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Analysis of biologically inspired Small-World networks

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Abstract. Small-World networks are highly clusterized networks with small distances between their nodes. There are some well known biological networks that present this kind of connectivity. On the other hand, the usual models of Small-World networks make use of undirected and unweighted graphs in order to represent the connectivity between the nodes of the network. These kind of graphs cannot model some essential characteristics of neural networks as, for example, the direction or the weight of the synaptic connections. In this paper we analyze different kinds of directed graphs and show that they can also present a Small-World topology when they are shifted from regular to random. Also analytical expressions are given for the cluster coefficient and the characteristic path of these graphs.

1 Introduction

Graph theory [1] provides the most adequate theoretical framework in order to characterize the anatomical connectivity of a biological neural network. To represent neural networks as graphs allows a complete structural description of the network and the comparison with different known connection patterns. The use of graph theory for modeling of neural networks has been used in theoretical neuroanatomy for the analysis of the functional connectivity in the cerebral cortex. In [2] it is shown that the connection matrices based on neuroanatomical data that describes the macaque visual cortex and the cat cortex, present structural characteristics that coincide best with graphs whose units are organized in densely linked groups that were sparsely but reciprocally interconnected. These kind of networks also provide the best support for the dynamics and high complexity measures that characterize functional connectivity.

There are some well known biological neural networks [3, 4] that present a clear clustering in their neurons but have small distances between each pair of neurons. These kind of highly clusterized, highly interconnected sparse networks are known as Small-World (SW) networks. SW topologies appear in many real life networks [6, 7], as a result of natural evolution [4] or a learning process [8]. In

[5] it is shown that on SW networks coherent oscillations and temporal coding can coexist in synergy in a fast time scale on a set of coupled neurons.

In [9] a method to study the dynamic behavior of networks when the network is shifted from a regular, ordered network to a random one is proposed. The method is based on the random rewiring with a fixed probability p for every edge in the graph. We obtain the original regular graph for $p = 0$, and a random graph for $p = 1$. This method shows that the characteristic path length (the average distance between nodes measured as the minimal path length between them) decreases with the increasing value of p much more rapidly than the clustering coefficient (that average number of neighbors of each node that are neighbors between them) does. It was found that there is a range of values of p where paths are short but the graph is highly clustered.

As initial substrate for the generation of SW, the use of undirected and unweighted ring-lattices or grids is usually proposed [9]. They are used because these graphs are connected, present a good transition from regular to random, there are not specific nodes on them and model a high number of real networks. Biological or artificial neural networks are not accurately represented by these models as neural networks present a clear directionality and a different coupling in their connections. For these reasons is necessary to develop new models of regular networks that take in account the direction and the weight of the neuronal synapses and to explore if these models can also present a SW area when they are shifted from regular to random.

2 Models of networks

In the existing literature about SW, only graphs that are connected, sparse, simple, undirected and unweighted are considered. Even though the graphs that conform the previous conditions can be accurate models for many networks, there are many other networks where relations are often directed or can have a value associated with the connection. In biological neural networks the information flows mainly from the presynaptic neuron to the postsynaptic neuron and the connections between different neurons have different coupling efficacies. As pointed in [9] and [2] it is necessary to explore models of networks where the undirected and unweighted conditions are relaxed. It is not clear, a priory, that relaxing those two conditions, the resulting networks present a similar SW area when they are shifted from regular to random.

As initial substrate we are going to consider directed rings-lattices and grids both weighted and unweighted. These substrates are selected since they are connected, regular and do not have special nodes. The directed unweighted ring-lattices will be explored for two different distributions of the neighbors of each node. The forward-backward distribution makes each node to connect to neighbors that are both at its left and right sides. In the forward networks each node only connects with nodes that are at the right side. Ring-lattices are depicted in fig. 1.

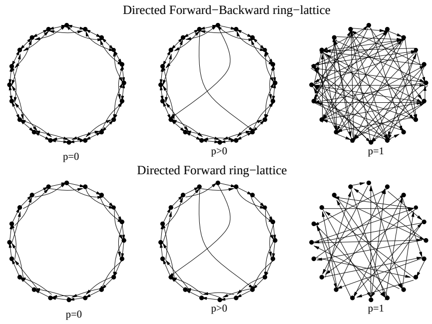

Fig. 1. Directed ring lattices

For grids we explore two different configurations. In the case of forward-backward distribution, each node connects with neighbors that are in each of the four possible directions in the grid. In the Forward networks each node only connect with nodes that are at its right and top sides.

In the case of weighted graphs we are going to consider a forward connection pattern. In our model the weight of each connection will follow an uniform random distribution w, with values $0 < w \leq 1$. Models of weighted graphs are depictured in fig. 2.

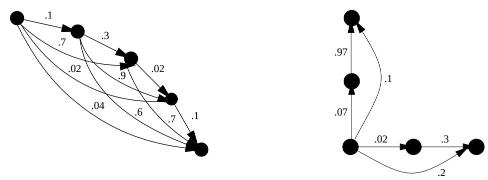

Fig. 2. Weighted graphs, left: Forward Ring lattice $k = 4$, right: Forward Grid $k = 4$

3 Length and Cluster scaling

In this section we give analytical expressions for both the cluster coefficient and the characteristic path for the regular graphs presented in the previous section.

We denote by n as the number of nodes in the graph and k as the number of neighbors of each node. For a directed graph we say that node b is a neighbor of node a if the edge (a, b) exists. Note that b neighbor of a does not imply a neighbor of b. The neighborhood of a node is the set of nodes that a given node is connected to.

Intuitively the cluster coefficient is the average number of neighbors of each node that are neighbors between them. More precisely, for a vertex v let us define $\Gamma(v)$ as the subgraph composed by the neighbors of v (not including v itself). Let us define the cluster coefficient for a given node v as $C_v = |E(\Gamma(v))|/(k_v(k_v-1))$ where $|E(\Gamma(v))|$ is the (weighted in the case of weighted graphs) number of edges in the neighborhood of v, k_v is the number of neighbors of v. The cluster coefficient for a given graph G can now be defined as

$$C = \frac{\sum_{i=1}^{n} C_i}{n}. \tag{1}$$

The characteristic path length of a graph indicates how far the nodes are among them. For a vertex v let us define its characteristic path length as $L_v = \sum_{i=1}^{n} d(v,i)/(n-1)$ where $d(v,i)$ indicates the (weighted in the case of weighted graphs) length of the shortest path connecting v and i. Using L_v we define the characteristic path length over a graph as

$$L = \frac{\sum_{i=1}^{n} L_i}{n}. \tag{2}$$

Simple counting for each type of graph provides with the following expressions for L and C:

– Unweighted Forward-Backward ring-lattices

$$L = \frac{n(n+k-2)}{2k(n-1)} = O(n) \quad C = \frac{3(k-2)}{2(k-1)} = O(1). \tag{3}$$

– Unweighted Forward ring-lattices

$$L = \frac{n(n+k-2)}{2k(n-1)} = O(n) \quad C = \frac{1}{2} = O(1). \tag{4}$$

– Unweighted Forward-Backward grids

$$L = \frac{2\sqrt{n}+k-4}{k} = O(n^{\frac{1}{2}}) \quad C = \frac{3(k-2)}{2(k-1)} = O(1). \tag{5}$$

– Unweighted Forward grids

$$L = \frac{2\sqrt{n}+k-4}{k} = O(n^{\frac{1}{2}}) \quad C = \frac{1}{2} = O(1). \tag{6}$$

– Weighted Forward ring-lattices

$$L = O(n) \quad C = O(1) \tag{7}$$

– Weighted Forward grids

$$L = O(n^{\frac{1}{2}}) \quad C = O(1) \tag{8}$$

The values of L for unweighted grids are obtained noting that $d(v_{ij}, v_{i'j'}) = d(v_{ij}, v_{ij'}) + d(v_{ij'}, v_{i'j'})$ and using the results for unweighted ring-lattices. The results for weighted graphs are due to the relation $\delta d_u(u,v) < d_w(u,v) < d_u(u,v)$ and $\delta > 0$ where d_w is the distance between two nodes in weighted graphs and

d_u is the distance between two nodes in unweighted graphs. We also use the fact that the weight for each connection has a value greater than 0.

In a random graph it can be shown that L scales as $log(n)$ and C scales to 0 as n tends to infinity [10]. This means that our models have a different scaling regime that the random graph model. This makes us expect that at some point, when we shift from these regular models to a random graph, there must be a phase change both in the value of L and C. If this change of phase is produced at different values of p for L and C we can build SW graphs using this regular substrates, and therefore, there exist SW graphs for these models.

4 Metrics behavior

In this section we explore the values of L and C when the models described in the previous sections are shifted from regular to random.

If we apply the procedure described in [11] to the models described in the previous sections we can observe that all the unweighted models present a clear SW area. For all of the unweighted models in consideration, there is a clear range of values of p where L is low but C maintains high. In figure 3 the normalized values of L and C are depictured for unweighted rings and grids.

In the case of weighted graphs, grids also present a clear SW area, but rings present a significative smaller range of values where L is low but C is high. This is due to the low number of edges rewired in the ring-lattice for low values of p and to the fact that in the case of weighted graphs a rewired edge can make paths longer. When a small amount of edges are rewired, path length keeps almost constant, with a very small decrement. Rings with a higher number of nodes produce a wider SW area. The normalized values of L and C for weighted directed ring-lattices and grids can be seen in fig. 3.

5 Conclusions

The previous results allow us to establish the following conclusions.

- There exist regular, directed and weighted substrates that present a similar scaling behavior that undirected unweighted equivalent substrates.
- These new substrates have the directionality present in biological and artificial neural networks.
- The directed substrates present a clear SW area when they are shifted from regular to random.
- Weighted substrates need a higher number of nodes in order to present a clear SW area for rings.

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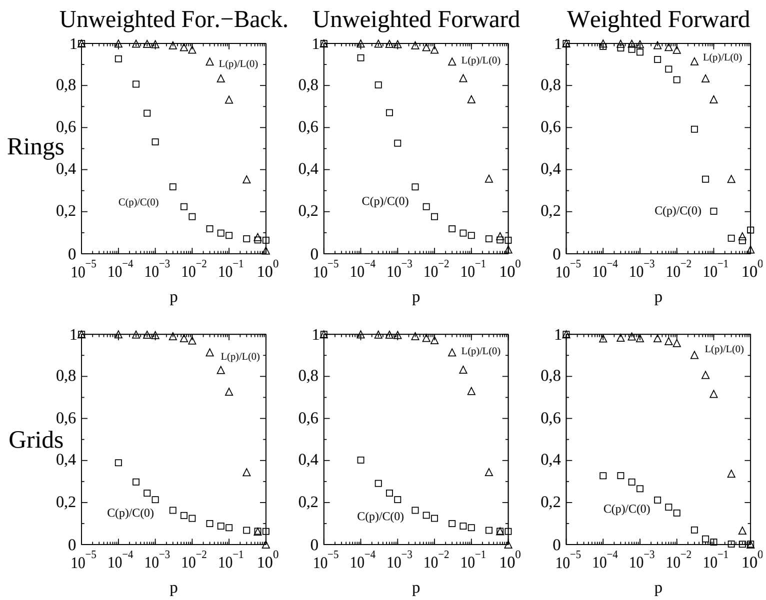

Fig. 3. Normalized values of L (squares) and C (triangles). Ring-lattice $n = 1000$, $k = 10$. Grids $n = 250000$, $k = 12$. Plots are the average of 100 experiments each

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