified torture of animals ought to be banned’ (proposition *b* or, if not, proposition ‘not *b*’);
- 3) In particular, ‘Bullfights should be abolished’ (proposition *c* or, if not, proposition ‘not *c*’);
- 4) ‘If they are not abolished, the cruelest parts, namely, placing barbed sticks in the bull and stabbing and killing the bull should be abolished’ (proposition *d* or, if not, proposition ‘not *d*’);
- 5) ‘The broadcasting of bullfights on television should be prohibited’ (proposition *e* or, if not, proposition ‘not *e*’);
- 6) ‘Children should be educated about this problem at school’ (proposition *f* or, if not, proposition ‘not *f*’).

Let us assume that the individual points of view are represented by the following three conjunctions, and that each of them is supported by 1/3 of individuals.<a

Table 3

Propositions expressing individual views	Support
(Radical view): <i>a</i> , and <i>b</i> , and <i>c</i> , and <i>d</i> , and <i>e</i> , and <i>f</i>	1/3 of persons
(Moderate view): not <i>a</i> , and <i>b</i> , and not <i>c</i> , and <i>d</i> , and <i>e</i> , and <i>f</i>	1/3 of persons
(Conservative view): not <i>a</i> , and not <i>b</i> , and not <i>c</i> , and not <i>d</i> , and not <i>e</i> , and not <i>f</i>	1/3 of persons

Applying the majority voting *en bloc*, eighteen propositions are supported by a majority of persons, namely: ‘*b* and *d*, and *e*, and *f*’, ‘*b* and *d*, and *e*’, ‘*b* and *d*, and *f*’, ‘*b* and *e*, and *f*’, ‘*d* and *e*, and *f*’, ‘not *a*, and not *c*’, ‘*b* and *d*’, ‘*b* and *e*’, ‘*d* and *e*’, ‘*b* and *f*’, ‘*d* and *f*’, ‘*e* and *f*’, ‘not *a*’, *b*, ‘not *c*’, *d*, *e*, and *f*. However, it can be assumed that in such a case the text ‘*b* and *d*, and *e*, and *f*’ would be chosen for expressing the association’s point of view and being made public.

Analogously, in the case depicted in Table 1 in the government example, there are three propositions accepted by some majority, namely *a*, ‘if *a* then *b*’, and ‘not *b*’. The proposition chosen to make the government point of view public is ‘not *b*’. In the case of Table 2, there are seven propositions accepted by some majority, and the proposition chosen for expressing the collective position is ‘if *a* then *b*; but not *a*; and not *b*’.³ Notice that the JA procedure is the same in both cases and does not need to be changed because the individual points of view change.

Examples of this sort suggest that such JA exercises can be modeled as processes consisting of two stages. In the first stage, the group carries out the aggregation of the individual judgments using HMV. This JA procedure gives the set of all the propositions addressing all or some of the issues at stake that are accepted by some majority in the group. This usually leads to obtaining several collective options. In the second stage, the collective option that best express the group’s point of view is chosen.⁴ In this paper, I introduce and analyze a model of this kind. In any case, it should be kept in mind that the analysis conducted in this paper focuses on the first stage, in which the judgment aggregation is carried out.⁵

As pointed out above, two salient and straightforward properties of HMV are that any proposition that obtains a majority of votes (1) is consistent, provided that individual judgments are consistent, and (2) is rejected by a minority of the individuals in the group at most. These facts guarantee that the proposition chosen for expressing the collective judgments is consistent and is not rejected by a majority of individuals in the group. This feature contrasts with the inability of the PMV (1) to guarantee the consistency of the collective judgments, and (2) to avoid that the collective judgments are rejected by everybody in the group.

³ The following propositions are accepted by some majority in the case of Table 2: (1) ‘not *a*, if *a* then *b*, and not *b*’; (2) ‘not *a*, and if *a* then *b*’; (3) ‘not *a*, and not *b*’; (4) ‘if *a* then *b*, and not *b*’; (5) ‘not *a*’; (6) ‘if *a* then *b*; and (7) ‘not *b*’.

⁴ The selection of a text for expressing the group’s point of view may be made in each case by different agents and in different ways, and may be far from straightforward or unproblematic. In particular, if the choice requires the aggregation of individual preferences, the familiar difficulties of preference aggregation may arise, and in addition, there is the risk of some sort of circularity or infinite regress because preference aggregation may be considered a special case of judgment aggregation. I owe this last remark to an anonymous referee.

⁵ The model proposed in this paper may also be interpreted as a JA procedure for solving JA problems such as those illustrated above.

It is known that PMV meets a broad set of appealing properties of JA rules, like those imposed on the conditions occurring in May's and Arrow's theorems. It can be easily verified that H MV meets the adapted variants of them.

In contrast with these facts, the press release case in the government example and the *Politeia* association example show that H MV may not always return a text that addresses all of the points at issue. When it does not, the requirement of full (logical) rationality is violated. Full rationality is conceived as the conjunction of logical consistency and completeness; in other words, a fully rational judgment aggregation procedure has to generate without fail consistent collective judgments that, in addition, address all of the issues at stake.

However, the requirement of completeness has been declining in importance compared to that of consistency. While it is true that completeness may be a compelling requirement in some kinds of situations, this is not always the case. Gärdenfors (2006), for instance, openly criticizes completeness as an unnatural requirement. Consequently, as Dietrich and List (2008) note, dropping the completeness requirement at the collective level is a frequently used means for avoiding the impossibility results obtained under the proposition-wise approach. In a similar vein, I do not consider completeness until Section 5.

In the next section, the basic notation is presented, the notion of H MV is formalized, and the choice function representing the second stage is introduced. Section 3 addresses the question of consistency. In Section 4, a characterization of H MV similar to that presented by May (1952) is given. In the same section, I present the adapted variants of some very well known properties of JA rules that are satisfied by H MV, and discuss which properties are transferred to the second-stage choice function. Section 5 focuses on the completeness requirement. It is shown there that if completeness is required of the collective outcome of the two-stage aggregation process, then impossibilities similar to those faced by PMV and proposition-wise JA rules arise. It is argued, in addition, that there are significant JA problems where completeness is expendable and H MV becomes a relevant JA rule. Section 6 is devoted to some concluding remarks.

2 Basic notions and notation

A judgment aggregation procedure generates collective judgments on the basis of the judgments made by the individuals in a group. Let $N = \{1, 2, \dots, n\}$ be a finite group of two or more persons.

2.1 Notation

Let us adopt standard propositional logic as the logical framework (for a more general logical framework, see Dietrich 2007). Specifically, the propositions in the agenda are represented in a language L , which contains (a) a given set of atomic propositions a, b, c, \dots , and (b) compound propositions with the logical connectives : \neg (not), \wedge (and), \vee (or), \rightarrow (if, then), \leftrightarrow (if and only if). Formally, L is the smallest set such that $a, b, c, \dots \in L$ and if $p, q \in L$, then $\neg p, (p \wedge q), (p \vee q), (p \rightarrow q), (p \leftrightarrow q) \in L$.

In this logical framework, a *truth-value assignment* is a function assigning the value 'true' or 'false' (or '0' and '1' respectively) to each proposition in L in a logically consistent way. A proposition $p \in L$ (a set of propositions Q), (a) is (*logically*) *consistent* if there exists a truth-value assignment for which p is true (for which any proposition in Q is true); (b) is (*logically*) *inconsistent* otherwise; (c) proposition p (the set of propositions Q) (*logically*) *entails* proposition q if q is also true for all truth-value assignments for which proposition p is true (for which any proposition in Q is true); (d) proposition p (the set of propositions Q) *entails* the set of propositions Q' if any proposition in Q' is true for all truth-value assignments for which proposition p is true (for which any proposition in Q is true); and (e) proposition p or the set of propositions Q , on one hand, and proposition q or the set of propositions Q' , on the other, are (*logically*) *equivalent*, if each of them entails the other.

The *agenda* $X = \{p^1, \neg p^1, p^2, \neg p^2, \dots, p^m, \neg p^m\}$ is a finite non-empty subset $X \subseteq L$ consisting of consistent non-negated propositions and their negations. We assume also that the agenda does not contain any twice-negated propositions (i.e., if a non-negated proposition p is in the agenda, then $\neg\neg p \notin X$), and that it is not trivial.⁶ Let us call $\sim p$ the *complementary* proposition of p , and p the *complementary* proposition of $\sim p$, where $\sim p = \neg p$ if p is not itself a negated proposition, and $\sim p = q$ if p is the negated proposition $\neg q$.

Usually, the agenda is conceived as the set of all the propositions under consideration on which judgments are to be made, and by means of which judgments are to be expressed. However, to enable that judgments can be aggregated *en bloc*, we need a broader set of propositions that contains conjunctions of agenda propositions. Let us introduce for this purpose the set Ω that contains (1) all the propositions in the agenda X , and (2) all the conjunctions p formed, in the order set up in the agenda, by any set of agenda propositions in such a way that (a) if an agenda proposition q occurs in p , then

⁶ An non-trivial agenda contains at least two propositions p and q such that p is neither logically equivalent to q nor to $\neg q$.

its complementary proposition $\sim q$ does not occur in p , and (b) no agenda proposition occurs twice in p . Let us address Ω as the *set of all the propositions under consideration*.

A proposition $p \in \Omega$ is *complete* if for any pair $q, \sim q \in X$, q or $\sim q$ occurs in p . Let \bar{X} be the set of all the complete propositions in Ω , and let $\tilde{X} \subseteq \bar{X}$ the set of those complete propositions that are consistent.

Given any set of propositions $Q \subseteq \Omega$, let us refer to the conjunction consisting, in the order established in the agenda, of all the agenda propositions that occur in some proposition in the set Q as the *conjunctive extension* of Q . It should be noted that for any set of propositions $Q \subseteq \Omega$, (1) its conjunctive extension is in Ω , and that (2) Q and its conjunctive extension are logically equivalent.

2.2 Holistic majority voting

It is usually assumed that individuals make judgments on every pair $p, \sim p$ of propositions in the agenda and that, in addition, those judgments are consistent with each other. In the same vein, let us suppose that the point of view of any person i in the group is expressed by a proposition $p_i \in \tilde{X}$, that is, by a proposition that is complete and consistent. We denote by Π the set, usually known as the *universal domain*, of all such individual judgment profiles $\pi = (p_1, p_2, \dots, p_n)$, where $p_i \in \tilde{X}$ for any $i \in N$. This way, given an individual judgment profile $\pi = (p_1, p_2, \dots, p_n) \in \Pi$ and a proposition under consideration $q \in \Omega$, $N_q = \{i \in N: p_i \text{ entails } q\}$ is the set of all group members who accept proposition q , and $|N_q|$ is the number of people in this set. A direct consequence of all this is that if $N_q \neq \emptyset$, then q is consistent.

To represent JA procedures that aggregate judgments *en bloc*, let us introduce a new kind of JA rules, which we call *JA correspondences* (JACs). A *JA correspondence* C assigns to each profile π in the universal domain Π a set of propositions $C(\pi) \subseteq \Omega$.⁷ In particular, *holistic majority voting* (HMV) is the JA correspondence C such that for any profile $\pi \in \Pi$, $C(\pi) = \{p \in \Omega: |N_p| \geq \lceil (n+1)/2 \rceil\}$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x . It should be noted that HMV is defined as a universal correspondence, that is, as a JAC with the universal domain.

Notice, in addition, that the outcome $C(\pi)$ returned by HMV is the set of all the propositions under consideration that the group decides to accept. Since Ω is the set of all the propositions under consideration, this implies that if $p \in \Omega$ is a proposition such that $p \notin C(\pi)$, then p is rejected by the group.

⁷ JACs are analogous aggregation rules to Dietrich's multi-valued aggregation rules (Dietrich 2011).

Issue-by-issue JA rules are usually represented as JA functions (JAFs). A *JA function* F assigns to each profile π in the universal domain Π a set of propositions $F(\pi) \subseteq X$. In this way, PMV is represented as the function F such that for any profile $\pi \in \Pi$, $F(\pi) = \{p \in X : |N_p| \geq \lceil (n+1)/2 \rceil\}$.

Given a JA correspondence C , consider the function C_X that assigns to each profile π in the universal domain Π the set of propositions $C(\pi) \cap X$. We call it the *restriction of C on the agenda*. Obviously, C_X is a JAF.

2.3 Choosing collective judgments

In the two-stage model, the aggregation stage is represented by a JA correspondence C , and the second stage of the aggregation process, that is, the *choice stage*, is represented by a *choice function* h that for any profile $\pi \in \Pi$, if the solution set $C(\pi)$ is not empty, then h chooses one proposition from that set; in symbols, if $C(\pi) \neq \emptyset$ then $h(\pi) \in C(\pi)$. Let us call h the *judgment choice function (JCF) based on the correspondence C* .

The question of which conditions this choice function should meet depends heavily on the aggregation problem at hand. This notwithstanding, we can assume that it meets some very general properties. In this vein, let us assume that choices are *not erratic*, in the sense that if $C(\pi) = C(\pi')$, then $h(\pi) = h(\pi')$. Let us assume also that choices are *coherent*, in a sense close to that of the weak axiom of revealed preference: if $C(\pi) \cap C(\pi') \neq \emptyset$, $p \in h(\pi)$ and $q \in h(\pi')$, then $p = q$ or $q \in C(\pi') \setminus C(\pi)$. Another criterion for choosing among collective judgments may be their *informative content*. In this regard, we may assume that for any proposition $p \in \Omega$, if there is another proposition $q \in C(\pi)$ such that q entails p and is not entailed by it, then $p \notin h(\pi)$. We do not go further in analyzing this choice stage.

3 The consistency question

The introduction of JACs raises the question of which rationality conditions may be reasonably imposed on them. As pointed out in the introduction, completeness is no longer universally recognized in the literature as an unconditionally binding requirement in every situation. For this reason, let us consider the aforementioned question in two steps, focusing first on consistency and deductive closure, and then considering completeness in Section 5.

The literature distinguishes between strong and weak consistency. Calling the condition of strong consistency usually required of JAFs '*strong consistency on the agenda X* ', let us refer to its adaptation to JACs as *strong consistency on the set Ω* .

Strong consistency on Ω (on X). Let C be a JAC. It is strongly consistent on the set Ω (on X) if, for any profile $\pi \in \Pi$, the set $C(\pi)$ (set $C(\pi) \cap X$) is consistent.

The doctrinal paradox illustrates that PMV and HMV fail to meet this requirement.

However, there is another consistency requirement that appears to be strongly significant under the holistic approach. The important condition here is that the proposition chosen in the second stage must be consistent. And this is guaranteed if any proposition in $C(\pi)$ is consistent. Let us address this weaker notion of strong consistency as *local consistency on Ω* .

Local consistency on Ω . Let g be a JAC or a JCF. Then g meets this condition if, for any profile $\pi \in \Pi$ in the domain of g , any $p \in g(\pi)$ is consistent.

Whether, in addition to meeting this condition, $C(\pi)$ is or is not consistent seems to be a less relevant question. Notice in addition that the conjunctive extension of the set C is not necessarily included in $C(\pi)$. This is the case, for instance, with the discursive dilemma. But if this is so, then demanding that the set $C(\pi)$ be consistent means requiring the consistency of a proposition that cannot be chosen in the second stage of the aggregation process. Moreover, if a JAC like HMV is used in such a case, then the conjunctive extension of $C(\pi)$ is rejected by the group.

With regard to deductive closure, let us call ‘*deductive closure on X* ’ the usual condition of deductive closure on JAFs, and let us call ‘*deductive closure on Ω* ’ its extension to the whole set of the propositions under consideration Ω .

Deductive closure on Ω (on X): Any JA correspondence C is deductively closed on the set Ω (on the agenda X) if, for any profile π in the domain of C , any set $Q \subseteq C(\pi)$ (any set $Q \subseteq C(\pi) \cap X$) and any proposition $p \in \Omega$ ($p \in X$), if Q entails p , then $p \in C(\pi)$.

Since $h(\pi)$ is a singleton, it makes no sense to require deductive closure of it. However, a relevant requirement is that if p is chosen to express the group’s point of view, then we know that any q under consideration that is entailed by p is also accepted by the group. Analogously, it is also a relevant requirement that if p is chosen for expressing the group’s point of view, then we know that any q inconsistent with p is rejected by the group. All this is guaranteed by the *condition of cross-consistency*, which in addition makes it unnecessary to use another condition for adapting the usual condition of weak consistency to this framework.

Cross-consistency on Ω (on the agenda X). Given a JA correspondence C , let $g=C$ (or let g be the JCF based on C). Then g meets this condition if, for any profile π in the domain of g , and any $p, q \in \Omega$ (any $p, q \in X$) such that $p \in g(\pi)$,

- (a) if p entails q , then $q \in C(\pi)$;
- (b) if, on the contrary, p entails $\neg q$, then $q \notin C(\pi)$.

Notice that cross-consistency implies local consistency.

Let us summarize. In addition to extending the usual conditions of strong consistency and deductive closure to the set Ω , two conditions—local consistency and cross-consistency—have been introduced. They are weaker than the former requirements, but under the holistic approach, they become the more suitable ones.

The important point is that in contrast to PMV, whose logical flaws have been emphasized repeatedly in the literature, HMV meets the two conditions introduced above as demonstrated by the following straightforward result.

Proposition 1. Let C be HMV, and let h be any JCF based on C . For any individual judgment profile $\pi = (p_1, p_2, \dots, p_n) \in \Pi$ and any propositions $p, q \in \Omega$,

- (a) if $p \in C(\pi)$, then p is consistent; thus, C and h are locally consistent on Ω ;
- (b) let $p \in C(\pi)$ or $p \in h(\pi)$; if p entails q , then $q \in C(\pi)$; and if p entails $\neg q$, then $q \notin C(\pi)$ and $q \notin h(\pi)$; thus, C and h are cross-consistent.

Proof of Proposition 1. All proofs are given in the appendix.

While HMV satisfies cross-consistency on Ω , it does not satisfy the stronger condition of deductive closure on Ω .

4 Characterization of holistic majority voting

May's famous theorem on preference aggregation states that the only universal and decisive preference aggregation function that is anonymous, neutral, and positively responsive is the majority method (May 1952).⁸ In regard to JA, Dietrich and List's (2010) Theorem 1 states that if a JAF is consistent, anonymous and acceptance/rejection-neutral (a property close to systematicity) on a somehow restricted domain, then it is PMV (restricted to that domain).⁹ In addition, it can easily be shown that PMV satisfies many other desirable properties.

However, it is also known that if the agenda is not extremely simplified, then PMV fails to meet strong consistency and deductive closure for some profiles of individual

⁸ May calls the preference aggregation functions 'group decision functions.'

⁹ Given a JA function F , Dietrich and List (2010) define the condition of *acceptance/rejection neutrality* in the following way: for any profiles π, π' in the domain of F and any proposition $p \in X$, if for any $i \in N$, $i \in N_p$ if and only if $i \in N'_p$, then $p \in F(\pi)$ if and only if $p \in F(\pi')$.

judgment sets.¹⁰ In contrast, I have shown that HMV is locally and cross-consistent. Moreover, I have argued that in situations like those depicted by our model, these conditions are more suitable requirements than the usual stronger conditions. Thus, the following question arises: Can HMV be characterized similarly to the characterization proposed by May or by Dietrich and List? Can it be shown that HMV also satisfies a large set of desirable properties? If this is the case, then the majority method should be acknowledged as a feasible and relevant *JA* method for aggregating judgments *en bloc*. This assessment runs contrary to the usual assessment of PMV.

Theorem 1 below is a variant of May's theorem.¹¹ It not only states that HMV satisfies some variants of the properties mentioned above in regard to May's theorem; it also states that HMV is the only one that satisfies them.

Anonymity on Ω (on X). Let C be a JAC or a JCF. For any proposition $p \in \Omega$ (for any $p \in X$) and any profiles π, π' in the domain of C that are permutations of each other, $p \in C(\pi)$ iff $p \in C(\pi')$.

Systematicity on Ω (on X). Let C be a JAC or a JCF. For any propositions $p, q \in \Omega$ (for any $p, q \in X$) and any profiles π, π' in the domain of C , if $N_p = N'_q$, then, $p \in C(\pi)$ iff $q \in C(\pi')$.

Notice that systematicity on Ω is a rather strong property because it states that propositions that may be completely different in all respects and therefore different in size should be treated the same if the subgroup of persons that support each of them is the same. However, the point here is that HMV satisfies it.

Positive responsiveness on X . Let C be a JAC. C satisfies this property, iff for any $p \in X$ and any profiles π, π' in the domain of C , if $N_p \subset N'_p$ and, in addition, $p \in C(\pi)$ or $\sim p \notin C(\pi')$, then $p \in C(\pi')$ and $\sim p \notin C(\pi')$.

Theorem 1. Let C be any JAC. C satisfies cross-consistency on X , anonymity on X , systematicity on Ω , and positive responsiveness on X , iff C is HMV.

In fact, HMV satisfies cross-consistency and anonymity on the set Ω . It also satisfies monotonicity on the set Ω , a property that the literature on *JA* generally prefers over positive responsiveness.

¹⁰ Specifically, PMV may always generate inconsistent collective judgments as soon as there is in the agenda an inconsistent set of three or more propositions such that any of its subsets is consistent.

¹¹ I follow May's approach instead of that of Dietrich and List because, while the latter leads to restricting the domain of the majority rule, the former allows us to exploit the fact that this rule meets the binding logical requirements in the whole universal domain.

Monotonicity on Ω (on X). Let C be a JAC or a JCF. For any $p \in \Omega$ (any $p \in X$), any $i \in N$ and any two i -variants profiles $\pi = (p_1, \dots, p_i, \dots, p_n)$, $\pi' = (p_1, \dots, p'_i, \dots, p_n)$ in the domain of C , if $i \notin N_p$, $i \in N'_p$ and $p \in C(\pi)$, then $p \in C(\pi')$, where two profiles are called i -variants if they coincide for all the individuals except possibly for i .

Analogously, it is easy to verify that HMV satisfies on Ω the adapted variants of the conditions in Arrow's theorem (*unanimity, independence, and non-dictatorship*).

Unanimity on Ω (on X). Let C be a JAC or a JCF. For any profile π in the domain of C and any proposition $p \in \Omega$ (any $p \in X$), if $N_p = N$, then $p \in C(\pi)$.

Independence on Ω (on X). Let C be a JAC or a JCF. For any proposition $p \in \Omega$ (any $p \in X$), and any profiles π, π' in the domain of C , if $N_p = N'_p$, then, $p \in C(\pi)$ iff $p \in C(\pi')$.

Non-dictatorship on Ω (on X). Let C be a JAC or a JCF. There is no individual $k \in N$ such that for any profile π in the domain of C and any proposition $q \in \Omega$ (any proposition $q \in X$), if $k \in N_q$, then $q \in C(\pi)$.

Since any dictator on Ω is a dictator on X , non-dictatorship on X implies non-dictatorship on Ω . Notice in addition that if C is systematic, then it is dictatorial on Ω iff it is dictatorial on X .

Besides the question of which properties HMV meets, it is also important to know which properties HMV transfers to any JC function h based on it. Notice in this respect that since HMV is anonymous, then h meets anonymity. Analogously, any JAC is dictatorial if some h based on it is dictatorial. Since HMV meets non-dictatorship, h meets this property as well.

In contrast, neither is h necessarily monotonic¹², nor does h necessarily meet unanimity.¹³ However, in regard to the latter, it happens that if $N_p = N$ and $q \in h(\pi)$, then q entails p , as can be easily verified. So, as soon as it is known that q is chosen for expressing the group's point of view, it becomes known that p is accepted by the group.

¹² The following example illustrates this fact. Let $X = \{p, \neg p, q, \neg q, r, \neg r, s, \neg s\}$ an agenda of contingent propositions such that any proposition in Ω is consistent. Imagine a group of five persons, with the following profile $\pi = (p_1, \dots, p_5)$, where $p_1 = (p \wedge \neg q \wedge r \wedge s)$, $p_2 = (p \wedge q \wedge \neg r \wedge \neg s)$, $p_3 = (p \wedge q \wedge \neg r \wedge \neg s)$, $p_4 = (\neg p \wedge q \wedge r \wedge s)$, and $p_5 = (\neg p \wedge \neg q \wedge \neg r \wedge s)$. Then, $C(\pi) = \{p, q, \neg r, s\}$. Assume that $h(\pi) = \{p\}$. Let $\pi' = (p'_1, \dots, p'_5)$ be a 5-variant of π such that for any $m = 1, 2, 3, 4$, $p'_m = p_m$, and $p'_5 = (p \wedge q \wedge r \wedge s)$. Then, $C(\pi') = \{r \wedge s, p, q, r, s\}$ and h may be such that $h(\pi') = \{r \wedge s\}$.

¹³ Let $X = \{p, \neg p, q, \neg q\}$ be an agenda of contingent propositions such that p and q are logically independent with each other, let $(p \wedge q)$ be supported by a non unanimous majority, and let $(p \wedge \neg q)$ be supported by the rest of the individuals in the group. Then $h(\pi) = \{p \wedge q\}$.

Analogously, h is not necessarily independent.¹⁴ Since systematicity implies independence, it follows that h is not necessarily systematic. However, it trivially satisfies this weaker version of systematicity (implying the analogous weaker version of independence): for any $p, q \in \Omega$ and any profiles π, π' in the domain of C , if $N_p = N'_q$ and $p \in h(\pi)$, then q is accepted by the group under the second profile, that is, $q \in C(\pi')$.

Should be any h based on HMV be systematic, or at least independent? In the two-stage model, two kinds of decisions are taken. First, propositions are accepted or rejected by the group. These collective decisions should depend on the degree of support reached by each proposition under consideration, so that conditions such as unanimity, systematicity, independence, and monotony become highly relevant. In the second stage, however, the aim of the decision is to properly express the collective point of view. Of course, it also seems suitable that the proposition chosen is accepted by the group. But other considerations may be also relevant, such as the informational content of the proposition chosen, for instance. The point is that the features of propositions that may be relevant in the second stage may not always go hand in hand with those focusing on their degree of support. Thus, requirements like independence, systematicity, monotonicity, and unanimity may be too strong with respect to JCFs.

Proposition 1, Theorem 1 and the remarks made on the properties that HMV and any h based on it additionally satisfy lead to a favorable assessment of HMV, provided that local and cross-consistency are accepted as the main logical restrictions on JACs. These results and remarks show that in addition to the aforementioned consistency requirements, and except for completeness, HMV satisfies the properties that are usually presented as desirable conditions on the JA rules. In addition, it has been shown that any JCF based on HMV meets anonymity, non-dictatorship and weaker variants of unanimity, independence and systematicity. These facts may explain why the majority method is frequently used for aggregating judgments *en bloc*. In any case, it may provide the basis for a favorable assessment of the feasibility of the majority method in contrast to the assessment of this method emerging from the proposition-wise JA approach.¹⁵

5 Demanding completeness

¹⁴ The following example illustrates this fact. Let $X = \{p, \neg p, q, \neg q, r, \neg r\}$ an agenda of contingent propositions such that any proposition in Ω is consistent. Imagine a group of five persons, with the following profile $\pi = (p_1, \dots, p_5)$, where $p_1 = (p \wedge q \wedge r)$, $p_2 = (p \wedge q \wedge \neg r)$, $p_3 = (p \wedge \neg q \wedge r)$, $p_4 = (\neg p \wedge \neg q \wedge \neg r)$, and $p_5 = (\neg p \wedge \neg q \wedge r)$. Then, $C(\pi) = \{p \wedge q, p, q, \neg r\}$. Imagine, in addition, that $h(\pi) = \{p \wedge q\}$. Let $\pi' = (p'_1, \dots, p'_5)$ be another profile such that $p'_1 = (p \wedge q \wedge \neg r)$, $p'_2 = (p \wedge q \wedge r)$, $p'_3 = (p \wedge \neg q \wedge \neg r)$, $p'_4 = (\neg p \wedge \neg q \wedge \neg r)$, and $p'_5 = (\neg p \wedge \neg q \wedge r)$. $C(\pi') = \{p \wedge q \wedge \neg r, p \wedge q, p \wedge \neg r, q \wedge \neg r, p, q, \neg r\}$. Given that $(p \wedge q \wedge \neg r)$ entails any other proposition in $C(\pi')$ and it is not entailed by any of them, then $h(\pi') = \{p \wedge q \wedge \neg r\}$, while $N_{p \wedge q} = N'_{p \wedge q}$.

¹⁵ On the results obtained under the proposition-wise approach weakening or dropping the completeness requirement, see List (2009, part. 5.2) or List and Puppe (2009: part 5.1).

Given the properties of any choice function based on a JA correspondence C , if there is a complete and consistent proposition p in $C(\pi)$, then p or any other proposition equivalent to it is chosen in the choice stage. The reason is that in that case, p implies any other proposition in $C(\pi)$. Thus, the most relevant completeness condition under our approach is that for any profile π in the domain of C , there is a complete proposition in $C(\pi)$. Let us call this condition *strong completeness*, because it is more demanding than the usual condition of completeness, which we call *completeness with respect to the agenda*.

Completeness with respect to X . Let C be a JAC or a JCF. For any profile π in the domain of C and any pair $p, \sim p \in X$, $p \in C(\pi)$ or $\sim p \in C(\pi)$.

Strong completeness. Let C be a JAC or a JCF. For any profile π in the domain of C , there is a complete proposition in $C(\pi)$.

The failure of HMV to meet strong completeness is its main fault. Notice in particular that if $|N|$ is odd, then PMV is complete, whereas HMV may not be strongly complete. In order for HMV to be strongly complete, it is necessary that a majority supports a complete proposition, and this may frequently not be the case.¹⁶

In addition, strong completeness leads to impossibility results on JACs similar to those that have been obtained on proposition-wise JAFs (on these latter results, see Dietrich and List, 2012).

A JAF is *fully rational on X* if it is strongly consistent on X and complete with respect to X , and is therefore deductively closed on X . In a similar way, let us say that a JAC is *sufficiently rational* if it meets local and cross-consistency on Ω and is, in addition, strongly complete.

Proposition 2. If C is sufficiently rational and C_X is its restriction on X , then C_X is fully rational.

Proposition 2 allows us to derive impossibility results on JACs from the impossibility results and the characterizations of impossibility agendas on proposition-wise fully rational JAFs. To illustrate this point, take, for instance, the following theorem of Nehring and Puppe (Theorem 3 in List and Puppe, 2009): There exist universal, fully rational JACs that meet unanimity on X , monotonicity on X , systematicity on X , and non-dictatorship on X , if and only if the agenda X has the median property, where X has

¹⁶ This does not mean that PMV behaves better than HMV in regard to completeness, because if PMV is complete with regard to the agenda, then HMV is also complete with regard to the agenda.

the median property if all minimal inconsistent subsets of the agenda X contain exactly two propositions.¹⁷

Assume now that a JA universal correspondence C is sufficiently rational and meets unanimity on Ω , monotonicity on Ω , and systematicity on Ω . Then, Proposition 1 implies that its restriction on the agenda C_X is fully rational. In addition, it can easily be verified that C_X meets unanimity on X , monotonicity on X , and systematicity on X . Then, according to Nehring and Puppe's theorem, if the agenda X does not meet the median property, there is a dictator on X under C_X . This means that there is a dictator k on X under correspondence C . Since C is systematic, then k is a dictator on Ω . Therefore C meets neither non-dictatorship on X nor non-dictatorship on Ω . Hence, we can conclude that if the agenda does not satisfy the median property, then there is no JA universal and sufficiently rational correspondence C that meets unanimity on Ω , monotonicity on Ω , systematicity on Ω , and non-dictatorship on X as well as on Ω .

Moreover, strong completeness allows us to obtain impossibility results on JACs that are similar to some results addressing JAFs, except that they are not conditional on the logical structure of the agenda. The following impossibility theorem illustrates this point.

Theorem 2. There is no universal JAC that meets cross-consistency on Ω , strong completeness, unanimity on Ω , systematicity on Ω , monotonicity on Ω , non-dictatorship on X and, therefore, non-dictatorship on Ω .

Given these results, the use of HMV would be justified only in those JA problems where strong completeness is not a compulsory requirement. Do such JA problems exist?

In some of the examples given in the introduction, such as the United Nations and the *Politeia* association, where the group must issue a public declaration, strong completeness is clearly expendable. However, these examples may raise the objection that the public declaration is of a rather rhetorical nature and does not have any normative force. The question is thus whether there are JA problems of a normative character where strong completeness is expendable.

Let us go back to the government example, which is of a normative character, from the beginning of the paper. The question is whether in cases like the one in Table 1 it is compulsory that the expression of the collective position takes place by means of a complete proposition, even if that proposition is rejected by all the ministers. I think that it is not, and that in such a case it is acceptable and would be accepted in practice that the government gives a press release like this: 'After deliberating on the issue, the government has decided not to increase the spending on health care'.

¹⁷ A minimal inconsistent set $Q \subseteq X$ is an inconsistent set such that all of its subsets are consistent.

Take another example. Imagine that the government assigns the parliament a law project that addresses the six issues in the *Politeia* example, substituting the word “will be” for the expression “should be” in propositions *c*, *d*, and *e*. Imagine in addition that the party’s positions are as in Table 4.

Table 4

Propositions expressing party’s views	Support
<i>a</i> , and <i>b</i> , and <i>c</i> , and <i>d</i> , and <i>e</i> , and <i>f</i>	1/2 of parliamentarians
not <i>a</i> , and <i>b</i> , and not <i>c</i> , and <i>d</i> , and <i>e</i> , and <i>f</i>	1/4 of parliamentarians
not <i>a</i> , and not <i>b</i> , and not <i>c</i> , and not <i>d</i> , and not <i>e</i> , and not <i>f</i>	1/4 of parliamentarians

Is it compulsory that if a law is passed, it must address the six issues at stake? It is not possible to fulfill this requirement without manipulation, even if a proposition-wise majority were used. Should the law project therefore be abandoned? I think that in such a case, the parliament would pass a law addressing a smaller set of issues, such as a law of this kind: ‘*b*, and *d*, and *e*, and *f*’. These examples strongly suggest that situations like those depicted by them, in which strong completeness is not a binding requirement, may be common.

In a similar vein, addressing the weaker condition of completeness with respect to the agenda, Dietrich and List (2009: 2) summarize the point in this way: ‘This [collective rationality] is a strong requirement, whose completeness part, in particular, has been criticized as being too demanding in many real-world decision-making settings (see especially Gärdenfors 2006, but also List and Pettit 2002, Dietrich and List 2007a, Goodin and List 2006). Often individuals and groups wish to abstain from making any judgment on certain proposition-negation pairs. Courts and expert panels, for example, may wish to abstain from making judgments on issues on which there is too much uncertainty, and legislatures and international decision-making bodies, such as the EU Council of Ministers or the UN Security Council, on issues on which there is too much disagreement.’

6 Concluding remarks

Like PMV, HMV satisfies many appealing properties. Any JCF based on HMV also satisfies some of these properties and weaker and acceptable variants of others. In addition, HMV and any JCF based on it also meet consistency requirements that, in JA settings like those depicted by the two-stage model introduced in this paper, are more suitable as conditions on JA rules than the usual conditions imposed on them.

In contrast, HMV and any JCF based on it fail to meet completeness. Moreover, if strong completeness is demanded, impossibility results on HMV and similar holistic JA

rules can be obtained, as in the case of proposition-wise JA rules. However, strong completeness is not always compulsory, even in JA settings of a normative character. Thus, HMV deserves to be treated as a feasible JA rule, which PMV does not.

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A Appendix

Proof of Proposition 1.

For (a). Since any individual only accepts consistent propositions, clause (a) holds whenever any proposition in $C(\pi)$ is accepted by at least one individual in the group. It is also straightforward that if C is HMV, then if $p \in C(\pi)$ then $N_p \neq \emptyset$. QED.

For (b). By definition, for any individual $i \in N$ and any individual judgment profile $\pi = (p_1, p_2, \dots, p_n) \in \Pi$, any individual i accepts a given proposition if and only if p_i entails it. Thus, if she accepts q , and in addition, q entails r , all this means that p_i entails q and q entails r . Therefore, p_i entails r by the transitivity of the entailment relation, and hence individual i accepts proposition r . Therefore, $N_q \subseteq N_r$. Thus, if C is HMV and h is a JCF based on it, and if $q \in C(\pi)$ or $q \in h(\pi)$, then $r \in C(\pi)$. If on the contrary, q entails $\neg r$, then $i \notin N_r$. Hence, $|N_r| \leq n - |N_q|$. Therefore, if C is HMV and h is a JCF based on it, and if $q \in C(\pi)$ or $q \in h(\pi)$, then $r \in C(\pi)$ and $r \in h(\pi)$. QED.

Proof of Theorem 1. The ‘ \Leftarrow ’ part is straightforward. Therefore, let us focus on the ‘ \Rightarrow ’ part.

Claim 1. The statement ‘for any profile $\pi \in \Pi^X$ and any proposition from the agenda $p \in X$, $p \in C(\pi)$ iff $|N_p| \geq \lceil (n+1)/2 \rceil$ ’ implies that for any profile $\pi' \in \Pi^X$ and any proposition under consideration $q \in \Omega$, $q \in C(\pi')$ iff $|N'_q| \geq \lceil (n+1)/2 \rceil$.

Proof of Claim 1. To verify this, let π' be any profile in the domain of C such that for some propositions $p \in X$ and $q \in \Omega$, $N_p = N'_q$. By universality, there are such profiles in the domain of C . Imagine that the statement ‘ $p \in C(\pi)$ iff $|N_p| \geq \lceil (n+1)/2 \rceil$ ’ holds. Given that $N_p = N'_q$, systematicity on Ω implies then that $q \in C(\pi')$ iff $|N'_q| \geq \lceil (n+1)/2 \rceil$. QED.

Thus, to prove the theorem, it suffices to prove that for any profile $\pi \in \Pi^X$ and any proposition $p \in X$, $p \in C(\pi)$ iff $|N_p| \geq \lceil (n+1)/2 \rceil$. This is done in Claim 6.

Claim 2. If $|N_p| = |N_{\sim p}|$, then, $p \notin C(\pi)$ and $\sim p \notin C(\pi)$.

Proof of Claim 2. If $|N_p| = |N_{\sim p}|$, then systematicity on Ω implies that $p \in C(\pi)$ iff $\sim p \in C(\pi)$. Thus, by cross-consistency on X , $p \notin C(\pi)$ and $\sim p \notin C(\pi)$. QED.

Claim 3. If $|N_p| > |N_{\sim p}|$, then $p \in C(\pi)$.

Proof of Claim 3. Take from the universal domain a profile π' such that $N'_p = N'_{\sim p}$ and $N'_p \subset N_p$. By Claim 2, $p \notin C(\pi')$ and $\sim p \notin C(\pi')$. Then by positive responsiveness on X , $p \in C(\pi)$ and $\sim p \notin C(\pi)$. QED.

Claim 4. $p \in C(\pi)$ and $\sim p \notin C(\pi)$, iff $|N_p| > |N_{\sim p}|$.

Proof of Claim 4. Notice that according to Claim 3, if $|N_p| > |N_{\sim p}|$, then $p \in C(\pi)$ and $\sim p \notin C(\pi)$. Let us prove, then, that if $p \in C(\pi)$ and $\sim p \notin C(\pi)$, then $|N_p| > |N_{\sim p}|$. Also by Claim 3, if $|N_{\sim p}| > |N_p|$, then $p \notin C(\pi)$ and $\sim p \in C(\pi)$. According to Claim 2, if $|N_p| = |N_{\sim p}|$, then $p \notin C(\pi)$ and $\sim p \notin C(\pi)$. Therefore, if $p \in C(\pi)$ and $\sim p \notin C(\pi)$, then $|N_p| > |N_{\sim p}|$. Hence, $p \in C(\pi)$ and $\sim p \notin C(\pi)$, iff $|N_p| > |N_{\sim p}|$. Q.E.D.

Claim 5. For any profile π and any proposition $p \in X$, $p \in C(\pi)$ iff $|N_p| \geq \lceil (n+1)/2 \rceil$.

Proof of Claim 5. If $p \in C(\pi)$, then cross-consistency on X implies that $\sim p \notin C(\pi)$. Thus, Claim 4 implies that $|N_p| > |N_{\sim p}|$. But $|N_p| + |N_{\sim p}| = n$, because p_1, p_2, \dots, p_n are all complete. Thus, we have $|N_p| \geq \lceil (n+1)/2 \rceil$. On the other hand, if $|N_p| \geq \lceil (n+1)/2 \rceil$ then $|N_p| > |N_{\sim p}|$. Hence, Claim 4 also implies that if $|N_p| \geq \lceil (n+1)/2 \rceil$ then $p \in C(\pi)$. QED.

Proof of Proposition 2.

Claim 1. If C is cross-consistent on Ω and strongly complete, then C_X is complete.

Proof of Claim 1. If C is strongly complete, then for any profile π there is in $C(\pi)$ a complete proposition p . Then C is complete because for any pair $q, \sim q \in X$, p implies one of them and, therefore, cross-consistency on Ω implies that $q \in C(\pi)$ or $\sim q \in C(\pi)$. QED.

Claim 2. If C is sufficiently rational, then C_X is strongly consistent on X .

Proof of Claim 2.

Let us assume, in contrast, that there is a set $Q \subseteq C_X(\pi) \cap X$ and a proposition $s \in C_X(\pi) \cap X$ such that $Q \cup \{s\}$ is an inconsistent set. By strong completeness, there is a complete and consistent proposition p in $C(\pi)$. Since p is a complete and consistent proposition, there is a proposition $t \in Q \cup \{s\}$ such that p entails $\sim t$, and does not entail t . Then, by deductive closure on Ω , $\sim t \in C(\pi)$, and by cross-consistency, $t \notin C(\pi)$. But then, it happens (1) that $s \notin C_X(\pi) \cap X$, or (2) that $C_X(\pi) \cap X$ does not include the set Q , contradicting the assumption that $Q \subseteq C_X(\pi)$ and $s \in C_X(\pi)$. QED.

Proof of Theorem 2.

Claim 1. $C(\pi) \cap \tilde{X}$ is a singleton.

Proof of Claim 1. By strong completeness, there is a complete proposition $p \in \tilde{X}$ such that $p \in C(\pi)$, for any profile π . In addition, cross-consistency on Ω implies that there is not another complete proposition in $C(\pi)$. QED.

Claim 2. Let Σ be the set of all the coalitions $S \subseteq N$ such that $S = \{i: p_i = q\}$ for a profile π , where $\{i: p_i = q\} \subseteq N$ is the coalition of individuals who accept the complete proposition $q \in C(\pi) \cap \tilde{X}$. If $S = \{i: p_i = q\}$ for a profile π , and there is another profile π' such that $p'_i = r$ for all $i \in S$, then $r \in C(\pi')$. Thus, Σ is the set of all the winning coalitions.

Proof of Claim 2. By systematicity on Ω . QED.

Claim 3. $N \in \Sigma$.

Proof of Claim 3. By unanimity on Ω . QED.

Claim 4. If $S \in \Sigma$, then $N \setminus S \notin \Sigma$.

Proof of Claim 4. There is a profile π in the universal domain, and two complete propositions $q \neq r$ such that for any $i \in S$, $p_i = q$, and for any $j \in N \setminus S$, $p_j = r$. Thus, $q \in C(\pi)$. But then, by Claim 1, $r \notin C(\pi)$ and, therefore, $N \setminus S \notin \Sigma$. QED.

Claim 5. $\emptyset \notin \Sigma$.

Proof of Claim 5. By claims 3 and 4. QED.

Claim 6. For any coalitions $S, S' \subseteq N$ such that $S \subseteq S'$, if $S \in \Sigma$ then $S' \in \Sigma$.

Proof of Claim 6. By monotonicity on Ω . QED.

Claim 7. If $S, S' \in \Sigma$, then $S \cap S' \neq \emptyset$.

Proof of Claim 7. Notice that if S and S' were disjoint, then $S' \subseteq N \setminus S$; thus, by Claim 6, $N \setminus S \in \Sigma$, contradicting Claim 4. QED.

Claim 8. If $S, S' \in \Sigma$, then $S \cap S' \in \Sigma$.

Proof of Claim 8. Let π be a profile such that any individual in $S \cap S'$ accepts the complete proposition $q^1 \in \tilde{X}$, any individual in $S \setminus S \cap S'$ accepts $q^2 \in \tilde{X}$, any individual in $S \setminus S \cap S'$ accepts $q^3 \in \tilde{X}$, and any individual in $N \setminus S \cup S'$ accepts $q^4 \in \tilde{X}$. It is possible that

$q^2=q^3=q^4$, or that $q^2=q^3 \neq q^4$, or that $q^2=q^4 \neq q^3$, or that $q^2 \neq q^3=q^4$. In any case, $q^1 \neq q^2$, $q^1 \neq q^3$, and $q^1 \neq q^4$.

Assume that $q^h \neq q^k$ for any $h, k=1, 2, 3, 4, h \neq k$. By strong completeness, there is a complete proposition $r \in \tilde{X}$ such that $r \in C(\pi)$. It should happen that $r=q^h$, for some $h=1,2,3,4$, because $\emptyset \notin \Sigma$. If $r=q^2$, then $S \setminus S' \in \Sigma$; but $S \setminus S'$ cannot be in Σ because $S' \cap S \setminus S' = \emptyset$ and Claim 7. If $r=q^3$, then $S' \setminus S \in \Sigma$; but $S' \setminus S$ cannot be in Σ because $S \cap S' \setminus S = \emptyset$ and Claim 7. If $r=q^4$, then $M \setminus S \cap S' \in \Sigma$; but $M \setminus S \cap S'$ cannot be in Σ because $S \cap M \setminus S \cap S' = \emptyset$. Therefore, $r=q^1$ and $S \cap S' \in \Sigma$. QED.

Claim 9. $\bigcap_{S \in \Sigma} S \in \Sigma$ and $\bigcap_{S \in \Sigma} S \neq \emptyset$.

Proof of Claim 9. The first part is a corollary of Claim 8. The second part follows from the first part and Claim 5. QED.

Claim 10. $\bigcap_{S \in \Sigma} S$ is a singleton and, therefore, the one individual in it is a dictator on Ω and, therefore, on X also.

Proof of Claim 10. Let us assume in contradiction that $\bigcap_{S \in \Sigma} S$ includes two or more individuals. Let be $\{S, S'\}$ a partition of N such that S includes some individual from $\bigcap_{S \in \Sigma} S$ and S' includes the rest of the individuals from $\bigcap_{S \in \Sigma} S$. Assume by universality that under profile π any individual in S accepts the complete proposition $p \in \tilde{X}$, while any individual in S' accepts the complete proposition $q \in \tilde{X}$, $p \neq q$. By strong completeness, $r \in C(\pi)$ for a complete proposition r . It must happen that $r=p$ or that $r=q$, because otherwise $\emptyset \in \Sigma$, contradicting Claim 5. Therefore, $p \in C(\pi)$ or $q \in C(\pi)$. If $p \in C(\pi)$, S becomes a winning coalition by systematicity on Ω ; but this cannot be the case because S does not include $\bigcap_{S \in \Sigma} S$. If $q \in C(\pi)$, the argument is analogous. Hence, $\bigcap_{S \in \Sigma} S$ is a singleton, and there is a dictator on Ω , who is also a dictator on X . QED.